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Duals of Cesàro sequence vector lattices, Cesàro sums of Banach lattices, and their finite elements

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Abstract. In this paper, we study the ideals of finite elements in special vector lattices of real sequences, first in the duals of Cesàro sequence spaces ces_p for $p \in \{0\} \cup [1, \infty)$ and, second, after the Cesàro sum $\mathrm{ces}_p(\mathfrak{X})$ of a sequence of Banach spaces is introduced, where $p = \infty$ is also allowed, we characterize their duals and the finite elements in these sums if the summed up spaces are Banach lattices. This is done by means of a remarkable extension of the corresponding result for direct sums.

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1. Introduction. In the first part of the paper, guided by Bennett's approach, we consider the dual spaces d_q of the Cesàro sequence spaces ces_p for $1 < p < \infty$ where $1/q + 1/p = 1$. Equipped with the appropriate norm and the coordinatewise order, they are Dedekind complete Banach lattices with order continuous norm. Then of interest are the finite elements in these spaces. In the last years, these classes of finite elements, which were introduced in [15], are studied thoroughly by many authors in different Banach lattices, in particular, in Cesàro sequence spaces by the authors in [11]. The spaces d_q do not possess order units and all kinds of finite elements in them coincide with c_{00}. In the second part, similarly to the classical (c_0-, ℓ_p-, and ℓ_∞-) direct sums of Banach lattices in [18, §3.3.3] and [8,9], we introduce the so-called Cesàro sum for a sequence of Banach spaces, study their dual space, and characterize the

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finite elements if the summed up spaces are Banach lattices. In this paper, the Theorem 4.3 essentially extends the corresponding result for direct sums in [9].

2. Preliminaries. The aim of this section is to provide some necessary definitions and facts. For unexplained terminology concerning the vector lattices theory, the reader can consult the books [2,4,5,13].

An element φ in an Archimedean vector lattice E is called *finite* whenever there exists a so-called *majorant* $z \in E$ such that for any $x \in E$, there is a number $c_x > 0$ with the property that $|x| \wedge n\,|\varphi| \leq c_x\, z$ holds for any $n \in \mathbb{N}$. If z is a finite element, then φ is called *totally finite* and if $|\varphi|$ itself is a majorant, the element φ is called *selfmajorizing*. The sets of all finite and totally finite elements of an Archimedean vector lattice E are denoted by $\Phi_1(E)$ and $\Phi_2(E)$, respectively. All positive selfmajorizing elements are denoted by $S_+(E)$ and $\Phi_3(E) := S_+(E) - S_+(E)$. The collections $\Phi_i(E)$ are order ideals in E for $i \in \{1, 2, 3\}$ (see [18, Chapt. 3]).

We need some more notions and facts the details of which can be found in [4,5,18].

Definition 2.1. (a) An element $u \in E_+$, $u \neq 0$, of a vector lattice E is called an *atom* whenever $0 \leq x \leq u$, $0 \leq y \leq u$, and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$.

(b) An element $u \in E_+$, $u \neq 0$, of a vector lattice E is called *discrete*, whenever $0 \leq v \leq u$ implies $v = \lambda u$ for some $\lambda \in \mathbb{R}_+$.

(c) A vector lattice E is said to be *atomic* if for each $x > 0$, there exists an atom u such that $0 < u \leq x$.

In an Archimedean vector lattice E, a positive element is an atom if and only if it is a discrete element. Also if u is an atom in E, then $\{\lambda u : \lambda \in \mathbb{R}\}$ (the vector space generated by u) is a projection band. Each atom of a vector lattice is a totally finite element. Even more, for two elements a and x, one has

$$\frac{1}{n}(|x| \wedge na) \leq a,$$

and if a is an atom, then $|x| \wedge na = \lambda_n a \leq |x|$ follows for some $\lambda_n \in \mathbb{R}_+$. The Archimedean property implies $r_a(|x|) = \sup\{\lambda \in \mathbb{R}_+ \colon \lambda a \leq |x|\} < \infty$ for the atom a, which finally yields $|x| \wedge na \leq r_a(|x|)\, a$, i.e., the element a is selfmajorizing.

For a normed vector lattice E, denote by Γ_E the set of all atoms of E with norm 1. Then Γ_E is a subset of $\Phi_3(E)$. It consists of pairwise disjoint elements, and forms a linearly independent system. According to [18, Theorem 3.18] and the remark above, we have the following theorem.

Theorem 2.2. *Let E be a Banach lattice with order continuous norm. Then*

(i) $\Phi_3(E) = \Phi_2(E) = \Phi_1(E) = \operatorname{span}(\Gamma_E)$;

(ii) *$\Phi_1(E)$ is closed in E if and only if Γ_E is a finite set. In particular, $\Phi_1(E) = E$ if and only if E is finite dimensional.*

The following results will be used in the sequel.

