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A note on strong conciseness in virtually nilpotent profinite groups

Eloisa Detomi

Abstract. A group word w is said to be strongly concise in a class $\mathcal{C}$ of profinite groups if, for every group G in $\mathcal{C}$ such that w takes less than $2^{\aleph_0}$ values in G, the verbal subgroup $w(G)$ is finite. In this paper, we prove that every group word is strongly concise in the class of virtually nilpotent profinite groups.

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1. Introduction. Let $w = w(x_1, \ldots, x_k)$ be a group word, that is, a nontrivial element of the free group on $x_1, \ldots, x_k$. The word w can be naturally viewed as a function of k variables defined on any group G. The subgroup generated by the set of w-values is called the verbal subgroup corresponding to the word w. We denote this by $w(G)$. In the context of topological groups G, we write $w(G)$ to denote the closed subgroup generated by all w-values in G.

The word w is said to be concise in the class of groups $\mathcal{C}$ if the verbal subgroup $w(G)$ is finite whenever w takes only finitely many values in a group $G \in \mathcal{C}$ (see for example [2,3], and references therein for results on conciseness of words).

For topological groups, especially profinite groups, variations of the classical notion of conciseness arise quite naturally (see [1,4]). Following [4], we say that w is strongly concise in a class of profinite groups $\mathcal{C}$ if the verbal subgroup $w(G)$ is finite in any group $G \in \mathcal{C}$ in which w takes less than $2^{\aleph_0}$ values. Several results on strong conciseness of group words can be found in [4]: in particular, it is proved that every group word is strongly concise in the class of nilpotent profinite groups.

In this paper, we apply the approach via parametrized words introduced in [4] to deal with generalized words, where automorphisms might appear, and we extend the above result on strong conciseness to virtually nilpotent profinite groups, proving the following:

Theorem 1.1. *Every group word w is strongly concise in the class of virtually nilpotent profinite groups.*

Recall that a profinite group is virtually $\mathcal{C}$ if it has an open normal subgroup belonging to the class $\mathcal{C}$. Note that for finitely generated profinite groups, a stronger result holds (see [6, Theorem 4.1.5]): given a word w and a finitely generated virtually abelian-by-nilpotent profinite group G, there exists an integer t such that every element of $w(G)$ can be written as a product of at most t w-values or inverses of w-values. Hence every word is strongly concise in the class of finitely generated virtually abelian-by-nilpotent profinite groups.

As a corollary of Theorem 1.1, we obtain the following:

Corollary 1.2. *Let G be a virtually soluble profinite group and let l be the length of a normal series with nilpotent factors of an open soluble normal subgroup of G. If w is a group word that takes less than $2^{\aleph_0}$ values in G, then $w^l(G)$ is finite.*

Here $w^l(G)$ is inductively defined as $w^1(G) = w(G)$ and $w^l(G) = w(w^{l-1}(G))$, when $l \geq 2$.

2. Generalized parametrized words. In this section, we expand definitions and results of [4, Section 5] in order to deal with 'generalized words', as defined in [6, Section 1.3]. Generalized words are, roughly speaking, group words in which the variables are twisted by automorphisms. All the definitions regarding "parametrized words" in [4, Section 5] have to be formally restated to include the possibility of automorphisms, but then the proofs in that section will work smoothly with these modifications, so we will only state the relative results.

Throughout this section, we fix a profinite group G, a set Ψ of automorphisms of G, a positive integer $r \in \mathbb{N}$, and a normal subgroup $\mathbf{G} \trianglelefteq G \times \cdots \times G$ of the direct product of r copies of G. A typical situation would be $\mathbf{G} = G_1 \times \cdots \times G_r$, where $G_1, \ldots, G_r = G$.

We consider two disjoint alphabets

$$X = \{x_1, x_2, \ldots\} \quad \text{and} \quad \Phi = \{\phi_\alpha \mid \alpha \in \Psi\}$$

and we set $F(X \cup \Phi)$ to be the free group on $X \cup \Phi$. A *generalized word* for G is an element w of $F_\Phi(X)$ where

$$F_\Phi(X) = \langle x^\phi \mid x \in X,\ \phi \in \Phi \rangle < F(X \cup \Phi)$$

(here Φ is in bijection with the subset Ψ of automorphisms of G).

