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# (1+)-complemented, (1+)-isomorphic copies of $L_1$ in dual Banach spaces

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**Abstract.** The present paper contributes to the ongoing programme of quantification of isomorphic Banach space theory focusing on the Hagler–Stegall characterisation of dual spaces containing complemented copies of $L_1$. As a corollary, we obtain the following quantitative version of the Hagler–Stegall theorem asserting that for a Banach space $X$, the following statements are equivalent:

- $X$ contains almost isometric contains almost isometric copies of $(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}$;
- for all $\varepsilon > 0$, $X^*$ contains a $(1+\varepsilon)$-complemented, $(1+\varepsilon)$-isomorphic copy of $L_1$;
- for all $\varepsilon > 0$, $X^*$ contains a $(1+\varepsilon)$-complemented, $(1+\varepsilon)$-isomorphic copy of $C[0,1]^*$.

Moreover, if $X$ is separable, one may add the following assertion:

- for all $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T\colon X \to C(\Delta)$ so that $T^*[C(\Delta)^*]$ is $(1+\varepsilon)$-complemented in $X^*$, where $\Delta$ is the Cantor set

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**1. Introduction.** In 1968, Pełczyński [17] showed that if a Banach space $X$ contains an isomorphic copy of $\ell_1$, then the dual space $X^*$ contains an isomorphic copy of $L_1$ and proved that the converse holds as well subject to a mild technical condition that was later removed by Hagler [7]. More precisely, the result stated that the isomorphic containment of $\ell_1$ is equivalent to the following assertions: $X^*$ contains a subspace isomorphic to $L_1$, $X^*$ contains a subspace isomorphic to $C[0,1]^*$. When $X$ is separable, these are further equivalent to the assertions: $X^*$ contains a subspace isomorphic to $\ell_1([0,1])$, and $C[0,1]$ is a quotient of $X$.

Shortly after, Hagler and Stegall [8] obtained a 'complemented' version of Pełczyński's aforementioned classical work:

**Theorem** (Hagler–Stegall). *Let $X$ be a Banach space. Then the following assertions are equivalent:*

(1) *$X$ contains a subspace isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_{\infty}^{n})_{\ell_1}$;*
(2) *$X^*$ contains a complemented subspace isomorphic to $L_1$;*
(3) *$X^*$ contains a complemented subspace isomorphic to $C[0,1]^*$;*
(4) *$X^*$ contains an infinite set $K$ such that $K$ is equivalent to the usual basis of $\ell_1(\Gamma)$ for some $\Gamma$, $[K]$ is complemented in $X^*$, and $K$ is dense in itself in the weak* topology on $X^*$.*

If, in addition, $X$ is separable, then the assertions (1)–(4) are equivalent to

(5) There exists a surjective operator $T\colon X \to C[0,1]$ such that $T^*[C[0,1]^*]$ is complemented in $X^*$.

The purpose of this note is to quantify the Hagler–Stegall theorem in the spirit of a large number of recent results on quantitative versions of various theorems on and properties of Banach spaces, such as quantitative versions of Krein's theorem [6], Gantmacher's theorem [2], James' compactness theorem [5], weak sequential completeness and the Schur property [11,12], the (reciprocal) Dunford–Pettis property [10,13], the Banach–Saks property [3], etc. More broadly speaking, the present paper contributes to the on-going programme of quantification of Banach space theory.

In the present paper, we quantify the Hagler–Stegall theorem by introducing the following three quantities denoted by lower-case Greek letters and defined as infima of certain sets (when the sets happen to be empty, we use the convention that the corresponding value is $\infty$).

Hereinafter $X$ and $Y$ will stand for Banach spaces; $\mathcal{B}(X,Y)$ is the space of (bounded, linear) operators from $X$ to $Y$. We then introduce the following quantities:

- $\alpha_Y(X) = \inf\{\mathrm{d}(Y,Z)\colon Z \text{ is a subspace of } X\}$, where $\mathrm{d}(Y,Z)$ is the Banach–Mazur distance between $Y$ and $Z$.