Theorem 2.3 ([18, Proposition 3.44]). *If a vector lattice E has an order unit, then $\Phi_i(E) = E$, $i \in \{1, 2, 3\}$.*

Theorem 2.4 ([18, Theorem 3.15]). *Let E be a Banach lattice and $\varphi \in E$. Then the following statements are equivalent*:

(i) *φ is a finite element.*
(ii) *The closed unit ball $B_{\{\varphi\}^{dd}}$ of $\{\varphi\}^{dd}$ is order bounded in E.*
(iii) *$\{\varphi\}^{dd}$ has a generalized order unit, i.e., there exists $0 \le z \in E$ such that for any $x \in \{\varphi\}^{dd}$, there is a real number $\gamma_x > 0$ with $|x| \le \gamma_x z$.*

Theorem 2.5 ([18, Theorem 3.28]). *Let H be a projection band in a vector lattice E, and P_H the band projection from E onto H. Then $P_H(\Phi_1(E)) = \Phi_1(E) \cap H = \Phi_1(H)$.*

3. The dual space of $\mathbf{ces}_p$. The vector space $\mathbb{R}^{\mathbb{N}}$ which is partially ordered by the coordinatewise order is a Dedekind complete vector lattice, where the lattice operations are given coordinatewise: $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots)$ and $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots)$. As usual (cf. [4, Chapt. 13]) denote by ℓ_∞, ℓ_p, c_0, and c_{00} the spaces of bounded, p-summable, null, and finite sequences in $\mathbb{R}^{\mathbb{N}}$, respectively.

The Cesàro operator $\mathcal{C} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is given by

$$\mathcal{C}x = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n \in \mathbb{N}}$$

for all $x = (x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$. By using Hardy's inequality, for $x \in \ell_p$ $(1 < p < \infty)$, we have

$$\|\mathcal{C}x\|_p \le \|\mathcal{C}|x|\|_p \le \frac{p}{p-1} \|x\|_p .$$

So, the Cesàro operator $\mathcal{C} : \ell_p \to \ell_p$ is linear and continuous with operator norm $\frac{p}{p-1}$. For $1 \le p \le \infty$, the Cesàro sequence spaces are defined as

$$\mathrm{ces}_p := \{x \in \mathbb{R}^{\mathbb{N}} \colon \mathcal{C}|x| \in \ell_p\}$$

with the norm $\|x\|_{\mathrm{ces}_p} := \|\mathcal{C}|x|\|_p$.

The space ces_0 is defined as

$$\mathrm{ces}_0 := \{x \in \mathbb{R}^{\mathbb{N}} \colon \mathcal{C}|x| \in c_0\}$$

with the norm $\|x\|_{\mathrm{ces}_0} := \|\mathcal{C}|x|\|_\infty$.

Actually, ces_p and ces_0 spaces are Banach lattices. These spaces have been thoroughly studied by Leibowitz [14] and Shiue [17]. In particular, they showed that ces_p is trivial if $p = 1$. Since they are order ideals in $\mathbb{R}^{\mathbb{N}}$, the Dedekind completeness of the latter implies that all these vector lattices are also Dedekind complete.

The characterization problem of the dual space ces_p' of the Banach lattice $(\mathrm{ces}_p, \|\cdot\|_{\mathrm{ces}_p})$ of Cesàro sequence spaces for $1 < p < \infty$, which was offered by the Dutch Mathematical Society as a *prijsvrage* [1], was solved by Jagers [12]. A second solution, given by Bennet [6], provides a simple explicit isomorphic identification. To present Bennett's approach, first consider for $0 < q < \infty$ the set

$$d_q := \left\{ a = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \colon \left(\sup_{k\geq n} |a_k| \right)_{n\in\mathbb{N}} \in \ell_q \right\}.$$

For $q = 0$ and $q = \infty$, the above constructed sequences $\left(\sup_{k\geq n} |a_k|\right)_{n\in\mathbb{N}}$ are supposed to belong to c_0 and ℓ_∞, respectively. A simple calculation shows that $d_0 = c_0$ and $d_\infty = \ell_\infty$ what is not of interest.

Each sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ produces another sequence $\hat{a} = (\hat{a}_n)_{n\in\mathbb{N}}$ with $\hat{a}_n := \sup_{k\geq n} |a_k|$ called the *least decreasing majorant* of a. Clearly, $a \in d_q$ if and only if $\hat{a} \in \ell_q$ if and only if $\hat{a} \in d_q$.

