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An inequality on polarized endomorphisms

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Abstract. We show that assuming the standard conjectures, for any smooth projective variety X of dimension n over an algebraically closed field, there is a constant $c > 0$ such that for any positive rational number r and any polarized endomorphism f of X, we have

$$\|G_r \circ f\| \leq c \deg(G_r \circ f),$$

where G_r is a correspondence of X so that for each $0 \leq i \leq 2n$, its pullback action on the i-th Weil cohomology group is the multiplication-by-r^i map. This inequality is known to imply the generalized Weil Riemann hypothesis and is a special case of a more general conjecture by the authors' work Hu and Truong (A dynamical approach to generalized Weil's Riemann hypothesis and semisimplicity. arXiv:2102.04405v3, 2021).

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1. Introduction. Let X be a smooth projective variety of dimension n over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic and let H_X be a fixed ample divisor on X. Fix a Weil cohomology theory $H^\bullet(X)$ with coefficients in a field $\mathbf{F}$ of characteristic zero (see [8, §1.2]). Let $r \in \mathbf{Q}_{>0}$ be a positive rational number. Let γ_r be the unique homological correspondence of X, i.e.,

$$\gamma_r \in H^{2n}(X \times X) \simeq \bigoplus_{i=0}^{2n} H^i(X) \otimes_{\mathbf{F}} H^{2n-i}(X) \simeq \bigoplus_{i=0}^{2n} \mathrm{End}_{\mathbf{F}}(H^i(X)),$$

such that its pullback γ_r^* on $H^i(X)$ is the multiplication-by-r^i map for each i. Note that γ_r commutes with all homological correspondences of X.

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If we assume that the standard conjecture C holds on X, then clearly the above γ_r is algebraic and can be represented by a (rational) correspondence $G_r := \sum_{i=0}^{2n} r^i \Delta_i$, i.e., $\gamma_r = \mathrm{cl}_{X\times X}(G_r)$ (see [4, Remark 1.8]). Note that the real vector space $\mathsf{N}^n(X \times X)_{\mathbf{R}}$ of numerical cycle classes of codimension n on $X\times X$ is finite dimensional (see [8, Theorem 3.5]); we thus endow it with a norm $\|\cdot\|$. We also fix a degree function deg on $\mathsf{N}^n(X \times X)_{\mathbf{R}}$ with respect to a fixed ample divisor $H_{X\times X} := \mathrm{pr}_1^* H_X + \mathrm{pr}_2^* H_X$ by setting $\deg(g) := g \cdot H^n_{X\times X}$. The main result of this note is an inequality concerning the norm and the degree of the composite correspondence $G_r \circ f$ of the above G_r and any polarized endomorphism f (viewed as a correspondence via its graph), assuming the standard conjectures. More precisely, we have:

Theorem 1. *Suppose that the standard conjecture B holds on X and the standard conjecture of Hodge type holds on $X \times X$. Then for any $r \in \mathbf{Q}_{>0}$, the above homological correspondence γ_r of X is algebraic and represented by a (rational) correspondence G_r of X; moreover, there exists a constant $c > 0$, depending only on the Betti numbers b_i of X, the dimension n of X, and the choices of norm and degree, but independent of r, so that for any polarized endomorphism f of X (i.e., $f^*H_X \sim qH_X$ for some $q \in \mathbf{N}_{>0}$), we have*

$$\|G_r \circ f\| \le c \deg(G_r \circ f). \tag{1.1}$$

Remark 2. In a letter to Weil, Serre [9] sketched an elegant proof of a Kähler analog of Weil's Riemann hypothesis, which involves the pullback actions of polarized endomorphisms on cohomology groups of compact Kähler manifolds. The positive-characteristic analog of this famous result is still conjectural, which we call generalized Weil's Riemann hypothesis and semisimplicity (see [4, Conjectures 1.4 and 1.5]).

In the 1960s, Bombieri and Grothendieck independently proposed the so-called standard conjectures, which would yield the above generalized Weil Riemann hypothesis and semisimplicity (see [8] for details). It was Deligne [3] who ingeniously solved Weil's Riemann hypothesis. However, his arguments do not seem to be able to solve the aforementioned generalized Weil Riemann hypothesis. As of today, the standard conjectures (and also the generalized Weil Riemann hypothesis) are still widely open. For instance, the standard conjecture D is only known in a few cases (including the divisor case and abelian varieties over finite fields [2][1]), and the standard conjecture of Hodge type is known only for surfaces, abelian fourfolds [1][2], and squares of $K3$ surfaces [7].

