Arch. Math. 120 (2023), 449–455

0003-889X/23/050449-7
published online March 19, 2023
https://doi.org/10.1007/s00013-023-01844-3

Archiv der Mathematik

On representations of direct products and the bounded generation property of branch groups

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Abstract. We prove that the minimal representation dimension of a direct product G of non-abelian groups $G_1, \ldots, G_n$ is bounded below by $n+1$ and thereby answer a question of Abért. If each G_i is moreover non-solvable, then this lower bound can be improved to be $2n$. By combining this with results of Pyber, Segal, and Shusterman on the structure of boundedly generated groups, we show that branch groups cannot be boundedly generated.

Mathematics Subject Classification. Primary 20E08; Secondary 20E26.

Keywords. Bounded generation, Branch group, Faithful representations of direct products.

Introduction. An infinite group G is called *just-infinite* if all of its proper quotients are finite. Obvious examples of just-infinite groups are virtually simple groups. Other examples arise from irreducible lattices in higher rank semisimple Lie groups, such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, after dividing out their centers, see [12, Chapter IV]. Such groups are in fact *hereditarily just-infinite*, which means that they are residually finite and all of their finite index subgroups are just-infinite. Grigorchuk's group [8] provided the first example of a just-infinite group that is *not* virtually a finite direct power of a simple or a hereditarily just-infinite group. Grigorchuk's group is a just-infinite *branch group*, which means that its commensurability classes of subnormal subgroups form a lattice that is isomorphic to the lattice of open and closed subsets of a Cantor set. By Wilson's classification [18], just-infinite groups fall into three classes. Every just-infinite group G is either a branch group or virtually a direct power of a simple or a hereditarily just-infinite group.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - 441848266.

Since its introduction by McCarthy [13] in the late 1960s, the class of just-infinite groups remained an active field of research. One reason might be that every finitely generated infinite group admits a just-infinite quotient. Thus whenever there is some finitely generated, infinite group G that admits a property $\mathcal{P}$ that is preserved under homomorphic images, then there is also a finitely generated just-infinite group with $\mathcal{P}$. Following [10], we call a property $\mathcal{P}$ that is preserved under homomorphic images an $\mathcal{H}$-property. Well-known examples of $\mathcal{H}$-properties include amenability, property (T), bounded generation, being a torsion group, having subexponential growth etc. In view of Wilson's classification, it is natural to investigate which of the three classes of just-infinite groups contain groups that satisfy a given $\mathcal{H}$-property $\mathcal{P}$. For the $\mathcal{H}$-property "being a torsion group", this question is settled. In this case, it is known that there are finitely generated simple groups [2], just-infinite branch groups [8], and hereditarily just-infinite groups [7] that are torsion. On the other hand, there are torsion-free, finitely generated, just-infinite groups that are simple [11], branch [3], and hereditarily just-infinite (e.g., $\mathbb{Z}$).

The purpose of this note is to study this question for the bounded generation property. Recall that a group G is *boundedly generated* if it contains a finite subset $\{g_1, \ldots, g_n\}$ such that every $g \in G$ can be written as $g = g_1^{k_1} \cdots g_n^{k_n}$ for appropriate $k_i \in \mathbb{Z}$. Since infinite torsion groups are not boundedly generated, it follows that each of the three classes of just-infinite groups contains a finitely generated group that does not have the bounded generation property. On the other hand, it was proven by Carter and Keller [6] that $\mathrm{PSL}_n(\mathbb{Z})$ is boundedly generated for $n \geq 3$, which provides an interesting boundedly generated hereditarily just-infinite group. The existence of boundedly generated, infinite, simple groups was established by Muranov [14], whose construction seems to be the only one available at present. It remains to study just-infinite branch groups. The question of existence of boundedly generated just-infinite branch groups was raised by Bartholdi, Grigorchuk, and Šunić [10, Question 12] and remained open to the best of our knowledge. The purpose of the paper is to show that the answer is negative for arbitrary branch groups (even without the assumption of being just-infinite).