Notice that d_q is a proper subset of ℓ_q for $1 \leq q < \infty$. For example, consider the sequence $a = (a_n)_{n\in\mathbb{N}}$, where

$$a_n = \begin{cases} \frac{1}{2^{k/q}} & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

For the case $q = 1$, there is $a = (0, \frac{1}{2^1}, 0, \frac{1}{2^2}, 0, 0, 0, \frac{1}{2^3}, \ldots)$ and $a \in \ell_1$ is clear. On the other hand, $\hat{a} = (\frac{1}{2^1}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots)$, i.e., $\hat{a}_n = \frac{1}{2^{m+1}}$ if $n > 2$ and satisfies $2^m < n \leq 2^{m+1}$, where $m \in \mathbb{N}$. Then it is clear that

$$\sum_{n=1}^{\infty} \hat{a}_n > \sum_{m=1}^{\infty} \left(2^m \frac{1}{2^{m+1}}\right) = \sum_{m=1}^{\infty} \frac{1}{2} = \infty, \text{ i.e.,} \, \hat{a} \notin \ell_1 \text{ and, hence } a \notin d_1.$$

For $q \geq 1$, under the norm

$$\|a\|_{d_q} := \|\hat{a}\|_q = \left(\sum_{n=1}^{\infty} \sup_{k\geq n} |a_k|^q \right)^{1/q},$$

d_q is a Banach space. In particular, $\|a\|_{d_q} = \|\hat{a}\|_q = \|\hat{a}\|_{d_q}$. Therefore, every eventually decreasing, non-negative sequence $a \in \ell_q$ is an element of d_q because $a_n = \hat{a}_n$ holds for all sufficiently large n. Hence, c_{00} is a proper subset of d_q, and so $c_{00} \subsetneqq d_q \subsetneqq \ell_q$.

The coordinatewise order makes the spaces d_q into Dedekind complete Banach lattices.

Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then there is a lattice isomorphism between ces_p' and d_q whose duality[1] with ces_p is stated explicitly via

$$\langle a, x \rangle := \sum_{n=1}^{\infty} a_n x_n, \quad x \in \mathrm{ces}_p, \; a \in d_q.$$

Moreover, the identification is even isometric when ces_p is endowed with another norm equivalent to $\|\cdot\|_{\mathrm{ces}_p}$. For details, see [6] and [7, Lemma 2.2 and Proposition 2.4]. Recently, Curbera and Ricker have proved the identification $\mathrm{ces}_0' = d_1$ with equality of norms [10, Lemma 6.2]. On the other hand, [3, Theorem 1] yields that $\mathrm{ces}_0'' = d_1' = \mathrm{ces}_\infty$ with equality of norms. Jagers [12] proved that ces_p is reflexive for $1 < p < \infty$. Therefore, by Pettis' theorem

[1] This will be denoted by $\mathrm{ces}_p' = d_q$.

[16, Corollary 1.11.17], d_q is reflexive as well, where $1 < q < \infty$. According to [4, Corollary 9.23], the norm of d_q is order continuous for $1 < q < \infty$. However, the norm of ces_∞ is not order continuous (see [11, Corollary 4.2]). Nevertheless, the following Lemma 3.2 shows that the norm of d_1 is order continuous.

Recently, the Banach lattice d_q has been investigated in detail by Bonet and Ricker ([7], see also the references therein).

Lemma 3.1. *Let $(x^{(n)})_{n\in\mathbb{N}}$ be a positive decreasing sequence and $1 \le p < \infty$. Then $x^{(n)} \downarrow \mathbf{0}$ in d_p if and only if $\hat{x}^{(n)} \downarrow \mathbf{0}$ in d_p.*

Proof. Let $x^{(n)} = (x_k^{(n)})_{k\in\mathbb{N}}$. The "if" part is clear from $|x_k^{(n)}| \le \hat{x}_k^{(n)}$ for all $n, k \in \mathbb{N}$. Next, we prove the "only if" part. To this end, let $x^{(n)} \downarrow \mathbf{0}$ in d_p. Then $x_k^{(n)} \downarrow 0$ in $\mathbb{R}$ for each $k \in \mathbb{N}$ since d_p is an ideal in $\mathbb{R}^{\mathbb{N}}$. It suffices to show that $\hat{x}_k^{(n)} \downarrow 0$ in $\mathbb{R}$ for each $k \in \mathbb{N}$. Since the sequence $(x^{(n)})_{n\in\mathbb{N}}$ is decreasing (to $\mathbf{0}$) in d_p, we have $\left(\sup_{i\ge k} |x_i^{(n)}|\right)_{k\in\mathbb{N}} \in \ell_p$, i.e., $(\hat{x}_k^{(n)})_{k\in\mathbb{N}} \in \ell_p$ and so $\sum_{k=1}^{\infty} |\hat{x}_k^{(n)}|^p < \infty$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \left|\hat{x}_k^{(n)}\right|^p = \sum_{k=N}^{\infty} \sup_{i\ge k} \left|x_i^{(n)}\right|^p \le \sum_{k=N}^{\infty} \sup_{i\ge k} \left|x_i^{(1)}\right|^p < \varepsilon^p$$

for all $n \in \mathbb{N}$. Moreover, for some $n_0 \in \mathbb{N}$, one has $0 \le x_k^{(n)} \le \varepsilon$ for $n \ge n_0$ and $k \in \{1, \ldots, N\}$. Therefore, $0 \le \hat{x}_k^{(n)} \le \varepsilon$ holds for all $n \ge n_0$ and $k \in \mathbb{N}$. This completes the proof. □

