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A note on polydegree $(n, 1)$ rational inner functions, slice matrices, and singularities

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Abstract. We analyze certain compositions of rational inner functions in the unit polydisk $\mathbb{D}^d$ with polydegree $(n, 1)$, $n \in \mathbb{N}^{d-1}$, and isolated singularities in $\mathbb{T}^d$. Provided an irreducibility condition is met, such a composition is shown to be a rational inner function with singularities in precisely the same location as those of the initial function, and with quantitatively controlled properties. As an application, we answer a d-dimensional version of a question posed in Bickel et al. (Am J Math 144: 1115–1157, 2022) in the affirmative.

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1. Introduction.

Background. This note is concerned with certain bounded holomorphic functions on the unit polydisk in $\mathbb{C}^d$,

$$\mathbb{D}^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \colon |z_j| < 1,\ j = 1, \ldots, d\},$$

called rational inner functions, and their singularities on the d-torus

$$\mathbb{T}^d = \{\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \colon |\zeta_j| = 1,\ j = 1, \ldots, d\};$$

here and throughout, $d \in \mathbb{N}$. By Fatou's theorem for polydisks (see e.g. [20, Chapter 3]), any bounded holomorphic function $\phi\colon \mathbb{D}^d \to \mathbb{C}$ has non-tangential boundary values $\phi^*(\zeta) = \angle \lim_{\mathbb{D}^d \ni z \to \zeta} \phi(z)$ at almost every point $\zeta \in \mathbb{T}^d$. If these boundary values satisfy $|\phi^*(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}^d$, we say that ϕ is an inner function.

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Inner functions of the form $\phi = q/p$, where $q, p \in \mathbb{C}[z_1, \ldots, z_d]$ and p has no zeros in $\mathbb{D}^d$, are called rational inner functions (RIFs). In one variable, RIFs are precisely the finite Blaschke products in the unit disk $\mathbb{D}$. Blaschke products play a central role in function theory, see for instance [11] for an overview of the very rich theory of these functions. In two and more variables, RIFs form a concrete class of bounded holomorphic functions that is amenable to detailed study [20, Chapter 5], and appears naturally in several settings, for instance in connections with interpolation problems [1].

A classical result of Rudin and Stout (see [20, Chapter 5]) states that any RIF in $\mathbb{D}^d$ admits a representation of the form

$$\phi(z) = e^{ia} z^m \frac{\tilde{p}(z)}{p(z)}, \tag{1.1}$$

where $a \in \mathbb{R}$, $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and $\tilde{p}$ is the reflection of a polynomial p with no zeros in $\mathbb{D}^d$ known as a stable polynomial. The reflection polynomial is defined as

$$\tilde{p}(z) = z_1^{n_1} \cdots z_d^{n_d} \overline{p\left(\frac{1}{\bar{z}_1}, \ldots, \frac{1}{\bar{z}_d}\right)}.$$

The vector $(n_1, \ldots, n_d)$ is referred to as the polydegree of p; each $n_j = \deg_{z_j}(p)$ is the degree of p in the variable z_j. In this note, we shall strip out monomial factors and consider RIFs $\phi = e^{ia}\tilde{p}/p$; this simplifies formulas and is not material for the problem we study.

RIFs as well as more general bounded rational functions in two or more variables have been considered by a number of authors in recent years, often in connection with stable polynomials, representation formulas, and operator-theoretic problems. We cannot give a full overview here, but a sampler of related work might include papers of Anderson, Dritschel, and Rovnyak [3]; Ball, Sadosky, and Vinnikov [4]; Knese [13–15]; and Kollár [17].

A series of recent papers with Bickel and Pascoe [7–9]; Bickel, Knese, and Pascoe [10]; and Tully–Doyle [21] deal with aspects of RIF theory that are particular to dimensions $d \geq 2$. Namely, unlike in one dimension, RIFs in two or more variables can have singularities on the d-torus, arising at points $\zeta \in \mathbb{T}^d$ where $p(\zeta) = 0$ and $\tilde{p}(\zeta) = 0$ without common factors that cancel out. A d-dimensional example (see [15, Section 5] and [9, Example 2.5]) is given by

$$\phi_d(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{d\prod_{k=1}^{d} z_k - \sum_{j \in J} z_{j_1} \cdots z_{j_{d-1}}}{d - \sum_{k=1}^{d} z_k} \quad (d \geq 2) \tag{1.2}$$

which has a single singularity at $(1, \ldots, 1) \in \mathbb{T}^d$. Here, $J = \{(j_1, \ldots, j_{d-1}) \in \mathbb{N}^d \colon 1 \leq j_1 < j_2 < \cdots < j_{d-1} \leq d\}$.