To such an element $w \in F_\Phi(X)$, say involving the variables $x_1, \ldots, x_r$, we associate a 'parametrized (generalized) word' with parameters coming from G where each x_i is intended to take values in G_i and where we formally distinguish repeated occurrences of the same variable, as follows. Let $\Omega = \Omega_{G,r}$ be the alphabet

$$\xi_h, \quad \eta_{1,i}, \eta_{2,i}, \ldots, \eta_{r,i} \qquad \text{for } h \in G \text{ and } i \in \mathbb{N},$$

where Ω and Φ are disjoint. Informally, we think of each free generator ξ_h as a 'parameter variable' that is to take the value h, each $\phi_\alpha \in \Phi$ as an 'automorphism variable' that is to take the value α, and each free generator $\eta_{q,i}$ as a 'free variable' that can be specialised to x_q, irrespective of the additional index i. Let

$$F_\Phi(\Omega) = \langle x^\phi \mid x \in \Omega,\ \phi \in \Phi \rangle < F(\Omega \cup \Phi)$$

where $F(\Omega \cup \Phi)$ is the free group on $\Omega \cup \Phi$. We refer to elements $\omega \in F_\Phi(\Omega)$ as *r-valent (generalized) parametrized words* for G or, since r is fixed throughout, simply as *parametrized words* for G. For $\mathbf{g} = (g_1, \ldots, g_r) \in \mathbf{G}$, we write

$$\underline{\omega}(\mathbf{g}) = \underline{\omega}(g_1, \ldots, g_r) \in G$$

for the ω-value that results from replacing each ξ_h by h, each $\eta_{q,i}$ by g_q, and each ϕ_α by α for all $h \in G$, $q \in \{1, \ldots, r\}$, $i \in \mathbb{N}$, and $\alpha \in \Psi$. In this way, we obtain a parametrized word map

$$\underline{\omega}(\cdot)\colon \mathbf{G} \to G.$$

The *degree* $\deg(\omega)$ of the parametrized word ω is the number of generators $\eta_{q,i}^{\phi_\alpha}$, with $q \in \{1, \ldots, r\}$, $i \in \mathbb{N}$, and $\alpha \in \Psi$, appearing in (the reduced form of) ω; here we care whether a generator $\eta_{q,i}^{\phi_\alpha}$ appears, but not whether it appears repeatedly.

Example 2.1. Let $w(x_1, x_2, x_3) = [[x_1^\alpha, x_2^\alpha, x_2^\beta], [x_2^\alpha, x_3^\alpha]]$, where α, β are automorphisms of G. We set $r = 3$, $\mathbf{G} = G \times G \times G$, and

$$\omega = [[\eta_{1,1}^{\phi_\alpha}, \eta_{2,1}^{\phi_\alpha}, \eta_{2,1}^{\phi_\beta}], [\eta_{2,2}^{\phi_\alpha}, \eta_{3,1}^{\phi_\alpha}]]$$

to model w in the sense that

$$\underline{\omega}(g_1, g_2, g_3) = w(g_1, g_2, g_3) = [[g_1^\alpha, g_2^\alpha, g_2^\beta], [g_2^\alpha, g_3^\alpha]] \quad \text{for all } g_1, g_2, g_3 \in G.$$

Then the 3-valent parametrized word ω has degree 5.

Having extended the definition of parametrized words in order to allow homomorphisms, all the other definitions in [4, Section 5] work smoothly without any further modification. So, for a fixed set $\mathfrak{E} = \mathfrak{E}_{G,r} \subseteq F_\Phi(\Omega)$ of r-valent (generalized) parametrized words for G, an r-valent $\mathfrak{E}$-*product* for $\mathbf{G}$ is a finite sequence $(\epsilon_t)_{t \in T}$, where $\epsilon_t \in \mathfrak{E}$ for each $t \in T$; we denote it by

$$\dot{\prod_{t \in T}} \epsilon_t,$$

where the dot indicates that we consider a formal product and not the parametrized word that results from actually carrying out the multiplication in $F_\Phi(\Omega)$.

