Arch. Math. 119 (2022), 613–622
0003-889X/22/060613-10
published online October 12, 2022
https://doi.org/10.1007/s00013-022-01786-2

Archiv der Mathematik

Lower bounds on the spectral gap of one-dimensional Schrödinger operators

Joachim Kerner

Abstract. We establish an explicit lower bound on the spectral gap of one-dimensional Schrödinger operators with non-negative bounded potentials and subject to Neumann boundary conditions. In addition, for a smaller class of potentials, we provide an improved lower bound which holds on large intervals.

Mathematics Subject Classification. 34L15, 34L40, 47E05, 47A10.

Keywords. Spectral gap, Schrödinger operators, Eigenvalues.

1. Introduction. In this paper, one aim is to derive a lower bound on the spectral gap for (deterministic) Schrödinger operators defined on an interval of length $L > 0$, subject to Neumann boundary conditions and with non-negative potentials in $L^{\infty}(\mathbb{R})$. Our bound will be valid for all values of $L > 0$ and will depend on the underlying potential in an explicit way. In addition to that, for certain compactly supported and bounded potentials, we shall also provide an improved lower bound to the spectral gap which holds on large intervals. As a matter of fact, for the potentials considered, we are able to prove that the spectral gap cannot close faster than $\sim L^{-4}$.

The spectral gap is defined as the distance between the first two eigenvalues and constitutes a classical quantity in the spectral theory of operators. For example, in [1,2,9] and more recently in [3,4], the spectral gap of the Laplacian (subject to certain self-adjoint boundary conditions) on a fixed interval was compared with the spectral gap of the Laplacian plus some additional potential on the same interval. It turns out that the spectral gap may increase or decrease, depending on specific properties of the potential considered; for example, as shown in [9], the spectral gap for the (Dirichlet or Neumann) Laplacian always increases given the added potential is convex. Unfortunately, since a generic potential fails, e.g., to be convex, it seems rather difficult to control the spectral gap for a Schrödinger operator on a fixed interval. Surprisingly, as discovered in a recent paper [8], it turns out that the asymptotic behaviour of the spectral gap can nevertheless be studied for a rather general class of potentials. A main finding of [8] was that the spectral gap converges to zero strictly faster than the spectral gap of the free (Dirichlet-) Laplacian for potentials that decay fast enough at infinity. This holds, in particular, for bounded potentials of compact support. Furthermore, in [8], the authors also put forward the conjecture that the spectral gap cannot close faster than $\sim L^{-3}$ for bounded compactly supported potentials and this is part of what motivates the present paper. At this point, let us also refer to [6] where lower bounds on the spectral gaps between consecutive negative eigenvalues for one-dimensional Schrödinger operators have been studied.

The paper is organized as follows: In Section 2, we describe the setting. In Section 3, we derive a Harnack-type inequality which then allows us to establish an explicit lower bound to the spectral gap that holds for a large class of potentials. Finally, in Section 4, we restrict ourselves to a smaller class of potentials and derive an improved lower bound on large intervals.

2. The model. On the interval $\Lambda_L = (-L/2, +L/2)$, we consider the Schrödinger operator

$$h_L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v$$

with real, non-negative potentials $v \in L^\infty(\mathbb{R})$ and Neumann boundary conditions at the endpoints $x = \pm L/2$. Standard operator theory tells us that h_L is self-adjoint with purely discrete spectrum. We denote its eigenvalues as $\lambda_0^v(L) \leq \lambda_1^v(L) \leq \cdots$; the normalized ground-state eigenfunction shall be denoted as $\varphi_0^{v,L} \in L^2(\Lambda_L)$. We also recall that $\varphi_0^{v,L}$ is a positive function. We set $k_0^v(L) := \sqrt{\lambda_0^v(L)}$.

The main object of interest in this paper is the spectral gap

$$\Gamma_v(L) := \lambda_1^v(L) - \lambda_0^v(L). \tag{2.1}$$

Since the ground state is non-degenerate [10], one has $\Gamma_v(L) > 0$ for every value $L > 0$. Furthermore, for potentials that decay fast enough at infinity (i.e. at least quadratically), $\Gamma_v(L)$ converges to zero like $\sim L^{-2}$ as $L \to \infty$; this holds, in particular, for potentials v of compact support. Note that this follows from an immediate generalization of [8, Theorem 2.1] to the case of Neumann boundary conditions.

