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The infinite product of contraction semigroups on $l^1(\mathbb{N})$ and $l^\infty(\mathbb{N})$

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Abstract. I provide an example of a family of commuting contraction semigroups $(e^{tB_n})_{n\in\mathbb{N}}$ defined on $l^1(\mathbb{N})$ such that the product semigroup $\prod_{n=1}^{\infty} e^{tB_n}$ exists and has bounded generator. The infinite product of the corresponding family of adjoint semigroups $(e^{tB_n^*})_{n\in\mathbb{N}}$ defined on $l^\infty(\mathbb{N})$ also exists and its generator is bounded. I give explicit formulae for these generators. The results follow from a general convergence theorem for such semigroups proved in Arendt et al. (J Funct Anal 160: 524–542, 1998).

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1. Introduction. In 1958, D. Blackwell gave an example of a continuous-time Markov chain with all states instantaneous. The nature of this chain allows to perform some explicit or semi-explicit calculations related to it, see e.g. [4,5,8,10].

In [9], I found an explicit formula for an (unbounded) operator A such that its closure is the generator of a strongly continuous semigroup of Markov operators associated with Blackwell's chain. Based on this article, I give here two examples of semigroups that fit in the framework of [1, Proposition 2.7]. In order to cite this proposition, which I call Theorem 1.1, a short introduction is needed. I use the convention that $\mathbb{N} = \{1, 2, 3, \dots\}$.

Suppose that $(B_n)_{n\in\mathbb{N}}$ is a sequence of generators of commuting contraction semigroups defined on a Banach space $(X, ||\cdot||)$. This means that for any $m, n \in \mathbb{N}$ and $x \in X$, the following identity holds

$$e^{tB_n}e^{sB_m}x = e^{sB_m}e^{tB_n}x, \quad t, s \geq 0.$$

Furthermore, $\lim_{t\to 0+} e^{tB_n}x = x$, $||e^{tB_n}x|| \leq ||x||$ for every $x \in X$, $n \in \mathbb{N}$, and $t \geq 0$. And finally, if n is fixed, then e^{tB_n} is a semigroup, i.e., $e^{(t+s)B_n} = e^{tB_n}e^{sB_n}$ for all $t, s \geq 0$. In general, the generators may not be bounded so $D(B_n)$ denotes the domain of B_n.

If we have a sequence of semigroups described above, then for any $n \geq 2$, the product

$$T_n(t)x = \prod_{k=1}^{n} e^{tB_k}x, \quad t \geq 0,$$

is also a strongly continuous contraction semigroup on X and its generator is the closure of A_n given by

$$A_n x = \sum_{k=1}^{n} B_k x, \quad x \in \bigcap_{k=1}^{n} D(B_k), \tag{1.1}$$

see [2, p. 24]. As in [1], we say that the product $\prod_{k=1}^{\infty} e^{tB_k}$ exists if $T(t)x := \lim_{n\to\infty} T_n(t)x$ converges uniformly on compact subsets of $[0, \infty)$ for every $x \in X$. Then again $T(t)$, $t \geq 0$, is a C_0 semigroup of contractions on X. The following theorem was proved in [1].

Theorem 1.1. *Let $(e^{tB_n})_{n\in\mathbb{N}}$ be a commuting family of contraction semigroups and suppose that*

$$D_1 = \left\{ x \in \bigcap_{k=1}^{\infty} D(B_k) : \sum_{k=1}^{+\infty} ||B_k x|| < \infty \right\} \tag{1.2}$$

is dense in X. Then the product $\prod_{k=1}^{\infty} e^{tB_k}$ exists. Moreover, define A by

$$Ax = \lim_{n\to+\infty} \sum_{k=1}^{n} B_k x, \quad x \in D_1.$$

Then A is closable and its closure is the generator of $\prod_{k=1}^{\infty} e^{tB_k}$.

2. The semigroups. Recall that $l^1(\mathbb{N})$ is the Banach space of all absolutely summable sequences $x = (\xi_i)_{i\in\mathbb{N}}$. This means that $\sum_{i\in\mathbb{N}} |\xi_i| < \infty$ and $||x||_{l^1(\mathbb{N})} = \sum_{i\in\mathbb{N}} |\xi_i|$.