One would like to describe RIF singularities in detail, and there are different ways of doing this. The papers [7–9], as well as [10], investigate for which $\mathfrak{p} \geq 1$ the partial derivative of a RIF has $\frac{\partial \phi}{\partial z_d} \in L^{\mathfrak{p}}(\mathbb{T}^d)$. Roughly speaking, the smaller the maximal $\mathfrak{p}$ for which integrability holds, the stronger the singularity of ϕ. For the example (1.2), the maximal integrability index is $\mathfrak{p} = \frac{1}{2}(d+1)$; see [9] and [10] for comprehensive discussions. The paper [7] and the work of Bergqvist [5] also consider other notions of derivative integrability corresponding to norms of Dirichlet type.

Overview of results. The purpose of this short note is to present some straightforward observations regarding d-variable RIFs of polydegree $(n, 1)$, $n = (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1}$, and their singularities. This restricted class of functions is often singled as more amenable to analysis, see for instance [6,9,13]. If $\hat{\zeta} = (\zeta_1, \ldots \zeta_{d-1}) \in \mathbb{T}^{d-1}$ is kept fixed and $\phi = \tilde{p}/p$ is a RIF in $\mathbb{D}^d$, the resulting one-variable function $\phi_{\hat{\zeta}}(z_d)$ is either a Möbius transformation mapping the unit disk onto itself, or else is a unimodular constant. By encoding this fact in a 2×2 matrix-valued function of $\hat{\zeta}$, and expressing the determinant of this matrix in terms of $\hat{\zeta}$-polynomials extracted from p and $\tilde{p}$, we are able to read off certain geometric characteristics of such ϕ.

This allows us to exhibit d-variable RIFs with prescribed singularity types, and hence derivative integrability properties, while keeping the z_d-degree of the resulting functions equal to 1. As a specific application, we are able to answer a stronger version of [9, Question 3] in the affirmative.

2. Preliminaries.

Polydegree $(n, 1)$ RIFs and their singularities. Let p be an irreducible stable polynomial in $\mathbb{D}^d$, the latter meaning that $\mathcal{Z}(p) = \{z \in \mathbb{C}^d : p(z) = 0\}$ does not intersect $\mathbb{D}^d$. We assume throughout that p has polydegree $(n, 1)$ where $n = (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1}$ and that p is atoral, which in this context means that p and $\tilde{p}$ share no common factor, see [9, Section 1.2]. Then we can decompose p as a sum

$$p(z) = p_1(z_1, \ldots, z_{d-1}) + z_d\, p_2(z_1, \ldots, z_{d-1}) \tag{2.1}$$

where $p_1(\hat{z})$ and $p_2(\hat{z})$ are in $\mathbb{C}[z_1, \ldots, z_{d-1}]$, and similarly

$$\tilde{p}(z) = \tilde{p}_2(\hat{z}) + z_d\, \tilde{p}_1(\hat{z}), \quad \tilde{p}_1, \tilde{p}_2 \in \mathbb{C}[z_1, \ldots, z_{d-1}]. \tag{2.2}$$

As we are interested in singular RIFs $\phi = \tilde{p}/p$, we assume there exists at least one $\zeta \in \mathbb{T}^d$ such that $p(\zeta) = 0$. A result of Pascoe [18, Corollary 1.7] shows that if we assume p is irreducible, then any zero of p on $\mathbb{T}^d$ gives rise to a singularity of ϕ. We restrict attention to the class of such p for which we have the additional property that $\mathcal{Z}(p) \cap \mathbb{T}^d$ is finite; we call the corresponding $\phi = \tilde{p}/p$ finite-singularity RIFs.

