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A note on measures related to compactness and the Banach–Saks property in l_1

Andrzej Kryczka

Abstract. Using the Bessaga–Pełczyński selection principle, we give an alternative and concise proof of the results obtained by Tu (Arch Math, 117:315–322, 2021) that several quantities defined for bounded subsets of Banach spaces and related to compactness, weak compactness, and the Banach–Saks property coincide in l_1.

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1. Introduction. In a recent work [9], Tu proved that several quantities, which describe the deviation of bounded subsets of Banach spaces from compactness, weak compactness, and the Banach–Saks property, coincide in l_1. In this short note, we show that the results proved in [9] are almost immediate and direct consequences of the Bessaga–Pełczyński selection principle and the Schur property.

Due to the character and purpose of this paper, we give only basic facts on the notions we use. For more details, we refer the reader to [1,7–9].

Let X be a Banach space, $\mathsf{B}(X)$ the open unit ball of X, and D a bounded subset of X. The cardinality of a set $A \subset \mathbb{N} = \{1, 2, 3, \ldots\}$ will be denoted by $|A|$. The Hausdorff measure of noncompactness is given by

$$\chi(D) = \inf \{t > 0\colon D \subset K + t\mathsf{B}(X)\},$$

where the infimum is taken over all compact sets $K \subset X$. If the infimum in this formula is taken over all weakly compact sets K, we obtain the De Blasi measure of weak noncompactness $\omega(D)$. If X has the Schur property (norm and weak convergences coincide), $\chi(D) = \omega(D)$.

Another quantity describing the deviation from compactness is the separation measure of noncompactness

$$\beta(D) = \sup\{\operatorname{sep}(x_n)\colon (x_n) \subset D\},$$

where

$$\operatorname{sep}(x_n) = \inf_{m\neq n} \|x_m - x_n\|.$$

A counterpart of β for the weak topology uses the convex separation of a sequence,

$$\operatorname{csep}(x_n) = \inf_m \operatorname{dist}\left(\operatorname{conv}\{x_n\}_{n=1}^m, \operatorname{conv}\{x_n\}_{n=m+1}^\infty\right),$$

which is based on James' criterion [6]. The respective measure of weak noncompactness is given by

$$\gamma(D) = \sup\{\operatorname{csep}(x_n)\colon (x_n) \subset D\}.$$

The measure γ was defined in [8] with the supremum over all sequences in $\operatorname{conv} D$. By the quantitative extension of Krein's theorem proved in [5] and [8, Theorem 2.5], the supremum can be restricted to all sequences in D.

A modification of Beauzamy's condition [2] on the Banach–Saks property leads to the arithmetic separation of (x_n),

$$\operatorname{asep}(x_n) = \inf \left\| \frac{1}{m}\left(\sum_{n\in A} x_n - \sum_{n\in B} x_n\right)\right\|,$$

the infimum being taken over all $m \in \mathbb{N}$ and all finite subsets $A, B \subset \mathbb{N}$ with $|A| = |B| = m$ and $\max A < \min B$. The measure of deviation from the Banach–Saks property introduced in [7] is given by

$$\varphi(D) = \sup\{\operatorname{asep}(x_n)\colon (x_n) \subset D\}.$$

Since $\operatorname{csep}(x_n) \leqslant \operatorname{asep}(x_n) \leqslant \operatorname{sep}(x_n)$, we have $\gamma(D) \leqslant \varphi(D) \leqslant \beta(D)$.

Some applications of the above measures, among others, in metric fixed point theory and interpolation theory, can be found in [1,8].

2. Equality of measures. If X has a basis (e_n), then for each $n \in \mathbb{N}$, we define operators $P_n, R_n\colon X \to X$ such that $P_n x = \sum_{i=1}^n x(i)e_i$ and $R_n x = x - P_n x$ for $x = \sum_{i=1}^\infty x(i)e_i$. It is well known (see [1, Remark 4.4]) that for every bounded set $D \subset l_1$ and R_n taken with respect to the unit vector basis of l_1,

$$\chi(D) = \lim_{n\to\infty}\left(\sup_{x\in D} \|R_n x\|\right). \tag{2.1}$$

Let (p_n), (q_n) with $1 \leqslant p_1 \leqslant q_1 < p_2 \leqslant q_2 < \cdots$ be sequences of natural numbers. A sequence (f_n), where $f_n = \sum_{i=p_n}^{q_n} \alpha_i e_i$ is a nontrivial linear combination, is called a block basic sequence. In l_1, for all scalars $\lambda_1, \ldots, \lambda_m$,

$$\left\|\sum_{n=1}^m \lambda_n f_n\right\| = \sum_{n=1}^m |\lambda_n|\, \|f_n\|.$$

The following theorem includes the results of [9, Theorems 1 and 2]. Its concise proof is based on the classical result of Bessaga and Pełczyński [3] on locating block basic sequences.