Suppose that α_n, β_n, $n \geq 1$, are positive numbers and denote

$$S_n^1 = \{1, 2, 3, \ldots, 2^{n-1}\}, \quad S_n^2 = \{0, 2^{n-1} + 1, \ldots, 2^n - 1\}.$$

For example, $S_1^1 = \{1\}$, $S_1^2 = \{0\}$, $S_2^1 = \{1, 2\}$, $S_2^2 = \{0, 3\}$, etc.

For $n \geq 1$, define B_n on $l^1(\mathbb{N})$ as follows

$$B_n x = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} -\beta_n\xi_i + \alpha_n\xi_{i+2^{n-1}} & \text{if } i \bmod 2^n \in S_n^1, \\ -\alpha_n\xi_i + \beta_n\xi_{i-2^{n-1}} & \text{if } i \bmod 2^n \in S_n^2. \end{cases} \tag{2.1}$$

It is clear that the B_n's are linear bounded operators and

$$||B_n x||_{l^1(\mathbb{N})} \leq 2(\alpha_n + \beta_n)||x||_{l^1(\mathbb{N})}.$$

In Lemma 2.1, I prove that the B_n's commute. It may be verified directly that

$$\sum_{i=1}^{\infty} \eta_i = 0 \tag{2.2}$$

and this property is a consequence of the fact that the B_n's are isomorphic images of generators of Markov semigroups, see [9, Corollary 2]. The property (2.2) is also true for A in Theorem 3.1 since the norm convergence in $l^1(\mathbb{N})$ implies the coordinate-wise convergence.

It is well known (see [3, p. 207]) that the dual space of $l^1(\mathbb{N})$ can be identified with $l^\infty(\mathbb{N})$, that is, with the space of all bounded sequences. If $x = (\xi_i)_{i\in\mathbb{N}}$ is an element of $l^\infty(\mathbb{N})$, then $||x||_{l^\infty(\mathbb{N})} = \sup_{i\in\mathbb{N}} |\xi_i|$.

Thus B_n induces a linear map $B_n^* : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ called the adjoint of B_n, see [6, p. 15.] In our case it is given by

$$B_n^* x = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} \beta_n(-\xi_i + \xi_{i+2^{n-1}}) & \text{if } i \bmod 2^n \in S_n^1, \\ \alpha_n(-\xi_i + \xi_{i-2^{n-1}}) & \text{if } i \bmod 2^n \in S_n^2. \end{cases} \tag{2.3}$$

Moreover

$$||B_n^* x||_{l^\infty(\mathbb{N})} \le 2\max\{\alpha_n, \beta_n\}||x||_{l^\infty(\mathbb{N})}.$$

Since the B_n's and the B_n^*'s are continuous, they are the generators of the following semigroups

$$e^{tB_n} = \sum_{k=0}^{\infty} \frac{(tB_n)^k}{k!}, \quad e^{tB_n^*} = \sum_{k=0}^{\infty} \frac{(tB_n^*)^k}{k!}, \quad t \ge 0.$$

By a direct computation, we find (see also [8, p.60])

$$e^{tB_n} x = \begin{cases} p_n(t)\xi_i + (1 - q_n(t))\xi_{i+2^{n-1}} & \text{if } i \bmod 2^n \in S_n^1, \\ q_n(t)\xi_i + (1 - p_n(t))\xi_{i-2^{n-1}} & \text{if } i \bmod 2^n \in S_n^2, \end{cases} \tag{2.4}$$

and

$$e^{tB_n^*} x = \begin{cases} p_n(t)\xi_i + (1 - p_n(t))\xi_{i+2^{n-1}} & \text{if } i \bmod 2^n \in S_n^1, \\ q_n(t)\xi_i + (1 - q_n(t))\xi_{i-2^{n-1}} & \text{if } i \bmod 2^n \in S_n^2, \end{cases} \tag{2.5}$$

where $p_n(t)$, $q_n(t)$, $t \ge 0$, are given by

$$p_n(t) = \frac{\alpha_n}{\alpha_n + \beta_n} + \frac{\beta_n}{\alpha_n + \beta_n} e^{-(\alpha_n+\beta_n)t},$$
$$q_n(t) = \frac{\beta_n}{\alpha_n + \beta_n} + \frac{\alpha_n}{\alpha_n + \beta_n} e^{-(\alpha_n+\beta_n)t}.$$

Notice that $0 < p_n(t) \le 1$, $0 < q_n(t) \le 1$ and as a result

$$||e^{tB_n} x||_{l^1(\mathbb{N})} \le ||x||_{l^1(\mathbb{N})}, \quad ||e^{tB_n^*} x||_{l^\infty(\mathbb{N})} \le ||x||_{l^\infty(\mathbb{N})}$$

meaning that e^{tB_n}, $e^{tB_n^*}$ are contractions.