Remark 3. The authors of this note conjectured the inequality (1.1) in a more general setting of correspondences (see [4, Conjecture G_r]), whose validity implies the generalized Weil Riemann hypothesis (see [4, Theorem 1.9 and Remark 1.10(1)]). See also the authors' related works [5,6,10]. We have also shown that the inequality (1.1) indeed holds for (all effective correspondences

[1] Indeed, Clozel [2] showed that for abelian varieties over finite fields $\mathbf{F}_{p^n}$, there are infinitely many primes $\ell \neq p$ such that the standard conjecture D holds for ℓ-adic étale cohomology.
[2] In fact, Ancona [1] proved a (weaker) numerical version of the standard conjecture of Hodge type for abelian fourfolds.

of) abelian varieties. It has then been argued in [4] that this inequality could be viewed as an alternative way towards the generalized Weil Riemann hypothesis, compared to the standard conjectures. Our Theorem 1 confirms that this is indeed the case: for polarized endomorphisms, our inequality (1.1) follows from the standard conjectures. We thus wonder if a general version of the inequality (1.1) for effective correspondences is also a consequence of the standard conjectures.

2. Proof of Theorem 1. Recall that X denotes a smooth projective variety of dimension n over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic and H_X is a fixed ample divisor on X. We also fix a Weil cohomology theory $H^\bullet(X)$ with a coefficient field $\mathbf{F}$ of characteristic zero (see [8, §1.2]). In particular, we have a cup product $\cup$, Poincaré duality, the Künneth formula, the cycle class map cl_X, the Lefschetz trace formula, the weak Lefschetz theorem, and the hard Lefschetz theorem. Examples of classical Weil cohomology theories include:

- de Rham cohomology $H^\bullet_{\mathrm{dR}}(X(\mathbf{C}), \mathbf{C})$ if $\mathbf{k} \subseteq \mathbf{C}$,
- étale cohomology $H^\bullet_{ét}(X, \mathbf{Q}_\ell)$ with $\ell \neq \mathrm{char}(\mathbf{k})$ if $\mathbf{k}$ is arbitrary,
- crystalline cohomology $H^\bullet_{\mathrm{crys}}(X/W(\mathbf{k})) \otimes \mathbf{K}$, where $\mathbf{K}$ is the field of fractions of the Witt ring $W(\mathbf{k})$.

For the fixed ample divisor H_X on X and for $0 \leq i \leq 2n-2$, we let

$$\begin{aligned} L\colon H^i(X) &\to \quad H^{i+2}(X), \\ \alpha \quad &\mapsto \mathrm{cl}_X(H_X) \cup \alpha, \end{aligned} \tag{2.1}$$

be the *Lefschetz operator*.

By the hard Lefschetz theorem, for any $0 \leq i \leq n$, the $(n-i)$-th iterate L^{n-i} of the Lefschetz operator L is an isomorphism

$$L^{n-i}\colon H^i(X) \xrightarrow{\sim} H^{2n-i}(X).$$

However, $L^{n-i+1}\colon H^i(X) \to H^{2n-i+2}(X)$ may have a nontrivial kernel. Denote by $P^i(X)$ the set of cohomology classes $\alpha \in H^i(X)$, called *primitive*, satisfying $L^{n-i+1}(\alpha) = 0$, namely,

$$P^i(X) := \mathrm{Ker}(L^{n-i+1}\colon H^i(X) \to H^{2n-i+2}(X)) \subseteq H^i(X). \tag{2.2}$$

This gives us the following primitive decomposition (a.k.a. Lefschetz decomposition):

$$H^i(X) = \bigoplus_{j \geq i_0} L^j P^{i-2j}(X), \tag{2.3}$$

where $i_0 := \max(i-n, 0)$.