Definition 1. Suppose $\phi = e^{ia}\tilde{p}/p$ is a finite-singularity RIF in $\mathbb{D}^d$ with a singularity at $\vec{1}_d = (1, \ldots, 1) \in \mathbb{T}^d$. We say $\mathfrak{p}^* \geq 1$ is a local z_d-derivative integrability index of ϕ if

$$\mathfrak{p}^* = \sup_{\mathfrak{p} \geq 1} \left\{ \mathfrak{p} : \frac{\partial \phi}{\partial z_d} \in L^{\mathfrak{p}}_{\text{loc}}(\mathbb{T}^d) \right\},$$

where each $L^{\mathfrak{p}}_{\text{loc}}(\mathbb{T}^d)$ is a standard local Lebesgue space of measurable functions f on the d-torus such that $|f|^p$ is locally integrable near $\vec{1}_d$.

The global z_d-derivative integrability index of ϕ is the maximum of all the local z_d-derivative integrability indices of the finite-singularity RIF ϕ.

Because of the argument principle, $\frac{\partial\phi}{\partial z_d}$ is integrable for any RIF so the assumption that $\mathfrak{p} \geq 1$ is justified; see [5,7] for details. In a similar way, we can define z_j-derivative indices. To keep this note as elementary as possible, we focus on the z_d-derivative integrability index of a $(n, 1)$ RIF.

It is not a straight-forward task to determine local or global z_j-derivative indices of a d-variable RIF. Two-dimensional RIFs are much better understood than their general d-dimensional counterparts: for instance, the z_1 and z_2-derivative indices of a RIF coincide when $d = 2$, but this is false when $d \geq 3$, and their values are determined by a geometric characteristic of p at its zeros. See [7] and [10] for comprehensive presentations of the two-variable theory.

As is explained in [9], the z_d-derivative integrability of a polydegree $(n, 1)$ RIF ϕ is controlled by the rate at which the zero set of $\tilde{p}$ approaches $\mathbb{T}^d$ from inside the polydisk. To make this statement precise, we return to the one-variable function $\phi_{\hat{\zeta}}$ and note that the $L^{\mathfrak{p}}$ norm of the derivative of a Möbius transformation is proportional to the distance to $\mathbb{T}$ of the point $\psi^0 \in \mathbb{D}$ for which $\phi_{\hat{\zeta}}(\psi^0) = 0$; see [8, Lemma 4.2]. Solving $\tilde{p}(\hat{\zeta}, \psi^0) = 0$ yields $\psi^0(\hat{\zeta}) = -\tilde{p}_2(\hat{\zeta})/\tilde{p}_1(\hat{\zeta})$, where $\tilde{p}_1, \tilde{p}_2$ are the polynomials from (2.2).

Therefore, we set

$$\rho_\phi(\hat{\zeta}) = 1 - |\psi^0(\hat{\zeta})|^2 = \frac{|\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2}{|\tilde{p}_1(\hat{\zeta})|^2}.$$

Note that since ϕ is assumed to be a finite-singularity RIF, the polynomial $\tilde{p}_1$ has no zeros in $\mathbb{T}^{d-1}$; otherwise $\mathcal{Z}(\tilde{p}) \cap \mathbb{T}^d$ would contain a vertical line [9, Section 3], which is impossible since zeros of $\tilde{p}$ on $\mathbb{T}^d$ are also zeros of p. Hence the vanishing of ρ_ϕ near a singularity is determined by the vanishing of its numerator. As a consequence of this discussion and [9, Theorem 2.1], we obtain the following criterion.

Theorem 1. *Suppose ϕ is a finite-singularity RIF with polydegree $(n, 1)$ and a singularity at $\vec{1}_d = (1, \ldots, 1) \in \mathbb{T}^d$. Then $\frac{\partial\phi}{\partial z_d} \in L^{\mathfrak{p}}_{\mathrm{loc}}(\mathbb{T}^d)$ at $\vec{1}_d$ if and only if*

$$\int_U \left[|\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2\right]^{1-\mathfrak{p}} dm(\hat{\zeta}) < \infty,$$

where $U \subset \mathbb{T}^{d-1}$ is any sufficiently small open set in $\mathbb{T}^{d-1}$ containing $\vec{1}_{d-1}$.