Then we define a *length function* on $\mathfrak{E}$ as a map $\ell\colon \mathfrak{E} \to \mathsf{W}$ from $\mathfrak{E}$ into a well-ordered set $\mathsf{W} = (\mathsf{W}, \leq)$ such that elements $\epsilon \in \mathfrak{E}$ whose length $\ell(\epsilon)$ is minimal with respect to $\leq$ also have minimal degree $\deg(\epsilon) = 0$. The length function ℓ induces a total pre-order $\preceq_\ell$ on the set of all r-valent $\mathfrak{E}$-products as follows:

$$\dot{\prod_{s\in S}}\widetilde{\epsilon}_s \ \preceq_\ell \ \dot{\prod_{t\in T}}\epsilon_t \qquad \text{if} \quad \max\{\ell(\widetilde{\epsilon}_s) \mid s\in S\} \leq \max\{\ell(\epsilon_t)\mid t\in T\}.$$

Friendly products are (recursively) defined as follows: an r-valent $\mathfrak{E}$-product $\dot{\prod}_{t\in T}\epsilon_t$ for $\mathbf{G}$ is *friendly* if, for every $\mathbf{b}\in\mathbf{G}$, there exists an r-valent $\mathfrak{E}$-product $\dot{\prod}_{s\in S(\mathbf{b})}\widetilde{\epsilon}_{\mathbf{b},s}$ such that

(F1) $\dot{\prod}_{s\in S(\mathbf{b})}\widetilde{\epsilon}_{\mathbf{b},s}$ is friendly and $\dot{\prod}_{s\in S(\mathbf{b})}\widetilde{\epsilon}_{\mathbf{b},s} \prec_\ell \dot{\prod}_{t\in T}\epsilon_t$ and
(F2) the parametrized words $\omega = \prod_{t\in T}\epsilon_t$ and $\nu_{\mathbf{b}} = \prod_{s\in S(\mathbf{b})}\widetilde{\epsilon}_{\mathbf{b},s}$ satisfy

$$\underline{\omega}(\mathbf{bg}) = \underline{\omega}(\mathbf{b})\cdot\underline{\omega}(\mathbf{g})\cdot\underline{\nu}_{\mathbf{b}}(\mathbf{g}) \qquad \text{for all } \mathbf{g}\in\mathbf{G}.$$

Finally, adapting the proofs of [4, Lemma 5.8 and Proposition 5.9] to these new definitions, we get the following:

Proposition 2.2. *Let $\ell\colon \mathfrak{E}\to \mathsf{W}$ be a length function and let $\omega = \prod_{t\in T}\epsilon_t$, where $\dot{\prod}_{t\in T}\epsilon_t$ is an ℓ-friendly r-valent $\mathfrak{E}$-product for $\mathbf{G}$. Suppose that $\mathbf{V}$ is a closed subgroup of G such that $\mathbf{V}_\omega = \{\underline{\omega}(\mathbf{v}) \mid \mathbf{v}\in\mathbf{V}\}$ has less than $2^{\aleph_0}$ elements. Then $\mathbf{V}_\omega$ is already finite.*

Now we use the terminology introduced above in the context of virtually nilpotent groups. So, let E be a virtually nilpotent profinite group and G an open normal subgroup of E which is nilpotent of class at most c. Let Ψ be the set of automorphisms induced on G by the elements of E.

Denote by $\mathfrak{E}$ the set of all left-normed repeated commutators in the free generators ξ_h and $\eta_{q,i}^{\phi_\alpha}$ of $F_\Phi(\Omega)$ subject to the restriction that each $\eta_{q,i}^{\phi_\alpha}$ appears at most once. In other words, $\mathfrak{E}$ consists of all γ_m-values, for $m\geq 2$, that result from replacing the m variables in $\gamma_m = [x_1,x_2,\ldots,x_m]$ by arbitrary free generators ξ_h and $\eta_{q,i}^{\phi_\alpha}$ of $F_\Phi(\Omega)$ subject to the restriction that each $\eta_{q,i}^{\phi_\alpha}$ appears at most once.

For instance, given some element $a\in G$ and automorphisms α,β of G,

$$\epsilon_1 = [\xi_a,\eta_{1,1}^{\phi_\alpha},\xi_a,\eta_{2,1}^{\phi_\beta},\eta_{2,2}^{\phi_\beta},\eta_{2,3}^{\phi_\beta}]\in\mathfrak{E},$$

whereas $\epsilon_2 = [\xi_a,\eta_{1,1}^{\phi_\alpha},\xi_a,\eta_{2,1}^{\phi_\beta},\eta_{2,2}^{\phi_\beta},\eta_{2,1}^{\phi_\beta}]$ does not lie in $\mathfrak{E}$, even though $\underline{\epsilon}_1(\cdot) = \underline{\epsilon}_2(\cdot)$.