Definition 4 (cf. [8, §1.4]). For any $\alpha \in H^i(X)$, we write

$$\alpha = \sum_{j \geq i_0} L^j(\alpha_j), \quad \alpha_j \in P^{i-2j}(X). \tag{2.4}$$

Then we define an operator $*$ as follows:

$$\begin{aligned} *\colon H^i(X) &\to \qquad H^{2n-i}(X), \\ \alpha \quad &\mapsto *\alpha := \sum_{j\geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} L^{n-i+j}(\alpha_j). \end{aligned} \tag{2.5}$$

It is easy to check that $*^2 = \mathrm{id}$. The standard conjecture $B(X)$ predicts that the above homological correspondence $*$ is algebraic (cf. [8, Proposition 2.3]).

For any homological correspondence g of X, denote by g' its adjoint with respect to the following nondegenerate bilinear form

$$\begin{aligned} H^i(X) \times H^i(X) &\longrightarrow \qquad \mathbf{F}, \\ (\alpha, \beta) \qquad &\mapsto \ \langle \alpha, \beta \rangle := \alpha \cup *\beta. \end{aligned} \tag{2.6}$$

In other words, we have $g' = * \circ g^{\mathsf{T}} \circ *$ by definition, where g^{T} denotes the canonical transpose of g by interchanging the coordinates.

For any $0 \leq k \leq n$, let $\mathsf{A}^k(X) \subseteq H^{2k}(X)$ denote the $\mathbf{Q}$-vector space of cohomology classes generated by algebraic cycles of codimension k on X under the cycle class map cl_X, i.e.,

$$\mathsf{A}^k(X) := \mathrm{Im}(\mathrm{cl}_X \colon \mathsf{Z}^k(X)_{\mathbf{Q}} \longrightarrow H^{2k}(X)).$$

The standard conjecture of Hodge type predicts that, when restricted to $\mathsf{A}^k(X)$, the bilinear form (2.6) is positive definite for all $k \leq n/2$ (see [8, §3] for details).

Lemma 5. *Let $\pi_i \in H^i(X) \otimes H^{2n-i}(X)$ be the i-th Künneth component of the diagonal class, which corresponds to the projection operator $H^\bullet(X) \to H^i(X)$ via the pullback. Then for any polarized endomorphism f of X (i.e., $f^*H_X \sim qH_X$ for some $q \in \mathbf{N}_{>0}$), we have*

$$(\pi_i \circ f) \circ (\pi_i \circ f)' = q^i \pi_i$$

as homological correspondences.

Proof. Note that for any $\alpha \in H^i(X)$ with the above primitive decomposition (2.4),

$$f^*\alpha = \sum_{j\geq i_0} L^j(q^j f^*\alpha_j) \text{ with } f^*\alpha_j \in P^{i-2j}(X)$$

is the primitive decomposition of $f^*\alpha$. It follows that

$$\begin{aligned}
((\pi_i \circ f) \circ (\pi_i \circ f)')^*(\alpha) &= * \circ (\pi_i \circ f)_* \circ * \circ (\pi_i \circ f)^*(\alpha) \\
&= * \circ (\pi_i \circ f)_* \circ * \circ f^*\alpha \\
&= * \circ \pi_{2n-i}^* \circ f_* \sum_{j \geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} L^{n-i+j}(q^j f^*\alpha_j) \\
&= * \sum_{j \geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} f_*(\mathrm{cl}_X(H_X^{n-i+j}) \cup q^j f^*\alpha_j) \\
&= * \sum_{j \geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} f_* \,\mathrm{cl}_X(H_X^{n-i+j}) \cup q^j \alpha_j \\
&= * \sum_{j \geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} q^{i-j} \,\mathrm{cl}_X(H_X^{n-i+j}) \cup q^j \alpha_j \\
&= * \sum_{j \geq i_0} (-1)^{\frac{(i-2j)(i-2j+1)}{2}} q^i L^{n-i+j}(\alpha_j) \\
&= q^i *^2 \alpha \\
&= q^i \alpha,
\end{aligned}$$

where π_i^* and $(\pi_i)_* = \pi_{2n-i}^*$ are projections to $H^i(X)$ and $H^{2n-i}(X)$, respectively, the third equality follows from the definition of the $*$ operator, the fifth one follows from the projection formula, and the last one follows from the fact that $*^2 = \mathrm{id}$. This yields the lemma. □