Lemma 3.2. *The norm in the Banach lattice d_p is order continuous for $1 \le p < \infty$.*

Proof. As mentioned above, the norm in the Banach lattice d_p is order continuous for $1 < p < \infty$. By [5, Theorem 4.9], to complete the proof for the case $p = 1$, we must show that $x_n \downarrow \mathbf{0}$ in d_1 implies $\|x_n\|_{d_1} \downarrow 0$. Let $x_n \downarrow \mathbf{0}$ in d_1. Then it follows from Lemma 3.1 that $\hat{x}_n \downarrow \mathbf{0}$ in d_1. Also, $\hat{x}_n \downarrow \mathbf{0}$ in ℓ_1 since d_1 is an ideal in ℓ_1. By order continuity of the norm in ℓ_1, $\|\hat{x}_n\|_1 \downarrow 0$. The proof follows immediately from $\|x_n\|_{d_1} = \|\hat{x}_n\|_1$. □

Remark 3.3. Further on, let e_n denote the sequence $(\underbrace{0, 0, \ldots, 0}_{n-1}, 1, 0, 0, \ldots)$. Note that $\hat{e}_n = (\underbrace{1, 1, \ldots, 1}_{n}, 0, \ldots) = \sum_{k=1}^{n} e_k$ and therefore[2], e_n is an element of d_p for each $n \in \mathbb{N}$ and $1 \le p < \infty$. It turns out that each e_n is an atom in d_p. Actually, there are no other atoms in d_p except the non-zero multiples of the coordinate sequences e_n. Therefore, $\Gamma_{d_p} = \{n^{-1/p} e_n \colon n \in \mathbb{N}\}$. In particular, d_p is an atomic vector lattice.

Theorem 3.4. *Let $1 \le p < \infty$. Then*

(i) $\Phi_3(d_p) = \Phi_2(d_p) = \Phi_1(d_p) = c_{00}$,
(ii) *the space d_p has no order unit.*

[2] One has $\|e_n\|_{d_p} = \|\hat{e}_n\|_p = (\sum_{k=1}^{n} 1^p)^{1/p} = n^{1/p}$.

Proof. (i) Since d_p is a Banach lattice with order continuous norm, it follows from Remark 3.3 and Theorem 2.2 that

$$\Phi_3(d_p) = \Phi_2(d_p) = \Phi_1(d_p) = \operatorname{span}(\Gamma_{d_p}) = \operatorname{span}\{e_n \colon n \in \mathbb{N}\} = c_{00}.$$

(ii) If d_p had an order unit, then $\Phi_1(d_p) = d_p$ by Theorem 2.3. Due to part (i), there would be $d_p = c_{00}$, a contradiction since c_{00} is a proper subspace of d_p.

4. Finite elements in Cesàro sums of Banach lattices. Denote by $\mathfrak{X}$ a sequence $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ of Banach spaces and let $p \in \{0\} \cup [1,\infty]$. Then, similar to the construction of directed sums[3] of Banach lattices in [18, §3.3.3], we define the *p-Cesàro sums* and *d_p-sums* of $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ as follows

$$\operatorname{ces}_p(\mathfrak{X}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \colon x_n \in X_n,\ \big(\|x_n\|_n \big)_{n\in\mathbb{N}} \in \operatorname{ces}_p \right\},$$
$$d_p(\mathfrak{X}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \colon x_n \in X_n,\ \big(\|x_n\|_n \big)_{n\in\mathbb{N}} \in d_p \right\}.$$

Further on, in order to simplify the notation, we write $\|\cdot\|$ instead of $\|\cdot\|_n$ and $\mathbf{0}$ for the zero vector in X_n for each $n \in \mathbb{N}$. Under the coordinatewise algebraic operations, these sets are vector spaces. With the norms defined by

$$|||x|||_{\operatorname{ces}_p}(\mathfrak{X}) = \Big\| \big(\|x_n\| \big)_{n\in\mathbb{N}} \Big\|_{\operatorname{ces}_p} \quad \text{and} \quad |||y|||_{d_p(\mathfrak{X})} = \Big\| \big(\|y_n\| \big)_{n\in\mathbb{N}} \Big\|_{d_p}$$

for $x \in \operatorname{ces}_p(\mathfrak{X})$ and $y \in d_p(\mathfrak{X})$, respectively, the spaces $\operatorname{ces}_p(\mathfrak{X})$ and $d_p(\mathfrak{X})$ are Banach spaces as well.