We set $\mathsf{W} = \mathbb{N}_0\times\mathbb{N}_0$, equipped with the lexicographic order $\leq$.

Every $\epsilon\in\mathfrak{E}$ belongs to $\gamma_2(F_\Phi(\Omega))$ by definition. We denote by $k(\epsilon)$ the maximal $j\in\{1,\ldots,c+1\}$ such that $\epsilon\in\gamma_j(F_\Phi(\Omega))$, and we define a length function on $\mathfrak{E}$ by associating to ϵ the length

$$\ell(\epsilon) = \big(c+1-k(\epsilon),\deg(\epsilon)\big)\in\mathsf{W}.$$

For instance, if $c=8$, then $\ell(\epsilon_1) = (8+1-6,4) = (3,4)$.

Now it is not difficult to check that the same arguments of [4, Lemmas 5.5, 5.6, and 5.7], applied to generalized words, yield the following:

Lemma 2.3. *Suppose that G is a nilpotent open normal subgroup of a profinite group E. Let $\omega \in \gamma_k(F_\Phi(\Omega))$, where $k \geq 2$, be such that $\underline{\omega}(\mathbf{1}) = 1$. Then there exists an ℓ-friendly $\mathfrak{E}$-product $\dot{\prod}_{t\in T} \epsilon_t$ such that*

$$\underline{\omega}(\mathbf{g}) = \prod_{t\in T} \underline{\epsilon}_t(\mathbf{g}) \qquad \textit{for all } \mathbf{g} \in \mathbf{G}.$$

3. Virtually nilpotent groups. In this section, we prove Theorem 1.1. We will need the following observation:

Lemma 3.1 ([4, Lemma 2.2]). *Let G be a profinite group and let $x \in G$. If the conjugacy class $\{x^g \mid g \in G\}$ contains less than $2^{\aleph_0}$ elements, then it is finite.*

A classical result of Turner-Smith states that every word is concise in the class of groups all of whose quotients are residually finite [7]. Thus every word is concise in the class of finitely generated virtually nilpotent groups. As mentioned in the introduction, every word is actually strongly concise in the class of finitely generated virtually abelian-by-nilpotent profinite groups (see [6, Theorem 4.1.5]).

Proof of Theorem 1.1. Suppose that the profinite group E has a normal open subgroup G which is nilpotent of class at most c. Let $v = v(x_1, \ldots, x_s)$ be a word such that the set of v-values $E_v = \{v(y_1, \ldots, y_s) \mid (y_1, \ldots, y_s) \in E^s\}$ has less than $2^{\aleph_0}$ elements.

Choose a transversal $\mathcal{A}$ of E/G and an element $\mathbf{a} = (a_1, \ldots, a_s) \in \mathcal{A}^s$.

Define the generalized word $v'_{\mathbf{a}}$ on G as follows (see also [6, Section 1.3]):

$$v'_{\mathbf{a}}(x_1, \ldots, x_s) = v(a_1x_1, \ldots, a_sx_s)v(a_1, \ldots, a_s)^{-1}.$$

Note that $v'_{\mathbf{a}} \in F_\Phi(\{x_1, \ldots, x_s\})$, where the automorphisms involved are induced by the conjugation action of E on G. Moreover $v'_{\mathbf{a}}(1, \ldots, 1) = 1$ and $|G_{v'_{\mathbf{a}}}| \leq |G_v|^2 < 2^{\aleph_0}$.

Consider the generalized word in $r = 2s$ variables

$$w_{\mathbf{a}}(x_1, \ldots, x_r) = v'_{\mathbf{a}}(x_1x_{s+1}, \ldots, x_sx_{2s})v'_{\mathbf{a}}(x_1, \ldots, x_s)^{-1}v'_{\mathbf{a}}(x_{s+1}, \ldots, x_r)^{-1},$$

and set $X = \{x_1, \ldots, x_r\}$. Clearly, for $\mathbf{g} = (g_1, \ldots, g_s), \mathbf{h} = (h_1, \ldots, h_s) \in G^s$,

$$w_{\mathbf{a}}(g_1, \ldots, g_s, h_1, \ldots, h_s) = v'_{\mathbf{a}}(\mathbf{gh})v'_{\mathbf{a}}(\mathbf{g})^{-1}v'_{\mathbf{a}}(\mathbf{h})^{-1},$$

and $w_{\mathbf{a}}(\mathbf{1}) = 1$. Moreover $|G_{w_{\mathbf{a}}}| < 2^{\aleph_0}$.