Lemma 2.1. *For the B_n's defined by (2.1) and $m, n \in \mathbb{N}$, we have*

$$B_n B_m x = B_m B_n x, \quad x \in l^1(\mathbb{N}). \tag{2.6}$$

This implies that the B_n^**'s also commute and in consequence*

$$e^{tB_m}e^{sB_n}x = e^{sB_n}e^{tB_m}x, \quad x \in l^1(\mathbb{N}), \tag{2.7}$$

and

$$e^{tB_m^*}e^{sB_n^*}x = e^{sB_n^*}e^{tB_m^*}x, \quad x \in l^\infty(\mathbb{N}),$$

for all $t, s \geq 0$ *and* $m, n \in \mathbb{N}$.

Proof. It is enough to prove (2.6) since the conditions (2.6) and (2.7) are equivalent if the operators B_n, B_m are bounded, see [7, p. 19.] In addition, the equality $(B_nB_m)^* = B_m^*B_n^*$ implies that the B_n^*'s also commute.

Suppose now that $1 \leq m < n$ and denote $x = (\xi_i)_{i\in\mathbb{N}}$, $(\eta_i)_{i\in\mathbb{N}} = B_m x$, $(\eta_i')_{i\in\mathbb{N}} = B_n(\eta_i)_{i\in\mathbb{N}}$. Then by definition

$$\eta_i = \begin{cases} -\beta_m\xi_i + \alpha_m\xi_{i+2^{m-1}} & \text{if } i \bmod 2^m \in S_m^1, \\ -\alpha_m\xi_i + \beta_m\xi_{i-2^{m-1}} & \text{if } i \bmod 2^m \in S_m^2, \end{cases}$$

and

$$\eta_i' = \begin{cases} -\beta_n\eta_i + \alpha_n\eta_{i+2^{n-1}} & \text{if } i \bmod 2^n \in S_n^1, \\ -\alpha_n\eta_i + \beta_n\eta_{i-2^{n-1}} & \text{if } i \bmod 2^n \in S_n^2. \end{cases}$$

So $(\eta_i')_{i\in\mathbb{N}} = B_nB_m(\xi_i)_{i\in\mathbb{N}}$ is given by

$$\eta_i' = \begin{cases} \beta_n\beta_m\xi_i - \beta_n\alpha_m\xi_{i+2^{m-1}} - \alpha_n\beta_m\xi_{i+2^{n-1}} + \alpha_n\alpha_m\xi_{i+2^{m-1}+2^{n-1}} \\ \qquad \text{if } i \bmod 2^m \in S_m^1 \text{ and } i \bmod 2^n \in S_n^1, \\ \beta_n\alpha_m\xi_i - \beta_n\beta_m\xi_{i-2^{m-1}} - \alpha_n\alpha_m\xi_{i+2^{n-1}} + \alpha_n\beta_m\xi_{i-2^{m-1}+2^{n-1}} \\ \qquad \text{if } i \bmod 2^m \in S_m^2 \text{ and } i \bmod 2^n \in S_n^1, \\ \alpha_n\beta_m\xi_i - \alpha_n\alpha_m\xi_{i+2^{m-1}} - \beta_n\beta_m\xi_{i-2^{n-1}} + \beta_n\alpha_m\xi_{i+2^{m-1}-2^{n-1}} \\ \qquad \text{if } i \bmod 2^m \in S_m^1 \text{ and } i \bmod 2^n \in S_n^2, \\ \alpha_n\alpha_m\xi_i - \alpha_n\beta_m\xi_{i-2^{m-1}} - \beta_n\alpha_m\xi_{i-2^{n-1}} + \beta_n\beta_m\xi_{i-2^{m-1}-2^{n-1}} \\ \qquad \text{if } i \bmod 2^m \in S_m^2 \text{ and } i \bmod 2^n \in S_n^2. \end{cases}$$