3. A general lower bound. In a first step, we derive a lower bound to the infimum of the ground-state eigenfunction as well as a Harnack-type inequality. This inequality will then be the key ingredient to establish our lower bound to the spectral gap.

Lemma 3.1. *Assume $v \in L^\infty(\mathbb{R})$ and $v \geq 0$. Then*

$$\inf_{x \in \Lambda_L} \varphi_0^{v,L}(x) \geq \frac{\mathrm{e}^{-2L\|v\|_{L^1(\Lambda_L)}}}{\sqrt{L}} \tag{3.1}$$

holds for all $L > 0$. Furthermore, for all $L > 0$, one has

$$\inf_{x\in\Lambda_L} \varphi_0^{v,L}(x) \geq \mathrm{e}^{-2L\|v\|_{L^1(\Lambda_L)}} \cdot \sup_{x\in\Lambda_L} \varphi_0^{v,L}(x). \tag{3.2}$$

.

Proof. We follow the strategy outlined in [5] and, in particular, the proof of [5, Theorem 1.2]: The eigenvalue equation for the ground state reads

$$-(\varphi_0^{v,L})''(x) + v(x)\varphi_0^{v,L}(x) = \lambda_0^v(L)\varphi_0^{v,L}(x).$$

One then defines the auxiliary function, for $x \in \Lambda_L$, by

$$w_L^v(x) := \frac{(\varphi_0^{v,L})'(x)}{\varphi_0^{v,L}(x)} + \int_{-\frac{L}{2}}^{x} q_L^v(t)\ \mathrm{d}t$$

where $q_L^v(x) := \lambda_0^v(L) - v(x)$. The eigenvalue equation then implies that w_L^v is monotonically decreasing and therefore, for all $x \in \Lambda_L$,

$$w_L^v\left(+\frac{L}{2}\right) \leq w_L^v(x) \leq w_L^v\left(-\frac{L}{2}\right).$$

Using the fact that the $\varphi_0^{v,L}$ satisfies Neumann boundary conditions at $\pm L/2$, one has for all $x \in \Lambda_L$,

$$\int_{-\frac{L}{2}}^{+\frac{L}{2}} q_L^v(t)\ \mathrm{d}t \leq w_L^v(x) \leq 0$$

and hence

$$\int_{x}^{+\frac{L}{2}} q_L^v(t)\ \mathrm{d}t \leq \frac{(\varphi_0^{v,L})'(x)}{\varphi_0^{v,L}(x)} \leq -\int_{-\frac{L}{2}}^{x} q_L^v(t)\ \mathrm{d}t.$$

This immediately gives, for all $x \in \Lambda_L$,

$$\left|\frac{(\varphi_0^{v,L})'(x)}{\varphi_0^{v,L}(x)}\right| \leq \|q_L^v\|_{L^1(\Lambda_L)}. \tag{3.3}$$

Since, by the minmax-principle (using $1/\sqrt{L}$ as a test function in the Rayleigh quotient),

$$\lambda_0^v(L) \leq \frac{\|v\|_{L^1(\Lambda_L)}}{L}, \tag{3.4}$$

one concludes $\|q\|_{L^1(\Lambda_L)} \leq 2\|v\|_{L^1(\Lambda_L)}$.

In a next step, one introduces the function

$$h_L^v(t) := \ln\left(\varphi_0^{v,L}\left(t(x-y)+y\right)\right)$$

for $0 \leq t \leq 1$ and $x, y \in \Lambda_L$. This implies

$$(h_L^v)'(t) = (x-y)\frac{(\varphi_0^{v,L})'\left(t(x-y)+y\right)}{\varphi_0^{v,L}\left(t(x-y)+y\right)}.$$