Note that $w_{\mathbf{a}} \in \gamma_2(F_\Phi(X))$. Thus we can find a generalized parametrized word $\omega_{\mathbf{a}} \in F_\Phi(\Omega)$ such that $\omega_{\mathbf{a}}$ belongs to $\gamma_2(F_\Phi(\Omega))$,

$$\underline{\omega}_{\mathbf{a}}(g_1, \ldots, g_r) = w_{\mathbf{a}}(g_1, \ldots, g_r) \quad \text{for every } \mathbf{g} = (g_1, \ldots, g_t) \in G^r,$$

and $\underline{\omega}_{\mathbf{a}}(\mathbf{1}) = 1$.

Using Lemma 2.3, we apply Proposition 2.2 to deduce that $G_{\omega_{\mathbf{a}}}$ is finite. So, $G_{w_{\mathbf{a}}} = G_{\omega_{\mathbf{a}}}$ is finite and thus we can find a finitely generated subgroup H of E such that $G_{w_{\mathbf{a}}} \subseteq v(H)$.

As every word is strongly concise in the class of finitely generated virtually abelian-by-nilpotent profinite groups, we deduce that $v(H)$ is finite. Moreover, every v-value has a finite number of conjugates by Lemma 3.1. From Dicman's lemma [5, 14.5.7], it follows that $v(H)$ is contained in a finite normal subgroup of E, and we can pass to the quotient over this subgroup. Thus we can assume that $G_{w_{\mathbf{a}}}$ is trivial.

Now we have

$$v'_{\mathbf{a}}(\mathbf{gh})v'_{\mathbf{a}}(\mathbf{g})^{-1}v'_{\mathbf{a}}(\mathbf{h})^{-1} = w_{\mathbf{a}}(\mathbf{gh}) = 1$$

for every $\mathbf{g}, \mathbf{h} \in G^s$, thus $v'_{\mathbf{a}}$ induces an homomorphism from G^s to G and so $G_{v'_{\mathbf{a}}}$ is a subgroup. Since $|G_{v'_{\mathbf{a}}}| < 2^{\aleph_0}$, we conclude that $G_{v'_{\mathbf{a}}}$ is actually finite.

Repeating the argument for each of the finitely many $\mathbf{a} \in \mathcal{A}^s$, we are reduced to the case where

$$E_v = \bigcup_{\mathbf{a} \in \mathcal{A}^s} \left(G_{v'_{\mathbf{a}}} \cdot v(a_1, \ldots, a_s) \right)$$

is finite. In particular, we may now assume that E is finitely generated. As E is virtually nilpotent and E_v is finite, we conclude that $v(E)$ is finite, as claimed. □

The proof of Corollary 1.2 is almost straightforward.

Proof of Corollary 1.2. Let R be an open soluble normal subgroup of G and let l be the length of a normal series of R with nilpotent factors. By assumption, $|G_w| < 2^{\aleph_0}$.

We argue by induction on l; for $l = 0$, the assertion holds trivially. Now suppose that $l \geq 1$ and let N be a normal nilpotent subgroup of G, contained in R, such that R/N has a normal series with nilpotent factors of length $l-1$. By induction, $w^{l-1}(G/N) = \{g_1N, \ldots, g_tN\}$ is finite. Set $H = \langle g_1, \ldots, g_t \rangle N$. Since H is virtually nilpotent and $|H_w| \leq |G_w| < 2^{\aleph_0}$, from Theorem 1.1, it follows that $w(H)$ is finite. Hence $w^l(G) \leq w(H)$ is finite. □

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Eloisa Detomi
Dipartimento di Ingegneria dell'Informazione
Università degli Studi di Padova
Via Gradenigo 6/b
35131 Padova
Italy
e-mail: eloisa.detomi@unipd.it

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