From $m < n$, we have $2^m \leq 2^{n-1}$. It can be seen now that the condition $i \bmod 2^n \in S_n^1$ does not depend on whether or not $i \bmod 2^m \in S_m^1$. Similarly the condition $i \bmod 2^n \in S_n^2$ is independent of $i \bmod 2^m \in S_m^1$. Thus if we calculate $B_mB_n(\xi_i)_{i\in\mathbb{N}}$, we get the same result as that of $B_nB_m(\xi_i)_{i\in\mathbb{N}}$. □

3. Main theorems. Similar calculations to those in the proof of Lemma 2.1 can be carried out to find formulae for $T_n(t) = \prod_{k=1}^n e^{tB_k}$, $n \geq 2$. These formulae become more and more complicated as n increases. However the generator A_n of $T_n(t)$ has a rather simple form. Denote $(\eta_i)_{i\in\mathbb{N}} = A_n(\xi_i)_{i\in\mathbb{N}}$. Then from (1.1), we have that $\eta_i = \sum_{k=1}^n \zeta_k$, where ζ_k, $k = 1, \ldots, n$, are given by

$$\zeta_k = \begin{cases} -\beta_k\xi_i + \alpha_k\xi_{i+2^{k-1}} & \text{if } i \bmod 2^k \in S_k^1, \\ -\alpha_k\xi_i + \beta_k\xi_{i-2^{k-1}} & \text{if } i \bmod 2^k \in S_k^2. \end{cases}$$

I show in Theorem 3.1 that A_n still has a manageable form even if $n = +\infty$. For this, we only need the following condition to be satisfied

$$\sum_{n=1}^{\infty} \max\{\alpha_n, \beta_n\} < \infty, \tag{3.1}$$

which is equivalent to $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) < \infty$ for positive α_n's and β_n's.

Before stating the first of two main theorems, notice that any natural number $n \geq 2$ can be expressed in a unique way as $2^l + m$ for some $l \geq 0$ with $m \in \{1, 2, \ldots, 2^l\}$. For example, $2 = 2^0 + 2^0$, $7 = 2^2 + 3$, $8 = 2^2 + 2^2$, etc.

Theorem 3.1. *Assume that α_n, β_n, $n \geq 1$, are positive numbers satisfying (3.1). Then the infinite product $T(t)$, $t \geq 0$, of semigroups given by (2.4) exists. Let A be the generator of $T(t)$ and denote $(\eta_i)_{i\in\mathbb{N}} = Ax$, where $x = (\xi_i)_{i\in\mathbb{N}} \in l^1(\mathbb{N})$. Then*

$$\eta_1 = \sum_{k=1}^{\infty}(-\beta_k \xi_1 + \alpha_k \xi_{1+2^{k-1}}), \tag{3.2}$$

and if $i = 2^l + m$ for $l \geq 0$ with $m \in \{1, 2, \ldots, 2^l\}$, we have

$$\eta_i = \sum_{k=1}^{l+1} \zeta_k + \sum_{k=l+2}^{\infty} (-\beta_k \xi_i + \alpha_k \xi_{i+2^{k-1}}), \tag{3.3}$$

where ζ_k, $k = 1, \ldots, l+1$, are given by

$$\zeta_k = \begin{cases} -\beta_k \xi_i + \alpha_k \xi_{i+2^{k-1}} & \text{if } i \bmod 2^k \in S_k^1, \\ -\alpha_k \xi_i + \beta_k \xi_{i-2^{k-1}} & \text{if } i \bmod 2^k \in S_k^2. \end{cases}$$

Proof. We use Theorem 1.1. In our case, $D_1 = l^1(\mathbb{N})$ because

$$\sum_{k=1}^{+\infty} ||B_k x||_{l^1(\mathbb{N})} \leq 2||x||_{l^1(\mathbb{N})} \sum_{k=1}^{\infty}(\alpha_k + \beta_k) < \infty$$

for every $x \in l^1(\mathbb{N})$. The norm convergence in $l^1(\mathbb{N})$ implies the coordinate-wise convergence, hence components of Ax are limits of components of $A_n x$, where $A_n = \sum_{k=1}^{n} B_k x$. Thus (3.2) and (3.3) follow and