Polydegree $(n, 1)$ rational inner functions and 2×2 matrices. Suppose $\phi = \tilde{p}/p$ is a finite-singularity RIF of polydegree $(n, 1)$, and consider, for $\hat{\zeta} \in \mathbb{T}^{d-1}$ fixed, the one-variable function

$$\phi_{\hat{\zeta}}(z_d) = \phi(\hat{\zeta}, z_d).$$

Then $\phi_{\hat{\zeta}}(z_d)$ is a rational function in $\mathbb{D}$, which attains unimodular boundary values at every point $\zeta_d \in \mathbb{T}$ by a theorem of Knese [16, Theorem C]. Hence $\phi_{\hat{\zeta}}$ is either a Möbius transformation of the unit disk, or else $\phi_{\hat{\zeta}}(z_d)$ is constant, and equal to some element of $\mathbb{T}$. The former obtains generically, but the latter possibility certainly occurs on some exceptional sets, as can be checked by considering $\phi_d(1, \ldots, 1, \zeta_d)$, where ϕ_d is the function in (1.2).

This discussion together with (2.1), (2.2) guides the next definition.

Definition 2. The slice matrix of ϕ is the function $M_\phi\colon \mathbb{T}^{d-1} \to M_{2,2}(\mathbb{C})$ given by

$$M_\phi(\hat{\zeta}) = \begin{pmatrix} \tilde{p}_1(\hat{\zeta}) & \tilde{p}_2(\hat{\zeta}) \\ p_2(\hat{\zeta}) & p_1(\hat{\zeta}) \end{pmatrix}.$$

The slice determinant of ϕ is the function $P_\phi\colon \mathbb{T}^{d-1} \to \mathbb{C}$ given by

$$P_\phi(\hat{\zeta}) = \det M_\phi(\hat{\zeta}).$$

Formally, the numerator and the denominator of $\phi_{\hat{\zeta}}(z_d)$ can be read off from $M_\phi(\hat{\zeta})(z_d,\ 1)^T$. The slice determinant allows us to detect singularities of ϕ as well as their finer properties.

Lemma 2. *The function $\phi_{\hat{\xi}}$ is constant if and only if $P_\phi(\hat{\xi}) = 0$, and this happens if and only if $(\hat{\xi}, \eta)$ is a singularity of ϕ for some value of $\eta \in \mathbb{T}$. Moreover, $\frac{\partial \phi}{\partial z_d} \in L^{\mathfrak{p}}_{\text{loc}}(\mathbb{T}^d)$ at $(\hat{\xi}, \eta)$ if and only if $\int_{B_\epsilon(\hat{\xi})} |P_\phi(\hat{\zeta})|^{1-\mathfrak{p}} dm(\hat{\zeta}) < \infty$ for sufficiently small $\epsilon > 0$.*

Proof. The first assertion is a direct consequence of the facts that, for a, b, c, d complex, $\mathfrak{m}(z) = (az+b)/(cz+d)$ furnishes a non-trivial Möbius transformation of the Riemann sphere if and only if $ad - bc \neq 0$; and if $ad - bc = 0$, then $\mathfrak{m}$ is constant. See [12] for a comprehensive treatment of Möbius transformations and their connections with matrix groups.

The second assertion is a consequence of the results in [6, Section 3.2], see in particular [6, Lemma 3.3].

The third assertion essentially amounts to a computation. Namely,

$$\det M_\phi(\hat{\zeta}) = \tilde{p}_1(\hat{\zeta}) p_1(\hat{\zeta}) - \tilde{p}_2(\hat{\zeta}) p_2(\hat{\zeta}).$$

Observing that $\zeta_j = 1/\bar{\zeta}_j$, $j = 1, \ldots, d-1$, and examining the definition of reflection polynomials, the expression on the right-hand side can be rewritten (in standard multi-index notation) as

$$\hat{\zeta}^n \bar{p}_1(\hat{\zeta}) p_1(\hat{\zeta}) - \hat{\zeta}^n \bar{p}_2(\hat{\zeta}) p_2(\hat{\zeta}) = \hat{\zeta}^n \left(|\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2 \right).$$

The result now follows after taking moduli and appealing to Theorem 1. □

3. Compositions and local properties of singularities. Given an $(n, 1)$ finite-singularity RIF, we define the following sequence of functions. See [21] for a fuller study of dynamical properties of mappings, especially skew-products, whose components are RIFs.