Note that the equality[4] $\operatorname{ces}_1 = \{\mathbf{0}\}$ implies that $\operatorname{ces}_1(\mathfrak{X})$ is trivial.

For $p = 0$ and $1 < p \le \infty$, define the map $J_j : X_j \to \operatorname{ces}_p(\mathfrak{X})$ by

$$J_j x = (x_n)_{n\in\mathbb{N}} = (\mathbf{0}, \dots, \mathbf{0}, \underbrace{x}_{j\text{-th term}}, \mathbf{0}, \dots) = \begin{cases} \mathbf{0}, & n \ne j, \\ x, & n = j, \end{cases}$$

for $x \in X_j$ and $j \in \mathbb{N}$, which is an isomorphism into the space $\operatorname{ces}_p(\mathfrak{X})$. After some calculation, one obtains

$$|||J_j x|||_{\operatorname{ces}_p(\mathfrak{X})} = \begin{cases} \|x\| \left(\sum\limits_{i=j}^{\infty} \frac{1}{i^p} \right)^{1/p}, & 1 < p < \infty, \\ \frac{1}{j}\|x\|, & p = 0 \text{ or } \infty, \end{cases} \qquad \text{for } x \in X_j.$$

Let $x = (x_n)_{n\in\mathbb{N}} \in \operatorname{ces}_p(\mathfrak{X})$ and $p \in \{0\} \cup (1,\infty)$. Observe that

$$\left|\left|\left| x - \sum_{n=1}^{N-1} J_n x_n \right|\right|\right|_{\operatorname{ces}_p}^p = \sum_{n=N}^{\infty} \left(\frac{1}{n} \sum_{k=N}^{n} \|x_k\| \right)^p \le \sum_{n=N}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \|x_k\| \right)^p \to 0$$

if $N \to \infty$ and

$$\left|\left|\left| x - \sum_{n=1}^{N-1} J_n x_n \right|\right|\right|_{\operatorname{ces}_0} = \sup_{n\ge N} \left\{ \frac{1}{n} \sum_{k=N}^{n} \|x_k\| \right\} \le \sup_{n\ge N} \left\{ \frac{1}{n} \sum_{k=1}^{n} \|x_k\| \right\} \xrightarrow[N\to\infty]{} 0,$$

which imply $x = \sum_{n=1}^{\infty} J_n x_n$.

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[3] For simplicity, we restrict our investigation to $\mathbb{N}$ – as a countable index set.

[4] See [14, 17].

Proposition 4.1. *Let $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces, $\mathfrak{X}' = (X'_n)_{n\in\mathbb{N}}$ the sequence of their dual spaces, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the mapping $x' = (x'_n)_{n\in\mathbb{N}} \mapsto f_{x'}$ from $d_q(\mathfrak{X}')$ to $\mathrm{ces}'_p(\mathfrak{X})$ defined by*

$$f_{x'}(x) := \sum_{n=1}^{\infty} \langle x'_n, x_n\rangle, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathrm{ces}_p(\mathfrak{X}),$$

is a linear isomorphism from $d_q(\mathfrak{X}')$ onto $\mathrm{ces}'_p(\mathfrak{X})$ and satisfies

$$\frac{1}{q} |||x'|||_{d_q(\mathfrak{X}')} \le \|f_{x'}\| \le (p-1)^{1/p}\ |||x'|||_{d_q(\mathfrak{X}')} \quad \text{for all } x' \in d_q(\mathfrak{X}'). \tag{4.1}$$

Similarly, we have $\mathrm{ces}'_0(\mathfrak{X}) = d_1(\mathfrak{X}')$ with equality of the norms, i.e., $\|f_{x'}\| = |||x'|||_{d_1(\mathfrak{X}')}$.

Proof. We first demonstrate that the mapping $x' \mapsto f_{x'}$ is well defined. Fix $x' = (x'_n)_{n\in\mathbb{N}} \in d_q(\mathfrak{X}')$. Then for each $x = (x_n)_{n\in\mathbb{N}} \in \mathrm{ces}_p(\mathfrak{X})$, we have $|\langle x'_n, x_n\rangle| \le \|x'_n\|\,\|x_n\|$ for any $n \in \mathbb{N}$ and so by referring to the estimation established after the proof of [6, Theorem 12.3], we get

$$\sum_{n=1}^{\infty} |\langle x'_n, x_n\rangle| \le \sum_{n=1}^{\infty} \|x'_n\|\,\|x_n\| \le (p-1)^{1/p}\ |||x'|||_{d_q(\mathfrak{X}')} |||x|||_{\mathrm{ces}_p(\mathfrak{X})}.$$

Thus, the formula $f_{x'}(x) = \sum_{n=1}^{\infty} \langle x'_n, x_n\rangle$ defines a continuous linear functional on $\mathrm{ces}_p(\mathfrak{X})$ satisfying the relations

$$\|f_{x'}\| \le (p-1)^{1/p}\ |||x'|||_{d_q(\mathfrak{X}')} \quad \text{for all } x' \in d_q(\mathfrak{X}'). \tag{4.2}$$

Clearly, the mapping $x' \mapsto f_{x'}$ is a linear operator.