Then, starting from the identity

$$\ln\left(\frac{\varphi_0^{v,L}(x)}{\varphi_0^{v,L}(y)}\right) = h_L^v(1) - h_L^v(0) = \int_0^1 (h_L^v)'(t)\ \mathrm{d}t,$$

a simple estimate using (3.3) shows that, for $x, y \in \Lambda_L$,

$$\frac{\varphi_0^{v,L}(x)}{\varphi_0^{v,L}(y)} \leq \mathrm{e}^{2L\|v\|_{L^1(\Lambda_L)}}. \tag{3.5}$$

Eq. (3.5) then immediately implies (3.2). Finally, since $\varphi_0^{v,L}$ has L^2-norm one, one concludes

$$\sup_{x\in\Lambda_L} \varphi_0^{v,L}(x) \geq \frac{1}{\sqrt{L}}$$

which implies (3.1). □

Remark 3.2. Eq.(3.1) in Lemma 3.1 is sharp: choosing $v \equiv 0$ yields the lower bound $\frac{1}{\sqrt{L}}$. On the other hand, for the zero-potential, the ground state is given by $\varphi_0^{v=0,L}(x) = \frac{1}{\sqrt{L}}$.

Using [7, Theorem 1.4] in combination with Lemma 3.1 then yields the following result.

Theorem 3.3 (Lower bound spectral gap I). *Assume $v \in L^\infty(\mathbb{R})$ and $v \geq 0$. Then*

$$\Gamma_v(L) \geq \mathrm{e}^{-4L\|v\|_{L^1(\Lambda_L)}} \cdot \frac{\pi^2}{L^2}$$

holds for all $L > 0$.

Proof. The strategy is to compare the spectral gap of the operator h_L with the spectral gap of the Neumann Laplacian (i.e., setting $v \equiv 0$) which is given by π^2/L^2. Such a comparison result has been provided in [7, Theorem 1.4]: more explicitly, taking into account that the normalized ground-state eigenfunction for the Neumann Laplacian is the constant function $1/\sqrt{L}$, one has

$$\Gamma_v(L) \geq \left(\frac{\inf_{x\in\Lambda_L} \varphi_0^{v,L}(x)}{\sup_{x\in\Lambda_L} \varphi_0^{v,L}(x)}\right)^2 \cdot \frac{\pi^2}{L^2}.$$

The result then readily follows with Lemma 3.1. □

Remark 3.4. Theorem 3.3 establishes a lower bound which is (for non-zero potentials) at least exponentially small in the interval length. Hence, this bound is still far away from a lower bound as established, for example, in [8, Proposition 2.9] for a symmetric step-potential. In this case, the lower bound reads αL^{-3} for some constant $\alpha > 0$ and all $L > 0$ large enough. However, due to the fact that a Harnack-type inequality has been used in the proof of Theorem 3.3, an exponential factor seems expectable.

Also, for the zero-potential $v \equiv 0$, the lower bound in Theorem 3.3 is sharp.

4. An improved bound for certain potentials. In this section, we shall study a smaller class of potentials for which we are going to derive another lower bound to the spectral gap which holds on large intervals. More explicitly, we shall make the following additional assumptions on the non-negative potential $v \in L^\infty(\mathbb{R})$:

i) $v(x) \neq 0 \quad \Leftrightarrow \quad x \in (-b, +b)$ for some $0 < b < \infty$,
ii) $v(x) = v(-x)$ (symmetry),
iii) v is strictly monotonically increasing on $[-b, -\varepsilon]$ for some $0 < \varepsilon < b$ and such that $\inf_{x \in [-\varepsilon, 0]} v(x) > \gamma$ for some $\gamma > 0$ (alternatively, it is enough to assume $\inf_{x \in [-b, 0]} v(x) > \gamma$ for some $\gamma > 0$),
iv) $b^2 \|v\|_{L^\infty(\mathbb{R})} < 1/2$ (smallness condition).

In a first result, we derive a lower bound to $\Gamma_v(L)$ which only depends on the asymptotic behaviour of $k_0^v(L)$. The important aspect here is that asymptotic behaviour of the gap is reduced to studying the asymptotic behaviour of the ground state eigenvalue which is usually more accessible.