Definition 3. Let $\phi = \tilde{p}/p$ be a finite-singularity RIF of polydegree $(n, 1)$. Then $\phi^2\colon \mathbb{D}^d \to \mathbb{C}$ is defined as

$$\phi^2(z) = (\phi_{\hat{\zeta}} \circ \phi_{\hat{\zeta}})(z_d), \quad (\hat{z}, z_d) \in \mathbb{D}^d.$$

For any $N \in \mathbb{N}$ with $N \geq 3$, ϕ^N is defined inductively as $\phi_{\hat{\zeta}} \circ \phi^{N-1}_{\hat{\zeta}}$.

The functions ϕ^N are clearly rational and holomorphic in $\mathbb{D}^d$. As can be seen from (2.1) and (2.2), the z_j-degree of ϕ^N is at most $N \cdot n_j$ for $j = 1, \ldots, d-1$, and $\deg_{z_d}(\phi^N) \leq 1$. If ϕ^N has maximum possible polydegree given the initial polydegree of ϕ, we say that ϕ^N has full polydegree. One complication that can arise is that the numerator and the denominator of the composite function may initially share a common factor. We always assume any such factors are cancelled, in which case the polydegree of ϕ^N is reduced. In this case, we say ϕ^N experiences a polydegree drop.

Lemma 3. *Suppose $\phi^N = \tilde{p}_N/p_N$ is as in Definition 3 and does not experience a polydegree drop. Then ϕ^N is a finite-singularity RIF with the same singularities as ϕ.*

Proof. Since ϕ maps $\mathbb{T}^d$ onto $\mathbb{T}$, Knese's theorem implies that each $(\phi^N)^*$ is unimodular. Hence ϕ^N is inner.

Next, for $\hat{\zeta} \in \mathbb{T}^{d-1}$, computing the slice matrix of $\phi^N_{\hat{\zeta}}$ amounts to taking the matrix power $M^N_\phi(\hat{\zeta}) = M_\phi(\hat{\zeta}) \cdots M_\phi(\hat{\zeta})$, see [12]. The assumption that ϕ^N has full polydegree implies there are no common factors in the matrices that would be cancelled in ϕ^N. Then, by multiplicativity of determinants, $\det M^N_\phi(\hat{\zeta})$ vanishes if and only if $\det M_\phi(\hat{\zeta})$ does. Thus, the $\hat{\zeta}$-coordinates of the singularities of ϕ^N are the same as those of ϕ. Since ϕ^N has degree 1 in z_d, and since ϕ has a singularity on the line $\{\hat{\zeta}\} \times \mathbb{T}$, each such $\hat{\zeta}$ determines a unique $\eta \in \mathbb{T}$ such that $(\hat{\zeta}, \eta) \in \mathbb{T}^d$ is a singularity of ϕ^N. □

The following example illustrates that common factors may be introduced or removed if ϕ is rotated by a factor e^{ia}, $a \in \mathbb{R}$, or in other words, if $\tilde{p}$ is replaced by $e^{ia}\tilde{p}$. Doing this only affects P_ϕ up to a unimodular factor.

Example 4. Consider $\phi = -(2z_1z_2 - z_1 - z_2)/(2 - z_1 - z_2)$. As is shown by induction in [21, Example 1], each ϕ^N, $N = 1, 2, 3, \ldots$, has bidegree $(1, 1)$.

Next, consider $\phi = (2z_1z_2 - z_1 - z_2)/(2 - z_1 - z_2)$. Then

$$\psi = \phi^2 = \frac{4z_1^2z_2 - z_1^2 - 3z_1z_2 - z_1 + z_2}{4 - 3z_1 - z_2 - z_1z_2 + z_1^2}$$

is a RIF that often features as a second example of a singular RIF on the bidisk; see for instance [2,7,8]. In particular, while $\frac{\partial \phi}{\partial z_2} \in L^{\mathfrak{p}}(\mathbb{T}^2)$ if and only if $\mathfrak{p} < 3/2$, it was shown by direct computation in [7, Example 2] that $\frac{\partial \psi}{\partial z_2} \in L^{\mathfrak{p}}(\mathbb{T}^2)$ if and only if $\mathfrak{p} < 5/4$. We now give a conceptual explanation for this finding.