Theorem 1. *There is no boundedly generated branch group.*

As a consequence, it follows from Wilson's classification of just-infinite groups that every boundedly generated infinite group has a quotient that is virtually a product of finitely many copies of a boundedly generated simple or hereditarily just-infinite group. The proof of Theorem 1 is a rather direct combination of results of Pyber and Segal [15], Shusterman [16], and Abért [1].

Abért proved that weakly branch groups are not linear over any field (for branch groups, this result is due to Grigorchuk and Delzant). More precisely, he defined for every field k, the natural number $\mathrm{mat}_k(n)$ to be the minimal r such that every graph on n vertices can be represented in the matrix algebra $\mathrm{M}_{r,r}(k)$ where the graph's edges encode non-commutation. Abért showed that $\sqrt{\lfloor n/2 \rfloor} \leq \mathrm{mat}_k(n) \leq 2(n - \lfloor \log_2(n) \rfloor + 1)$ and asked for a linear lower bound

[1, Question 4]. The following theorem answers this question by considering the graph T_n that consists of $2n$ vertices and n disjoint edges. The choice of the graph T_n was motivated by an argument of Abért [1, Proposition 5] that tells us that the question can be reduced to that case.

Theorem 2. *Let k be a field and let $r \geq 1$. Suppose that there are $(r \times r)$-matrices $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathrm{M}_{r,r}(k)$ such that all pairwise commutators are trivial except for $[a_i, b_i] = a_i b_i - b_i a_i$ for all $i \in \{1, \ldots, n\}$. Then $r \geq n+1$.*

The lower bound in Theorem 2 is sharp. Let $\lambda \in k^{\times}$. Consider the matrices $a_i = I + E_{1,i+1}$, $b_i = I - \lambda E_{i+1,i+1} \in \mathrm{M}_{n+1,n+1}(k)$ for $i = 1, \ldots, n$, where I is the identity matrix and $E_{i,j}$ denotes the elementary matrix whose (i,j)-entry is 1 and all other entries are 0. Then a_i, b_i satisfy the assumptions of Theorem 2. If $\lambda \neq 1$, then a_i, b_i are invertible. If $|k| > 2$, this shows with [1, Prop. 5] that

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \mathrm{mat}_k(n) \leq n + 1.$$

The non-linearity of weakly branch groups follows since these groups contain infinite products of non-abelian groups. Let $\mu_k(G)$ denote the minimal dimension of a faithful, finite dimensional representation of a group G over a field k (we write $\mu_k(G) = \infty$ if G is not linear over k). Theorem 2 directly implies a lower bound $\mu_k(G_1 \times \cdots \times G_n) \geq n + 1$ for direct products of non-abelian groups. Similarly, Theorem 2 provides lower bounds for representations of products non-commutative (Lie) algebras. If the factors G_i are assumed to be non-solvable, the lower bound can be improved further.

Theorem 3. *Let k be a field, let $G_1, \ldots, G_n$ be groups, and let $G = G_1 \times \cdots \times G_n$ denote their direct product.*

(1) If the groups $G_1, \ldots, G_n$ are non-abelian, then $\mu_k(G) \geq n + 1$.
(2) If the groups $G_1, \ldots, G_n$ are non-solvable, then $\mu_k(G) \geq 2n$.

Both lower bounds in Theorem 3 are sharp. Let $a_i = I + E_{1,i+1}$, $b_i = I - 2E_{i+1,i+1} \in \mathrm{GL}_{n+1}(\mathbb{Q})$ be as above. Setting $G_i = \langle a_i, b_i \rangle$, we can therefore deduce that $\mu_{\mathbb{Q}}(G_1 \times \cdots \times G_n) = n+1$. Suppose that the groups G_i in Theorem 3 are non-solvable subgroups of $\mathrm{GL}_2(k)$ for some field k. Then each G_i can be embedded in a 2×2-diagonal block in $\mathrm{GL}_{2n}(k)$, which gives us an embedding of $G = G_1 \times \cdots \times G_n$ in $\mathrm{GL}_{2n}(k)$. Together with Theorem 3, this implies $\mu_k(G) = 2n$. In particular, this applies to the case where each G_i is a non-abelian free group and thereby recovers [5, Theorem 3] in the SL_n-case.