Theorem. *Let D be a bounded subset of l_1. Then $\gamma(D) = \varphi(D) = \beta(D) = 2\chi(D) = 2\omega(D)$.*

Proof. Clearly, $\gamma(D) \leqslant \varphi(D) \leqslant \beta(D) \leqslant 2\chi(D)$. Since l_1 has the Schur property, $\chi(D) = \omega(D)$. Thus, if $\chi(D) = 0$, the equalities are established.

Assume that $\chi(D) > \varepsilon > 0$. By (2.1), there is a sequence $(x_n) \subset D$ such that $\chi(D) - \varepsilon \leqslant \|R_n x_n\|$ for all n. Let $x_n = (x_n(i))$. There exist a number $x(1)$ and a subsequence $(x_n^{(1)})$ of (x_n) such that $|x(1) - x_n^{(1)}(1)| < \varepsilon 2^{-1}$ for all n. Proceeding inductively for $i > 1$, we can find $x(i)$ and a subsequence $(x_n^{(i)})$ of $(x_n^{(i-1)})$ with $|x(i) - x_n^{(i)}(i)| < \varepsilon 2^{-i}$ for all n. We put $x = (x(i))$. Since D is bounded, $x \in l_1$. Fix $N \geqslant 1$ such that $\|R_N x\| < \varepsilon$ and let (x_{n_i}) be a subsequence of (x_n) such that $x_{n_i} = x_{N+i-1}^{(N+i-1)}$ for $i \geqslant 1$.

Let $v_i = P_{N+i-1} x_{n_i}$ and $z_i = R_{N+i-1} x_{n_i}$. Since $N + i - 1 \leqslant n_i$, we have $\|x - v_i\| < 2\varepsilon$ and $\|z_i\| \geqslant \inf_n \|R_n x_n\| > 0$ for each i. By the Bessaga–Pełczyński selection principle (see [4, p. 46]), there exist a subsequence (z_n') of (z_n) and a block basic sequence (f_n) taken with respect to the unit vector basis of l_1 such that $\|z_n' - f_n\| \leqslant \varepsilon$ for all n. Denote by (v_n') the corresponding subsequence of (v_n) and put $x_n' = v_n' + z_n'$ for all n. Then (x_n') is a subsequence of (x_n).

Let A, B be finite subsets of $\mathbb{N}$ with $\max A < \min B$. If $\sum_{n\in A} \alpha_n = \sum_{n\in B} \beta_n = 1$ and $\alpha_n, \beta_n \geqslant 0$ for all $n \in A \cup B$, then

$$\begin{aligned}
&\left\| \sum_{n\in A} \alpha_n x_n' - \sum_{n\in B} \beta_n x_n' \right\| \\
&\geqslant \left\| \sum_{n\in A} \alpha_n z_n' - \sum_{n\in B} \beta_n z_n' \right\| - \left\| \sum_{n\in A} \alpha_n v_n' - x \right\| - \left\| x - \sum_{n\in B} \beta_n v_n' \right\| \\
&\geqslant \left\| \sum_{n\in A} \alpha_n f_n - \sum_{n\in B} \beta_n f_n \right\| - 6\varepsilon = \sum_{n\in A} \alpha_n \|f_n\| + \sum_{n\in B} \beta_n \|f_n\| - 6\varepsilon \\
&\geqslant 2 \inf_n \|f_n\| - 6\varepsilon \geqslant 2 \inf_n \|z_n\| - 8\varepsilon \geqslant 2 \inf_n \|R_n x_n\| - 8\varepsilon \geqslant 2\chi(D) - 10\varepsilon.
\end{aligned}$$

It follows that $\operatorname{csep}(x_n') \geqslant 2\chi(D) - 10\varepsilon$. Consequently, $\beta(D) \geqslant \varphi(D) \geqslant \gamma(D) \geqslant 2\chi(D)$, which completes the proof. $\square$

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Andrzej Kryczka
Institute of Mathematics
Maria Curie-Skłodowska University
Lublin 20-031
Poland
e-mail: andrzej.kryczka@umcs.pl

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