$$||A|| \leq 2 \sum_{k=1}^{\infty}(\alpha_k + \beta_k).$$

This completes the proof. □

Any $x \in l^1(\mathbb{N})$ can be written as $\sum_{i\in\mathbb{N}} \xi_i e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard Schauder basis in $l^1(\mathbb{N})$, i.e., $e_i = (\ldots, 0, 1, 0, \ldots)$ with 1 in the i-th coordinate. From Theorem 3.1, we have for example

$$Ae_1 = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} -\sum_{k=1}^{\infty} \beta_k, & i = 1, \\ \beta_k, & \text{for } i = 1 + 2^{k-1} \text{ and } k \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$Ae_2 = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} \alpha_1, & i = 1, \\ -\alpha_1 - \sum_{k=2}^{\infty}\beta_k, & i = 2, \\ \beta_k, & \text{for } i = 2 + 2^{k-1} \text{ and } k \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. *Assume that α_n, β_n, $n \geq 1$, are positive numbers satisfying (3.1). Then the infinite product $T^*(t)$, $t \geq 0$, of semigroups given by (2.5) exists. Let A^* be the generator of $T^*(t)$ and denote $(\eta_i)_{i\in\mathbb{N}} = A^*x$, where $x = (\xi_i)_{i\in\mathbb{N}} \in l^\infty(\mathbb{N})$. Then*

$$\eta_1 = \sum_{k=1}^{\infty} \beta_k(-\xi_1 + \xi_{1+2^{k-1}}), \tag{3.4}$$

and if $i = 2^l + m$ for $l \geq 0$ with $m \in \{1, 2, \ldots, 2^l\}$, we have

$$\eta_i = \sum_{k=1}^{l+1} \zeta_k + \sum_{k=l+2}^{\infty} \beta_k(-\xi_i + \xi_{i+2^{k-1}}) \tag{3.5}$$

where ζ_k, $k = 1, \ldots, l+1$, are given by

$$\zeta_k = \begin{cases} \beta_k(-\xi_i + \xi_{i+2^{k-1}}) & \text{if } i \bmod 2^k \in S_k^1, \\ \alpha_k(-\xi_i + \xi_{i-2^{k-1}}) & \text{if } i \bmod 2^k \in S_k^2. \end{cases}$$

Proof. The proof is analogous to that of Theorem 3.1. For every $x \in l^\infty(\mathbb{N})$, we have

$$\sum_{k=1}^{+\infty} ||B_k^* x||_{l^\infty(\mathbb{N})} \leq 2||x||_{l^\infty(\mathbb{N})} \sum_{k=1}^{\infty} \max\{\alpha_k, \beta_k\} < \infty.$$

So $D_1 = l^\infty(\mathbb{N})$ and the operator A^* is bounded. The norm convergence in $l^\infty(\mathbb{N})$ implies the coordinate-wise convergence, thus components of A^*x are limits of A_n^*x, where $A_n^* = \sum_{k=1}^{n} B_k^* x$ and

$$||A^*|| \leq 2\sum_{k=1}^{\infty} \max\{\alpha_k, \beta_k\}.$$

This completes the proof. □

4. Remarks. I proved in [9] that if α_n, β_n, $n \geq 1$, are positive numbers satisfying the following conditions (introduced by D. Blackwell in [4] to secure the existence of a Markov process with countably many states all of which are instantaneous)

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n + \beta_n} < \infty, \qquad \sum_{n=1}^{\infty} \beta_n = \infty, \tag{4.1}$$

then the infinite product $T(t)$, $t \geq 0$, of semigroups given by (2.4) exists and is composed of Markov operators associated with Blackwell's chain. However then the generator of $T(t)$ is unbounded and is the closure of A given by (3.2)–(3.3). In this case, A is defined on a dense subset $D(A)$ of $l^1(\mathbb{N})$ and interestingly $e_i \notin D(A)$ for every $i \in \mathbb{N}$. To see it, suppose that $i = 2^l + m$ for some $l \geq 0$ with $m \in \{1, 2, \ldots, 2^l\}$. Then by (2.1) and (4.1),

$$\sum_{k=1}^{+\infty} ||B_k e_i||_{l^1(\mathbb{N})} \geq \lim_{n \to +\infty} \sum_{k=l+2}^{n} \beta_k = +\infty.$$

As a result, any $x = (\xi_i)_{i \in \mathbb{N}}$ with a finite number of non-zero components does not belong to $D(A)$.

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