Now, let $f \in \mathrm{ces}'_p(\mathfrak{X})$. Define for any $n \in \mathbb{N}$, the linear functionals $x'_n : X_n \to \mathbb{R}$ by $\langle x'_n, x_n\rangle = f(J_n x_n)$. Then $x'_n \in X'_n$ and, moreover, $f(x) = \sum_{n=1}^{\infty} \langle x'_n, x_n\rangle$ holds for all $x = (x_n)_{n\in\mathbb{N}} \in \mathrm{ces}_p(\mathfrak{X})$ since $x = \sum_{n=1}^{\infty} J_n x_n$ is true in $\mathrm{ces}_p(\mathfrak{X})$ (as was mentioned just before the proposition). Fix $\alpha > 1$. For each n, pick some $y_n \in X_n$ with $\|y_n\| = 1$ and $\|x'_n\| \le \alpha\langle x'_n, y_n\rangle$. Observe that for each k, we have

$$\begin{aligned}\sum_{n=1}^{k} \|x'_n\|\,\|x_n\| &\le \sum_{n=1}^{k} \alpha\langle x'_n, y_n\rangle\,\|x_n\| = \alpha f\left(\sum_{n=1}^{k} J_n(\|x_n\|\, y_n)\right) \\ &\le \alpha\,\|f\| \left|\left|\left|\sum_{n=1}^{k} J_n(\|x_n\|\, y_n)\right|\right|\right|_{\mathrm{ces}_p(\mathfrak{X})} \le \alpha\,\|f\|\,|||x|||_{\mathrm{ces}_p(\mathfrak{X})}.\end{aligned}$$

Taking the limits as $k \to \infty$ and $\alpha \downarrow 1$, we see that

$$\sum_{n=1}^{\infty} \|x'_n\|\,\|x_n\| \le \|f\|\,|||x|||_{\mathrm{ces}_p(\mathfrak{X})}.$$

If $(\|x'_n\|)_{n\in\mathbb{N}} \in d_q$, i.e., $x' \in d_q(\mathfrak{X}')$, then from [6, Corollary 12.17], it follows that $f = f_{x'}$ and $|||x'|||_{d_q(\mathfrak{X}')} \le q\,\|f_{x'}\|$. Therefore, this together with (4.2) implies (4.1) and $x' \mapsto f_{x'}$ is a linear isomorphism from $d_q(\mathfrak{X}')$ onto $\mathrm{ces}'_p(\mathfrak{X})$. It remains to show that the relation $(\|x'_n\|)_{n\in\mathbb{N}} \in d_q$ holds[5]: Define the functional $\lambda\colon \mathrm{ces}_p \to \mathbb{R}$ by $\lambda\big((a_n)_{n\in\mathbb{N}}\big) = \sum_{n=1}^{\infty} \|x'_n\|\, a_n$. By the above inequality, we have

$$\begin{aligned}\big|\lambda\big((a_n)_{n\in\mathbb{N}}\big)\big| &\leq \sum_{n=1}^{\infty} \|x'_n\|\,|a_n| = \sum_{n=1}^{\infty} \|x'_n\| \left\| \frac{a_n}{\|x_n\|} x_n \right\| \\ &\leq \|f\| \left\|\left\| \left(\frac{a_n}{\|x_n\|} x_n \right)_{n\in\mathbb{N}} \right\|\right\|_{\mathrm{ces}_p(\mathfrak{X})} = \|f\|\, \|(a_n)_{n\in\mathbb{N}}\|_{\mathrm{ces}_p}.\end{aligned}$$

Therefore, $\lambda \in (\mathrm{ces}_p)'$, i.e., $(\|x'_n\|)_{n\in\mathbb{N}} \in d_q$.

By the same method and [10, Lemma 6.2], one can show $\mathrm{ces}'_0(\mathfrak{X}) = d_1(\mathfrak{X}')$ even with equality of norms. □

Remark 4.2. If $\mathrm{ces}_p(\mathfrak{X})$ is endowed with a special norm equivalent to $\|\cdot\|_{\mathrm{ces}_p}$, then by using [6, Theorem 4.5 and Corollary 12.17], it can be established that the mapping defined in Proposition 4.1 is an isometry.