Lemma 4.1. *Consider h_L with a potential $v \in L^\infty(\mathbb{R})$, $v \geq 0$, that also fulfils conditions i)-iv). Then*

$$\Gamma_v(L) \geq (1 - 2b^2\|v\|_{L^\infty(\mathbb{R})})^2 \cdot \frac{\pi^2}{L^2} \cdot \cos^2\left(k_0^v(L)(L/2 - b)\right)$$

holds for all $L > 0$ large enough.

Proof. First note that, for L large enough, the assumptions on the potential guarantee that $\varphi_0^{v,L}$ attains its maximum at the boundary of $(-L/2, +L/2)$ and the minimum at $x = 0$: To see this, we first note that $\varphi_0^{v,L} \in C^1(\overline{\Lambda_L})$ since h_L is self-adjoint on a domain contained in $H^2(\Lambda_L)$ which itself is a consequence of v being bounded. Furthermore, assumption ii) implies that the ground state is a symmetric function and hence its derivative vanishes at $x = 0$. Also, the eigenvalue equation $-(\varphi_0^{v,L})''(x) + v(x)\varphi_0^{v,L}(x) = \lambda_0^v(L)\varphi_0^{v,L}(x)$ together with the fact that $\lambda_0^v(L) \to 0$ as $L \to \infty$ (see (3.4) or the remark below (2.1)) and iii) imply that the second derivative of $\varphi_0^{v,L}$ changes sign exactly once in $(-L/2, 0)$ for all L large enough. Since $(\varphi_0^{v,L})'(-L/2) = 0$ and $(\varphi_0^{v,L})''$ is negative in a neighbourhood around $x = -L/2$, the maximum is assumed at $x = -L/2$ and the minimum at $x = 0$. In addition, we conclude that $(\varphi_0^{v,L})'(x) \leq 0$ for $x \in [-L/2, 0]$.

Now, taking assumption i) into account, the restriction of the ground state eigenfunction $\varphi_0^{v,L}$ to $(-L/2, -b)$ is given by

$$(\varphi_0^{v,L}|_{(-L/2,-b)})(x) = A\cos(k_0^v(L)(x + L/2))$$

with $A = \varphi_0^{v,L}(-L/2) = \sup_{x \in \Lambda_L} \varphi_0^{v,L}(x) > 0$. On the other hand, using the eigenvalue equation and the symmetry of v with respect to $x = 0$ (assumption ii)), we obtain

$$(\varphi_0^{v,L})'(x) = \int_x^0 (\lambda_0^v(L) - v(x))\varphi_0^{v,L}(x)\, \mathrm{d}x, \quad x \in [-b, 0], \tag{4.1}$$

and this yields, for $x \in [-b, 0]$,

$$\begin{aligned}|(\varphi_0^{v,L})'(x)| &\leq b\|\lambda_0^v(L) - v\|_{L^\infty(\mathbb{R})} \cdot \sup_{x\in[-b,0)} \varphi_0^{v,L}(x)\\ &= b\|\lambda_0^v(L) - v\|_{L^\infty(\mathbb{R})} \cdot A\cos\left(k_0^v(L)(L/2-b)\right)\\ &\leq 2Ab\|v\|_{L^\infty(\mathbb{R})} \cdot \cos\left(k_0^v(L)(L/2-b)\right)\end{aligned}$$

for all $L > 0$ large enough. Hence,

$$\inf_{x\in\Lambda_L} \varphi_0^{v,L}(x) \geq (1 - 2b^2\|v\|_{L^\infty(\mathbb{R})}) \cdot A\cos\left(k_0^v(L)(L/2-b)\right)$$

which is a useful bound assuming iv).

Finally, as in the proof of Theorem 3.3, [7, Theorem 1.4] yields (taking into account that the ground state eigenfunction of the Neumann Laplacian with zero potential is the constant function $1/\sqrt{L}$)

$$\Gamma_v(L) \geq \left(\frac{\inf_{x\in\Lambda_L} \varphi_0^{v,L}(x)}{\sup_{x\in\Lambda_L} \varphi_0^{v,L}(x)}\right)^2 \cdot \frac{\pi^2}{L^2},$$

and hence the statement follows readily. □

In a next step, we need to investigate the asymptotic behaviour of $k_0^v(L)$ as $L \to \infty$.