Theorem 5. *Suppose $\phi = \tilde{p}/p$ is a finite-singularity RIF of polydegree $(n, 1)$, with a singularity at $\vec{1}_d$, and suppose the local z_d-derivative integrability index of ϕ at $\vec{1}_d$ is equal to $\mathfrak{p}^* = 1 + \mathfrak{q}^*$, where $\mathfrak{q}^* \geq 0$.*

If $N \in \mathbb{N}$ and ϕ^N has full polydegree, then the RIF $\phi^N = \tilde{p}_N/p_N$ has local z_d-derivative integrability index equal to $1 + \mathfrak{q}^*/N$ at $\vec{1}_d$.

Proof. By Lemma 3, ϕ^N is a RIF with the same singularities as ϕ, and in particular, ϕ^N has a singularity at $\vec{1}_d$. Since $\tilde{p}_N$ and p_N have no common factors that can be cancelled, the slice matrix of ϕ^N is equal to M^N_ϕ, the

N-fold power of the slice matrix of ϕ. Hence the order of vanishing of the slice determinant of ϕ^N is equal to N times the order of vanishing of the slice determinant of ϕ. In other words, $\frac{\partial \phi}{\partial z_d} \in L^{\mathfrak{p}}(\mathbb{T}^d)$ precisely when

$$\int_U (|\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2)^{N(1-\mathfrak{p})} dm(\hat{\zeta})$$

is finite for $U \supset \vec{1}_{d-1}$ sufficiently small. By our assumption on ϕ, this holds if $N(1-\mathfrak{p}) > -\mathfrak{q}^*$ and fails when $N(1-\mathfrak{p}) < -\mathfrak{q}^*$, and the result follows. □

When $d = 2$ and ϕ has a singularity at $(e^{i\eta_1}, e^{i\eta_2})$, it can be shown that $1 - |\psi^0(e^{i\theta_1})|^2 \asymp (\theta_1 - \eta_1)^{2K}$ for some $K \in \mathbb{N}$. The number $2K$ is called the z_2-contact order of ϕ at $(e^{i\eta_1}, e^{i\eta_2})$; see [7,8] for definitions and proofs. The assumption that ϕ has finitely many singularities becomes superfluous in two variables by Bézout's theorem, and we obtain the following.

Corollary 6. *Suppose $\phi = \tilde{p}/p$ is a bidegree $(n_1, 1)$ RIF in $\mathbb{D}^2$ with s singularities having associated contact orders $\{2K_1, \ldots, 2K_s\}$, and suppose $\phi^N = \tilde{p}_N/p_N$ has full polydegree. Then ϕ^N has s singularities with contact orders $\{2NK_1, \ldots, 2NK_s\}$.*

4. Applications.

Finding extraneous zeros of two-variable RIF denominators. In [19], Pascoe presents a way of constructing two-variable RIFs with at least one singularity where the local contact order can be prescribed to take any value $2K$, $K \in \mathbb{N}$. (Strictly speaking, the construction is given in the setting of the bi-upper half-plane, but it can readily be transferred to the bidisk by means of conjugation by a suitable Möbius map. See [8, Section 7].) In particular, any positive even integer is the contact order of some RIF in $\mathbb{D}^2$.

However, Pascoe's construction may produce additional singularities in ϕ and, to the author's knowledge, does not appear to give any immediate information about their location or nature. In principle, this can be addressed by finding all zeros of the two-variable denominator p, and then using the techniques in [8,10] to determine the associated contact orders. By examining the matrix-valued function $\zeta_1 \mapsto M_\phi(\zeta_1)$, we can detect any such extraneous singularities and determine their contact orders in a fairly simple way. First, we compute $P_\phi(\zeta_1) = \det M_\phi(\zeta_1)$ and find the zeros $\{\zeta_1^1, \ldots, \zeta_1^s\}$ of the one-variable polynomial P_ϕ that are located on the unit circle. Plugging these values into p, we find the point $\zeta_2 \in \mathbb{T}$ at which the polynomial $p(\zeta_1^j, z_2)$ vanishes as a function of z_2. Finally, the order of vanishing of P_ϕ gives us the z_2-contact order of ϕ at each singularity. By [8, Section 4], this is equal to the z_1-contact order of ϕ as well, allowing us to read off the derivative integrability of ϕ at each singularity.