Turn now to the case that each X_n is a Banach lattice. Then with the coordinatewise defined lattice operations, the spaces $\mathrm{ces}_p(\mathfrak{X})$ and $d_p(\mathfrak{X})$ are even Banach lattices, J_j is a lattice isomorphism from X_j to $\mathrm{ces}_p(\mathfrak{X})$, and for any $j \in \mathbb{N}$, the set J_jX_j is a projection band in $\mathrm{ces}_p(\mathfrak{X})$. The latter implies $\Phi_1(J_jX_j) = J_j\Phi_1(X_j)$. Denote by $P_j : \mathrm{ces}_p(\mathfrak{X}) \to J_jX_j$ the band projection from $\mathrm{ces}_p(\mathfrak{X})$ onto J_jX_j, where $P_j\big((x_n)_{n\in\mathbb{N}}\big) = J_jx_j$. By Theorem 2.5 and in view of the just mentioned behaviour of $J_j\Phi_1(X_j)$, there hold the equalities

$$P_j\Big(\Phi_1\big(\mathrm{ces}_p(\mathfrak{X})\big)\Big) = \Phi_1\big(\mathrm{ces}_p(\mathfrak{X})\big) \cap J_jX_j = \Phi_1(J_jX_j) = J_j\Phi_1(X_j). \tag{4.3}$$

The mapping $x' \mapsto f_{x'}$ defined in Proposition 4.1 is now an onto lattice isomorphism since both the map and its inverse are positive operators.

As the characterization of the finite elements in the Banach lattices ces_p, where $1 < p \leq \infty$ or $p = 0$, we get a quite direct generalization of the results for the classical cases $X_n = c_0$, ℓ_p, and ℓ_∞, see [18, Theorem 3.33].

Theorem 4.3. *The following statements hold*:

(i) *For $p = 0$ and $1 < p < \infty$, the element $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is finite in $\mathrm{ces}_p(\mathfrak{X})$ if and only if $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$ and $\varphi_n = \mathbf{0}$ for all but finitely many $n \in \mathbb{N}$.*

(ii) *The element $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is finite in $\mathrm{ces}_\infty(\mathfrak{X})$ if and only if there exist $w_n \in X_n^+$ such that $(n\,\|w_n\|)_{n\in\mathbb{N}} \in \mathrm{ces}_\infty$ and*

$$B_{\{\varphi_n\}^{dd}} \subset [-w_n,\ w_n].$$

In particular, $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$.

Proof. (i) Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ be such that $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$ and $\varphi_n = \mathbf{0}$ for all but finitely many $n \in \mathbb{N}$. Clearly, $\varphi \in \mathrm{ces}_p(\mathfrak{X})$. For all $n \in \mathbb{N}$, $P_n\varphi = J_n\varphi_n \in J_n\Phi_1(X_n)$ and (4.3) yield $P_n\varphi \in \Phi_1\big(\mathrm{ces}_p(\mathfrak{X})\big)$. From the linearity of

[5]Here we use the short proof suggested by the referee.

the space $\Phi_1\big(\operatorname{ces}_p(\mathfrak{X})\big)$, it follows that $\varphi = \sum_{n=1}^{\infty} P_n\varphi \in \Phi_1\big(\operatorname{ces}_p(\mathfrak{X})\big)$. Observe that this sum has only finitely many non-zero terms.

For the converse, assume that $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is a finite element in $\operatorname{ces}_p(\mathfrak{X})$. Then $J_n\varphi_n = P_n\varphi \in P_n\Big(\Phi_1\big(\operatorname{ces}_p(\mathfrak{X})\big)\Big)$ holds for all $n \in \mathbb{N}$. Thus, using (4.3) again, one has $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$. Next we claim that $\varphi_n = \mathbf{0}$ for all but finitely many $n \in \mathbb{N}$. To see this, assume by way of contradiction that $\varphi_{n_k} \neq \mathbf{0}$ holds for some increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$. Consider $\psi = (\psi_n)_{n\in\mathbb{N}}$, where

$$\psi_n = \begin{cases} \frac{1}{\|\varphi_n\|}|\varphi_n| & \text{if } n = n_k \text{ for some } k,\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Note that $J_{n_k}\psi_{n_k} = \Big(\mathbf{0}\,,\dots,\mathbf{0}\,,\frac{|\varphi_{n_k}|}{\|\varphi_{n_k}\|},\mathbf{0}\,,\dots\Big)$ and consequently

$$|||J_{n_k}\psi_{n_k}|||^p_{\operatorname{ces}_p(\mathfrak{X})} = \sum_{i=n_k}^{\infty}\frac{1}{i^p} \quad\text{and}\quad |||J_{n_k}\psi_{n_k}|||_{\operatorname{ces}_0(\mathfrak{X})} = \frac{1}{n_k}$$