Proposition 4.2. *Consider h_L with a potential $v \in L^\infty(\mathbb{R})$, $v \geq 0$, that also fulfils conditions i)-iv). Then*

$$\lim_{L\to\infty} \left|\frac{\pi}{2} - k_0^v(L)L\left(\frac{1}{2} - \frac{b}{L}\right)\right| = 0. \tag{4.2}$$

Furthermore, there exists a constant $\delta > 0$ such that

$$\frac{\delta}{L} \leq \left(\frac{\pi}{2} - k_0^v(L)L\left(\frac{1}{2} - \frac{b}{L}\right)\right) \tag{4.3}$$

for all $L > 0$ large enough.

Proof. For the proof, we introduce the characteristic function $\mathbb{1}_A(\cdot)$ of a measurable set $A \subset \mathbb{R}$. The idea is to compare $k_0^v(L)$ with the square root of the lowest eigenvalue for the Laplacian with a step-potential $\tilde{v}(x) := \tilde{v} \cdot \mathbb{1}_{[-c,+c]}(x)$ with suitable $\tilde{v}, c > 0$. The square root of such an eigenvalue shall be denoted by $\tilde{k}_0(L)$ (for notational simplicity, we do not state the dependence of $\tilde{k}_0(L)$ on $\tilde{v}$ and c explicitly in the following).

As shown in the appendix, the quantitation condition (meaning the relation determining $\tilde{k}_0(L)$) for the step-potential $\tilde{v}$ with Neumann boundary conditions and large enough $L > 0$ reads

$$\frac{M_0(L)}{\tilde{\omega}_0(L)} \tanh(M_0(L)l_2(L)) = \tan(\tilde{\omega}_0(L)l_1(L)) \tag{4.4}$$

where $\tilde{\omega}_0(L) := \tilde{k}_0(L)L$, $M_0(L) := \sqrt{L^2\tilde{v} - \tilde{\omega}_0^2(L)}$, $l_1(L) := \frac{1}{2} - \frac{c}{L}$, and $l_2(L) := \frac{c}{L}$.

Now, to prove (4.2), we pick two step potentials, one with $c := \varepsilon$ and $\tilde{v} := \gamma$ and the other one with $c := b$ and $\tilde{v} := \|v\|_{L^\infty(\mathbb{R})}$. Let $\tilde{k}_0^{(1)}(L)$ denote the square root of the lowest eigenvalue for the first step-potential and $\tilde{k}_0^{(2)}(L)$ the square root of the lowest eigenvalue for the second step-potential. Consequently, by an operator-bracketing, we have

$$\tilde{k}_0^{(1)}(L)L \leq k_0^v(L)L \leq \tilde{k}_0^{(2)}(L)L$$

and the statement readily follows from the fact that both, the left-hand as well as the right-hand side, converge to π as $L \to \infty$. This, on the other hand, follows from the quantization condition (4.4).

On the other hand, in order to prove (4.3), we again choose the step-potential with $c := b$ and $\tilde{v} = \|v\|_{L^\infty(\mathbb{R})}$. Hence, $\tilde{k}_0^{(2)}(L) \geq k_0^v(L)$ and it is therefore enough to prove (4.3) with $\tilde{k}_0^{(2)}(L)$ instead; note there that, by (4.4),

$$\frac{\pi}{2} - \tilde{k}_0^{(2)}(L)L\left(\frac{1}{2} - \frac{b}{L}\right) > 0.$$

Now, for $\tilde{\omega}_0(L)l_1(L) < \pi/2$ close enough to $\pi/2$ (hence for $L > 0$ large enough), one obtains

$$\begin{aligned} \tan(\tilde{\omega}_0(L)l_1(L)) &\geq \frac{1}{2\cos(\tilde{\omega}_0(L)l_1(L))} \\ &\geq \frac{1}{2(-\tilde{\omega}_0(L)l_1(L) + \pi/2)}. \end{aligned} \tag{4.5}$$