Example 7 ([8, Example 7.4]). Consider the two-variable RIF

$$\phi(z_1, z_2) = \frac{4z_1^3 z_2 + z_1^3 - z_1^2 + 3z_1 + 1}{4 + z_2 - z_1 z_2 + 3z_1^2 z_2 + z_1^3 z_2},$$

which is obtained from Pascoe's method. His construction guarantees that ϕ has a singularity at $(-1,-1)$ with contact order equal to 4. The slice matrix associated with ϕ is

$$M_\phi(\zeta_1) = \begin{pmatrix} 4\zeta_1^3 & \zeta_1^3 - \zeta_1^2 + 3\zeta_1 + 1 \\ \zeta_1^3 + 3\zeta_1^2 - \zeta_1 + 1 & 4 \end{pmatrix}$$

and has determinant

$$P_\phi(\zeta_1) = \det M_\phi(\zeta_1) = -(\zeta_1 - 1)^2(\zeta_1 + 1)^4.$$

We immediately discern that ϕ has an additional singularity at $(1,-1)$, with contact order equal to 2, as was checked in an ad hoc way in [8].

Further derivative integrability cutoffs of d-variable RIFs. In [9], a glueing construction from [8, Section 7] was adapted to three variables and was used to exhibit a three-variable RIF with a single isolated singularity and worse derivative integrability properties than the three-variable instance of (1.2). The motivation for this was to demonstrate that bad derivative integrability in higher dimensions does not necessarily require a large singularity set. The drawback of that example is that the RIF so constructed has tridegree $(2,2,2)$, which in turn causes the verification of its claimed derivative integrability cutoff to involve lengthy computations. Thus, [9, Question 3] asked whether there exist tridegree $(n_1, n_2, 1)$ RIFs manifesting the same phenomenon. The example below answers this in the affirmative, in all dimensions $d \geq 3$.

Example 8. For $d \geq 2$ fixed and $N \in \mathbb{N}$, we consider the RIF in (1.2) and its associated compositions RIFs $\phi_d^N = \tilde{p}_{d,N}/p_{d,N}$, all of degree 1 in z_d. Reading off the slice matrix of ϕ_d from (1.2), we check that $M_{\phi_d}(\vec{1})^N$ has non-zero entries. This means there are no common factors vanishing at $\vec{1}$ to cancel in ϕ_d^N, and we can proceed as in Theorem 5.

In [9, Example 2.5], it was shown that near $\vec{1}_{d-1} \in \mathbb{T}^{d-1}$,

$$1 - |\psi^0(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2 \asymp \sum_{k=1}^{d-1} \theta_k^2,$$

and hence

$$\left[|\tilde{p}_1(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2 - |\tilde{p}_2(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2\right]^N \asymp \left(\sum_{k=1}^{d-1} \theta_k^2\right)^N.$$

Then $\frac{\partial \phi_{d,N}}{\partial z_d} \in L^{\mathfrak{p}}(\mathbb{T}^d)$ if and only if

$$\int_{B_\epsilon(\vec{0})} \left(\sum_{k=1}^{d-1} \theta_k^2\right)^{N(1-\mathfrak{p})} dm$$

converges for small $\epsilon > 0$. In $d-1$-dimensional polar coordinates, this corresponds to the convergence of

$$\int_0^1 r^{2N(1-\mathfrak{p})} r^{d-2} dr = \int_0^1 r^{2N+d-2-2N\mathfrak{p}} dr.$$

This integral is finite if and only if $-1 < 2N + d - 2 - 2N\mathfrak{p}$, and hence we obtain the z_d-derivative integrability cutoffs

$$\mathfrak{p}^*(d, N) = 1 + \frac{d-1}{2N}. \tag{4.1}$$

This extends the work in [9, Example 2.5], where it was shown that $\mathfrak{p}^*(d, 1) = 1 + (d-1)/2 = (d+1)/2$. Next, since $\mathfrak{p}^*(2, N) = 1 + 1/(2N)$, we observe that any contact order is realized by a RIF with a unique singularity at $(1, 1) \in \mathbb{T}^2$.