Proof of Theorem 1. Since the standard conjecture $B(X)$ implies the standard conjecture $C(X)$, the algebraicity of γ_r follows by taking $G_r := \sum_{i=0}^{2n} r^i \Delta_i$, where $\Delta_i \in \mathsf{Z}^n(X \times X)_{\mathbf{Q}}$ represents the i-th Künneth component π_i of the diagonal class. Also, by assumption, the bilinear form (2.6) is a Weil form; see [8, Theorem 3.11]. In particular, if we let f_i denote the composite correspondence $\Delta_i \circ f$, then the square root of

$$\mathrm{Tr}((f_i \circ f_i')^*|_{H^\bullet(X)}) = \mathrm{Tr}((f_i \circ f_i')^*|_{H^i(X)}) \in \mathbf{Q}_{>0}$$

gives us a norm $\|\cdot\|$ of $f^*|_{H^i(X)}$. On the other hand, it follows from Lemma 5 that

$$\mathrm{Tr}((f_i \circ f_i')^*|_{H^i(X)}) = q^i b_i,$$

where $b_i := \dim_{\mathbf{F}} H^i(X)$ is the i-th Betti number of X. Putting together, we thus obtain that

$$\left\| f^*|_{H^i(X)} \right\| = b_i^{1/2} q^{i/2}.$$

Now, we let g_r denote $G_r \circ f$. By assumption, the standard conjecture D holds on $X \times X$ (see [8, Corollaries 3.9, 2.5, and 2.2]). Hence the cycle class map induces an injective map

$$\mathsf{N}^n(X \times X) \otimes_{\mathbf{Z}} \mathbf{F} \hookrightarrow H^{2n}(X \times X);$$

see [8, Proposition 3.6]. It thus follows that

$$\|g_r\| \lesssim \|\,\mathrm{cl}_{X \times X}(g_r)\|,$$

where the right-hand side denotes a norm on $H^{2n}(X\times X) \simeq \bigoplus_{i=0}^{2n} \mathrm{End}_{\mathbf{F}}(H^i(X))$, equivalent to

$$\max_{0\le i\le 2n} \left\| g_r^*|_{H^i(X)} \right\|.$$

Note that the above equivalence part depends on the choices of norms. Also, by the definitions of G_r and f, we have that $g_r^*|_{H^i(X)} = r^i f^*|_{H^i(X)}$ and

$$\deg(g_r) = g_r \cdot H^n_{X\times X} = \sum_{k=0}^{n} \binom{n}{k} g_r \cdot \mathrm{pr}_1^* H_X^{n-k} \cdot \mathrm{pr}_2^* H_X^k.$$

For simplicity, we denote

$$\deg_k(g_r) := g_r \cdot \mathrm{pr}_1^* H_X^{n-k} \cdot \mathrm{pr}_2^* H_X^k = g_r^* H_X^k \cdot H_X^{n-k} = r^{2k} q^k H_X^n.$$

If $i = 2k$ is even, then we have that

$$\left\| g_r^*|_{H^i(X)} \right\| = r^{2k} \left\| f^*|_{H^{2k}(X)} \right\| = r^{2k}\, b_{2k}^{1/2}\, q^k = b_{2k}^{1/2} \deg_k(g_r)/H_X^n.$$

When $i = 2k+1$ is odd, similarly, one also has that

$$\begin{aligned}
\left\| g_r^*|_{H^i(X)} \right\| &= r^{2k+1} \left\| f^*|_{H^{2k+1}(X)} \right\| \\
&= r^{2k+1}\, b_{2k+1}^{1/2}\, q^{(2k+1)/2} \\
&\le b_{2k+1}^{1/2} (r^{2k} q^k + r^{2k+2} q^{k+1})/2 \\
&\le b_{2k+1}^{1/2} \max\{r^{2k} q^k, r^{2k+2} q^{k+1}\} \\
&= b_{2k+1}^{1/2} \max\{\deg_k(g_r), \deg_{k+1}(g_r)\}/H_X^n.
\end{aligned}$$

So overall, there is a constant $c > 0$ depending only on the Betti numbers b_i of X, the dimension n of X, and the choices of norm and degree, but independent of f and r, such that

$$\|g_r\| \le c \max_{0\le k\le n} \deg_k(g_r) \le c \deg(g_r).$$

We thus proved Theorem 1. □

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