Furthermore, setting $A(L) := M_0(L)l_1(L)\tanh(M_0(L)l_2(L))$, (4.4) and (4.5) imply

$$\begin{aligned} \tilde{\omega}_0(L)l_1(L) &\leq \frac{\pi A(L)}{(1 + 2A(L))} \\ &= \frac{\pi}{2(1 + \frac{1}{2A(L)})} \end{aligned} \tag{4.6}$$

for L large enough. This now yields, for some constant $\delta > 0$ and all L large enough,

$$\begin{aligned} \left(\frac{\pi}{2} - \tilde{\omega}_0(L)l_1(L)\right) &\geq \frac{\pi}{2}\left(\frac{1}{2A(L)(1 + \frac{1}{2A(L)})}\right) \\ &\geq \frac{\pi}{8A(L)} \\ &\geq \frac{\delta}{L}. \end{aligned}$$

□

Combining Lemma 4.1 and Proposition 4.2 then yields the following statement.

Theorem 4.3 (Lower bound spectral gap II). *Consider h_L with a potential $v \in L^\infty(\mathbb{R})$, $v \geq 0$, that also fulfils conditions i)-iv). Then there exists a constant $\beta > 0$ such that*

$$\Gamma_v(L) \geq \frac{\beta}{L^4}$$

for all $L > 0$ large enough.

Proof. We establish a lower bound to

$$\cos^2\left(k_0^v(L)(L/2-b)\right) = \sin^2\left(\frac{\pi}{2} + k_0^v(L)(L/2-b)\right)$$

for large $L > 0$. Since $|\sin(x)| \geq |-\frac{1}{2}x + \frac{\pi}{2}|$ in a small neighbourhood around $x = \pi$, Proposition 4.2 yields

$$\begin{aligned}\left|\sin\left(\frac{\pi}{2} + k_0^v(L)(L/2-b)\right)\right| &\geq \left|-\frac{1}{2}\left(\frac{\pi}{2} + k_0^v(L)(L/2-b)\right) + \frac{\pi}{2}\right| \\ &= \left|\frac{1}{2}\left(\pi - \frac{\pi}{2} - k_0^v(L)(L/2-b)\right)\right|\end{aligned}$$

for all large enough $L > 0$. Taking the square, we get, for L large enough,

$$\sin^2\left(\frac{\pi}{2} + k_0^v(L)(L/2-b)\right) \geq \frac{1}{4}\left(\frac{\pi}{2} - k_0^v(L)L\left(\frac{1}{2} - \frac{b}{L}\right)\right)^2.$$

The statement then follows with Proposition 4.2 and Lemma 4.1. □

Appendix. In this appendix, we shall derive relation (4.4). Recall that $\mathbb{1}_A(\cdot)$ denotes the characteristic function of a measurable set $A \subset \mathbb{R}$. We first remark that the Schrödinger operator

$$h_L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{v} \cdot \mathbb{1}_{[-c,+c]}(x)$$

with $\tilde{v}, c > 0$ and defined on $L^2(\Lambda_L)$ is unitarily equivalent to the operator $L^{-2}\tilde{h}_L$ with

$$\tilde{h}_L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + L^2\tilde{v} \cdot \mathbb{1}_{[-c/L,+c/L]}(x) \tag{A.1}$$

defined on $L^2(\Lambda_1)$ (for a proof, we refer to [8, Proposition 2.4]). Hence, given $\tilde{\omega}_0(L)$ denotes the square root of the lowest eigenvalue of $\tilde{h}_L$, one has $\tilde{\omega}_0(L) = \tilde{k}_0(L)L$ where $\tilde{k}_0(L)$ is the square root of the lowest eigenvalue of h_L. Furthermore, we set $M_0(L) := \sqrt{L^2\tilde{v} - \tilde{\omega}_0^2(L)}$, $l_1(L) := \frac{1}{2} - \frac{c}{L}$, and $l_2(L) := \frac{c}{L}$.

Equation (4.4) now follows from suitable matching conditions when constructing the ground state eigenfunction of $\tilde{h}_L$ (for large $L > 0$). More explicitly, we identify the interval $(-1/2, -c/L)$ with $I_1 := (0, l_1(L))$ and the interval $(-c/L, 0)$ with $I_2 := (0, l_2(L))$. Note that, since the ground state eigenfunction of h_L and $\tilde{h}_L$ are symmetric with respect to $x = 0$, it is sufficient to construct the ground state on $(-L/2, 0)$ or $(-1/2, 0)$, respectively.