1. Branch groups are not boundedly generated. There are several characterizations of branch groups. The following one, which is a slight reformulation of [10, Definition 1.1], does not involve a rooted tree which makes it rather abstract. However, it suits well for our purposes. A more geometric definition can be found in [10, Definition 1.13].

Definition. A group G is called a *branch group* if it admits a decreasing sequence of subgroups $(H_i)_{i \in \mathbb{N}_0}$ with $H_0 = G$ and $\bigcap_{i \in \mathbb{N}_0} H_i = 1$, and a sequence of integers $(k_i)_{i \in \mathbb{N}_0}$ with $k_0 = 1$ such that for each i, the following hold:

(1) H_i is a normal subgroup of finite index in G.
(2) H_i splits as a direct product $H_i = H_i^{(1)} \times \cdots \times H_i^{(k_i)}$, where the factors are pairwise isomorphic.
(3) The quotient $m_{i+1} := k_{i+1}/k_i$ is an integer with $m_{i+1} \geq 2$, and the product decomposition of H_{i+1} refines the product decomposition of H_i in the sense that each factor $H_i^{(j)}$ of H_i contains the factors $H_{i+1}^{(\ell)}$ of H_{i+1}, where ℓ satisfies $(j-1) \cdot m_{i+1} + 1 \leq \ell \leq j \cdot m_{i+1}$.
(4) G acts transitively by conjugation on the set of factors $H_i^{(j)}$ of H_i.

As indicated in the introduction, not every branch group is just-infinite. In fact, there is no need for finitely generated branch groups to admit a just-infinite quotient that is a branch group. See [17, Theorem 2] for an example of a finitely generated branch group that maps homomorphically onto $\mathbb{Z}$, which is of course just-infinite and boundedly generated. As a consequence, to prove that branch groups cannot be boundedly generated, it is not sufficient to consider the just-infinite case, in which the claim turns out to be a direct consequence of results of Abért [1], Pyber and Segal [15].

Proof of Theorem 1. Suppose there is a branch group G that is boundedly generated. Then [15, Corollary 1.6] tells us that G admits an epimorphism $\pi\colon G \to Q$, where Q is an infinite linear group. However, by [1, Corollary 7], branch groups are not linear over any field. Thus Q is a proper quotient of G. As such, Q is virtually abelian by [17, Proposition 6]. Since G, being a boundedly generated group, is finitely generated, it follows that Q has a (non-trivial) free abelian finite index subgroup Q_0. We can therefore consider the finite index subgroup $G_0 := \pi^{-1}(Q_0)$ of G, which by construction maps onto $\mathbb{Z}$. Let us now fix an arbitrary number $n \in \mathbb{N}$. From the definition of a branch group, it follows that G contains a finite index subgroup of the form $H_i = H_i^{(1)} \times \cdots \times H_i^{(k_i)}$, where the factors are pairwise isomorphic and $k_i \geq n$. Then $H_i \cap G_0$ is a finite index subgroup of H_i. In this case, it can be easily seen that there are pairwise isomorphic finite index subgroups $K_i^{(j)} \leq H_i^{(j)}$ such that $K_i := K_i^{(1)} \times \cdots \times K_i^{(k_i)} \leq H_i \cap G_0$. In particular, we see that K_i has finite index in G_0, which implies that it maps onto $\mathbb{Z}$. Thus some, and hence every, factor $K_i^{(j)}$ maps onto $\mathbb{Z}$. We can therefore deduce that the torsion-free part of the abelianization of K_i has rank at least $k_i \geq n$. As a consequence, this holds for every finite index subgroup of K_i. In particular, this tells us that there is no finite index subgroup of K_i that can be generated with less than n elements. Since $n \in \mathbb{N}$ was arbitrary, this contradicts a result of Shusterman [16, Theorem 1.1], which tells us that for every boundedly generated group H, there is a constant $C > 0$ such that every finite index subgroup of H contains a finite index subgroup that can be generated by at most C elements. $\square$

2. Lower bounds for the minimal representation dimension of directs products. Let us now prove the results concerning the minimal representation dimensions.