if $1 < p < \infty$ and $p = 0$, respectively. Therefore, if we define $t_{n_k} := \big(\sum_{i=n_k}^{\infty}\frac{1}{i^p}\big)^{1/p}$ for $1 < p < \infty$ and $t_{n_k} := \frac{1}{n_k}$ for $p = 0$, then $\frac{1}{t_{n_k}}J_{n_k}\psi_{n_k} \in B_{\{\varphi\}^{dd}} = B_{\operatorname{ces}_p(\mathfrak{X})}\cap\{\varphi\}^{dd}$ for $p = 0$ and $1 < p < \infty$. According to Theorem 2.4, there exists a positive $z = (z_n)_{n\in\mathbb{N}} \in \operatorname{ces}_p(\mathfrak{X})$ such that $B_{\{\varphi\}^{dd}} \subset [-z,\ z]$, which implies that $0 \le \frac{1}{t_{n_k}}\psi_{n_k} \le z_{n_k}$, and so $\frac{1}{t_{n_k}} \le \|z_{n_k}\|$ for all $k \in \mathbb{N}$. To complete the proof, it is enough to show that the sequence $a = (a_n)_{n\in\mathbb{N}}$ defined by

$$a_n = \begin{cases} \frac{1}{t_{n_k}} & \text{if } n = n_k \text{ for some } k,\\ 0 & \text{otherwise,}\end{cases}$$

does not belong to $\operatorname{ces}_p(\mathfrak{X})$ for $p = 0$ and $1 < p < \infty$, which contradicts $z \in \operatorname{ces}_p(\mathfrak{X})$ since ces_p is an order ideal in $\mathbb{R}^{\mathbb{N}}$. For $p = 0$, by the definition of a, one has $a_{n_k} = n_k$ which yields $(\mathcal{C}|a|)_{n_k} \ge 1$ for all $k \in \mathbb{N}$, i.e., a is not an element of ces_0, whereas $\mathcal{C}(\|z_{n_k}\|) \in c_0$.

For $1 < p < \infty$, there exists $c > 0$ such that $t_{n_k} = \big(\sum_{i=n_k}^{\infty}\frac{1}{i^p}\big)^{1/p} \le c\,n_k^{-1/q}$ for each $k \in \mathbb{N}$ (with $\frac{1}{p}+\frac{1}{q} = 1$). The latter estimation holds for each $n \in \mathbb{N}$ because of

$$\frac{1}{n^p} \le \frac{1}{n^{p-1}} \quad\text{and}\quad \sum_{i=n+1}^{\infty}\frac{1}{i^p} \le \int_n^{\infty}\frac{dx}{x^p} = \frac{1}{(p-1)n^{p-1}}.$$

Therefore, $\sum_{i=n}^{\infty}\frac{1}{i^p} \le \frac{1}{n^{p-1}}\cdot q$, and, if n is replaced by n_k, one has

$$n_k^{1/p}\,(\mathcal{C}\,|a|)_{n_k} = n_k^{1/p}\left(\frac{1}{n_k}\sum_{i=1}^{n_k}a_i\right) \ge \frac{n_k^{1/p}}{n_k t_{n_k}} \ge \frac{n_k^{1/p}\;n_k^{1/q}}{cn_k} = \frac{1}{c} > 0.$$

Thus, $\big((n_k)^{1/p}\,(\mathcal{C}|a|)_{n_k}\big)_{k\in\mathbb{N}}$ does not converge to 0, and so it follows from [10, Proposition 2.3 (iii)] that a is not an element of $\operatorname{ces}_p(\mathfrak{X})$.

(ii) Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ be a finite element in $\mathrm{ces}_\infty(\mathfrak{X})$. Again by Theorem 2.4, there is a positive $z = (z_n)_{n\in\mathbb{N}} \in \mathrm{ces}_\infty(\mathfrak{X})$ such that $B_{\{\varphi\}^{dd}} \subset [-z,\ z]$. For each $n \in \mathbb{N}$, put $w_n = \frac{z_n}{n}$, and note that $w_n \in X_n^+$ and $(n\,w_n)_{n\in\mathbb{N}} \in \mathrm{ces}_\infty(\mathfrak{X})$. Let $y \in B_{\{\varphi_n\}^{dd}}$. Then, clearly, $J_n y \in \{\varphi\}^{dd}$ and $\|J_n y\| = \frac{\|y\|}{n} \le \frac{1}{n}$, from where $J_n y \in \left[-\frac{z}{n},\ \frac{z}{n}\right]$ follows for all $n \in \mathbb{N}$. Thus, $y \in [-w_n,\ w_n]$, i.e., $B_{\{\varphi_n\}^{dd}} \subset [-w_n,\ w_n]$.

For the converse, observe that

$$B_{\{\varphi\}^{dd}} \subset \{(x_n)_{n\in\mathbb{N}} \in \mathrm{ces}_\infty(\mathfrak{X}) \colon x_n \in nB_{\{\varphi_n\}^{dd}} \text{ for all } n \in \mathbb{N}\}.$$

Since $nB_{\{\varphi_n\}^{dd}}$ is order bounded and the order in $\mathrm{ces}_\infty(\mathfrak{X})$ is coordinatewise, the proof of the theorem is completed by Theorem 2.4. □

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