Finally, the fact that $\mathfrak{p}^*(3, 2) = 3/2$ shows that $\phi_{3,2}$, with denominator

$$p_{3,2}(z) = 9 - 6z_1 - 6z_2 - 3z_3 + z_1^2 + z_2^2 + 3z_1z_2 + 2z_1z_3 + 2z_2z_3 - 3z_1z_2z_3,$$

provides an example of a RIF giving a positive answer to [9, Question 3].

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References

[1] Agler, J., McCarthy, J.E.: Pick Interpolation and Hilbert Function Spaces. Graduate Studies in Mathematics, 44. American Mathematical Society, Providence, RI (2002)

[2] Agler, J., McCarthy, J.E., Young, N.J.: A Carathéodory theorem for the bidisk via Hilbert space methods. Math. Ann. **352**, 581–624 (2012)

[3] Anderson, J.M., Dritschel, M.A., Rovnyak, J.: Schwarz-Pick inequalities for the Schur-Agler class on the polydisk and the unit ball. Comput. Methods Funct. Theory **8**, 339–361 (2008)

[4] Ball, J.A., Sadosky, C., Vinnikov, V.: Scattering systems with several evolutions and multidimensional input/state/output systems. Integral Equations Operator Theory **52**, 323–393 (2005)

[5] Bergqvist, L.: Rational inner functions and their Dirichlet-type norms. Comput. Methods Funct. Theory, to appear. https://doi.org/10.1007/s40315-022-00465-1

[6] Bickel, K., Cima, J.A., Sola, A. A.: Clark measures for rational inner functions. Michigan Math. J., to appear. https://doi.org/10.1307/mmj/20216046

[7] Bickel, K., Pascoe, J.E., Sola, A.: Derivatives of rational inner functions: geometry of singularities and integrability at the boundary. Proc. Lond. Math. Soc. **116**, 281–329 (2018)

[8] Bickel, K., Pascoe, J.E., Sola, A.: Level curve portraits of rational inner functions. Ann. Sc. Norm. Sup. Pisa Cl. Sc. (5) **21,** 451–494 (2020)

[9] Bickel, K., Pascoe, J.E., Sola, A.: Singularities of rational inner functions in higher dimensions. Amer J. Math. **144**, 1115–1157 (2022)

[10] Bickel, K., Knese, G., Pascoe, J.E., Sola, A.: Local theory of stable polynomials and bounded rational functions of several variables. arXiv:2109.07507 (2021)

[11] Garcia, S.R., Mashreghi, J., Ross, W.T.: Finite Blaschke Products and Their Connections. Springer, Cham (2018)

[12] Jones, G.A., Singerman, D.: Complex Functions. An Algebraic and Geometric Viewpoint. Cambridge University Press, Cambridge (1987)

[13] Knese, G.: Schur-Agler rational inner functions on the tridisk. Proc. Amer. Math. Soc. **139**, 4063–4072 (2011)

[14] Knese, G.: Rational inner functions in the Schur-Agler class of the polydisk. Publ. Mat. **55**, 343–357 (2011)

[15] Knese, G.: Stable symmetric polynomials and the Schur-Agler class. Illinois J. Math. **55**, 1603–1620 (2011)

[16] Knese, G.: Integrability and regularity of rational functions. Proc. Lond. Math. Soc. **111**, 1261–1306 (2015)

[17] Kollár, J.: Bounded meromorphic functions on the 2-disc. arXiv:2202.04043 (2022)

[18] Pascoe, J.E.: A wedge-of-the-edge theorem: analytic continuation of Pick functions. Bull. London Math. Soc. **49**, 926–925 (2017)

[19] Pascoe, J.E.: An inductive Julia-Carathéodory theorem for Pick functions in two variables. Proc. Edinburgh Math. Soc. **61**, 647–660 (2018)

[20] Rudin, W.: Function Theory in Polydisks. W. A. Benjamin Inc., New York-Amsterdam (1969)

[21] Sola, A., Tully-Doyle, R.: Iteration of low-degree rational inner skew-products on $\mathbb{T}^2$. Ann. Polon. Math. **128**, 249–273 (2022)

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