On the interval I_1, the potential is zero and hence we make the ansatz

$$\tilde{\varphi}_{1,L}(x) := A\cos(\tilde{\omega}_0(L)x)$$

for some non-zero $A \in \mathbb{R}$. On I_2, the potential is constant equal to $L^2\tilde{v}$ and this yields, for $L > 0$ large enough (i.e. one requires that $L^2\tilde{v} - \tilde{\omega}_0^2(L) > 0$), the ansatz

$$\tilde{\varphi}_{2,L}(x) := B\sinh(M_0(L)x) + C\cosh(M_0(L)x)$$

for some non-zero $B, C \in \mathbb{R}$ to be determined. Since the ground state eigenfunction is symmetric and differentiable, we obtain the matching conditions

$$\begin{aligned} \tilde{\varphi}_{1,L}(l_1(L)) &= \tilde{\varphi}_{2,L}(0), \\ \tilde{\varphi}'_{1,L}(l_1(L)) &= \tilde{\varphi}'_{2,L}(0), \\ \tilde{\varphi}'_{2,L}(l_2(L)) &= 0. \end{aligned}$$

The first condition implies

$$\cos(\tilde{\omega}_0(L) l_1(L)) = \frac{C}{A},$$

the second condition gives

$$\sin(\tilde{\omega}_0(L) l_1(L)) = -\frac{B M_0(L)}{A \tilde{\omega}_0(L)},$$

and the third condition yields

$$\tanh(M_0(L) l_2(L)) = -\frac{B}{C}.$$

Combining them, we eventually arrive at (4.4).

Acknowledgements. JK would like to thank the Bergische Universität Wuppertal for its kind hospitality while being on leave from the FernUniversität in Hagen. We dedicate this paper to the memory of Dr. Uwe Pittelkow (University of Ulm).

Funding Information Open Access funding enabled and organized by Projekt DEAL.

References

[1] Ashbaugh, M.S., Benguria, R.: Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials. Proc. Amer. Math. Soc. **105**(2), 419–424 (1989)

[2] Abramovich, S.: The gap between the first two eigenvalues of a one-dimensional Schrödinger operator with symmetric potential. Proc. Amer. Math. Soc. **111**(2), 451–453 (1991)

[3] Andrews, B., Clutterbuck, J., Hauer, D.: The fundamental gap for a one-dimensional Schrödinger operator with Robin boundary conditions. Proc. Amer. Math. Soc. **149**, 1481–1493 (2021)

[4] Ashbaugh, M.S., Kielty, D.: Spectral gaps of 1-D Robin Schrödinger operators with single-well potentials. J. Math. Phys. **61**(9), 091507 (2020)

[5] Berhanu, S., Mohammed, A.: A Harnack inequality for ordinary differential equations. Amer. Math. Monthly **112**(1), 32–41 (2005)

[6] Kirsch, W., Simon, B.: Universal lower bounds on eigenvalue splittings for one dimensional Schrödinger operators. Comm. Math. Phys. **97**, 453–460 (1985)

[7] Kirsch, W., Simon, B.: Comparison theorems for the gap of Schrödinger operators. J. Funct. Anal. **75**(2), 396–410 (1987)

[8] Kerner, J., Täufer, M.: On the spectral gap of one-dimensional Schrödinger operators on large intervals. arXiv:2012.09060 (2021)

[9] Lavine, R.: The eigenvalue gap for one-dimensional convex potentials. Proc. Amer. Math. Soc. **121**(3), 815–821 (1994)

[10] Lieb, E.L., Loss, M.: Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI (2001)

Joachim Kerner
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
58084 Hagen
Germany

and

Fakultät für Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
42119 Wuppertal
Germany
e-mail: Joachim.Kerner@fernuni-hagen.de;
kerner@uni-wuppertal.de

Received: 26 June 2021

Revised: 18 July 2022

Accepted: 5 September 2022.