Proof of Theorem 2. For the proof, we combine ideas from [1] and [4]. Extending scalars, we may assume that k is an infinite field. Recall that $I \in \mathrm{M}_{r,r}(k)$

denotes the identity matrix. We claim that $I, a_1, \ldots, a_n, b_1, \ldots, b_n$ are linearly independent. This follows along the lines of [1, Proof of Thm. 3]. Suppose that $cI + \sum_j \lambda_j a_j + \sum_j \lambda'_j b_j = 0$ for $c, \lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n \in k$. Taking commutators with a_i (resp. b_i) shows $\lambda_i = 0$ (resp. $\lambda'_i = 0$); since $I \neq 0$, the last remaining coefficient c vanishes as well.

Let $V = k^r$ and let C denote the linear span of $\{I, a_1, \ldots, a_n, b_1, \ldots, b_n\}$ in $\mathrm{M}_{r,r}(k)$. Consider the linear map $\Psi\colon C \to V$ defined by $\Psi(X) = Xv$ for some $v \in V$. We will see that the image of Ψ has dimension at least $n+1$ if v is chosen appropriately. As the commutators $z_i = [a_i, b_i]$ are non-trivial, the kernel of each z_i is a proper subspace of V. However, V cannot be covered by a finite union of proper subspaces (as k is infinite). Thus there is a vector $v \in V$ such that $z_i v \neq 0$ for all $i \in \{1, \ldots, n\}$. Let $\alpha\colon V \to k$ be a linear form such that $\alpha(v) \neq 0$ and $\alpha(z_i v) \neq 0$ for all $i \in \{1, 2, \ldots, n\}$ (such a linear form α exists, as the dual space V^* cannot be covered by finitely many proper subspaces). Now $\beta\colon C \times C \to k$ defined by $\beta(x, y) = \alpha([x, y](v))$ is an alternating form on C. It is not difficult to see that β is non-degenerate on the subspace $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle \subseteq C$ (e.g., the matrix representation has full rank). Let us observe that $kI + \ker(\Psi)$ is an isotropic subspace since for $x, y \in kI + \ker(\Psi)$, we have $[x, y](v) = xyv - yxv = 0$. As $v \neq 0$, we have $I \notin \ker(\Psi)$ and thus $\dim_k \ker(\Psi) + 1 \leq n+1$. This allows us to conclude that

$$r \geq \dim_k(\mathrm{im}(\Psi)) = 2n + 1 - \dim_k \ker(\Psi) \geq n + 1.$$

□

Proof of Theorem 3. The first assertion follows immediately from Theorem 2. Assume now that each G_i is non-solvable. If G is not linear, there is nothing to show. Assume that (ρ, V) is a finite dimensional faithful representation over k. By extension of scalars, we may assume that k is algebraically closed. Let $V^1, \ldots, V^t$ denote the composition factors of V considered as G-modules. Since k is algebraically closed, the composition factor V^j is isomorphic to a tensor product

$$V^j = V_1^j \otimes_k V_2^j \otimes_k \cdots \otimes_k V_n^j$$

where V_i^j is an irreducible G_i-representation; see, e.g., [9, Prop. 2.3.23]. The composition factors of $V|_{G_i}$ are the irreducible representations $V_i^1, \ldots, V_i^t$ each one possibly occurring several times. Suppose for a contradiction that V_i^j is one-dimensional for all j. Then there is a basis of V such that $\rho(G_i)$ is represented by upper triangular matrices. This gives a contradiction since G_i is not solvable.

For each j, let $S_j \subseteq \{1, \ldots, n\}$ be the set of i such that $\dim_k V_i^j \geq 2$. By the observation above, each $i \leq n$ belongs to at least one of the sets S_j. This implies

$$\dim_k V = \sum_{j=1}^{t} \prod_{i=1}^{n} \dim_k V_i^j \geq \sum_{j=1}^{t} 2^{|S_j|} \geq \sum_{j=1}^{t} 2|S_j| \geq 2n.$$

□

Funding Information Open Access funding enabled and organized by Projekt DEAL.

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Received: 12 January 2023

Revised: 24 January 2023

Accepted: 7 February 2023