The quantity $\alpha_Y(X)$, being directly related to the Banach–Mazur distance, measures how well $Y$ is from being isomorphically embeddable into $X$. Obviously, $\alpha_Y(X) = 1$ if and only if $X$ contains almost isometric copies of $Y$, that is, for every $\varepsilon > 0$, $X$ contains a subspace $(1+\varepsilon)$-isomorphic to $Y$.

- $\beta_Y(X) = \inf\{\|A\|\|B\|\colon A \in \mathcal{B}(X,Y), B \in \mathcal{B}(Y,X),\ AB = I_Y\}$.

The quantity $\beta_Y(X)$ measures how well $Y$ is from being isomorphic to a complemented subspace of $X$. It is easy to see that $\beta_Y(X) = 1$ if and only if for every $\varepsilon > 0$, there exists a subspace $M$ of $X$ so that $M$ is $(1+\varepsilon)$-isomorphic to $Y$ and $(1+\varepsilon)$-complemented in $X$.

- $\theta_Y(X) = \inf\{\|A\|\|S\|\colon A \in \mathcal{B}(X,Y), S \in \mathcal{B}(X^*,Y^*), SA^* = I_{Y^*}\}$.

The quantity $\theta_Y(X)$ measures how well $Y$ is isomorphic to a quotient of $X$ and its dual $Y^*$ is isomorphic to a complemented subspace of $X^*$. We see that $\theta_Y(X) = 1$ if and only if, for every $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T\colon X \to Y$ so that $T^*[Y^*]$ is $(1+\varepsilon)$-complemented in $X^*$.

A straightforward argument shows that

$$\beta_{Y^*}(X^*) \leqslant \theta_Y(X) \leqslant \beta_Y(X). \tag{1.1}$$

By using the aforementioned three quantities, we quantify the Hagler–Stegall theorem as follows:

**Theorem A.** *Let $X$ be a Banach space. Then*

$$\alpha_{(\oplus_{n=1}^{\infty}\ell_\infty^n)_{l_1}}(X) = \beta_{C[0,1]^*}(X^*) = \beta_{L_1}(X^*).$$

*If, in addition, $X$ is separable, then*

$$\theta_{C(\Delta)}(X) = \beta_{L_1}(X^*).$$

The following $(1+\varepsilon)$-version of the Hagler–Stegall theorem follows from Theorem A.

**Corollary 1.1.** *Let $X$ be a Banach space. Then the following assertions are equivalent:*

(1) *$X$ contains almost isometric copies of $(\bigoplus_{n=1}^{\infty}\ell_\infty^n)_{l_1}$;*
(2) *$X^*$ contains a $(1+\varepsilon)$-complemented subspace that is $(1+\varepsilon)$-isomorphic to $L_1$ for every $\varepsilon > 0$;*
(3) *$X^*$ contains a $(1+\varepsilon)$-complemented subspace that is $(1+\varepsilon)$-isomorphic to $C[0,1]^*$ for every $\varepsilon > 0$.*

*If, in addition, $X$ is separable, then*

(4) *for every $\varepsilon > 0$, there exists a $(1+\varepsilon)$-quotient map $T\colon X \to C(\Delta)$ so that $T^*[C(\Delta)^*]$ is $(1+\varepsilon)$-complemented in $X^*$.*

**2. Preliminaries.** Our notation and terminology are standard and mostly inline with [1,16]. Throughout the paper, all Banach spaces can be considered either real or complex. We work with real scalars but the results can be easily amended to the complex too. By a *subspace* we understand a closed, linear subspace and by an *operator* we understand a bounded, linear map. If $X$ is a Banach space, we denote by $B_X$ the closed unit ball of $X$, by $I_X$ the identity operator on $X$, and, for a subset $K \subseteq X$, by $[K]$ the closed linear span of $K$. For a surjective operator $T : X \to Y$, we set

$$\mathrm{co}(T) = \inf\{c > 0 : B_Y \subseteq c \cdot TB_X\}.$$

For $\lambda \geqslant 1$, we say that a surjective operator $T\colon X \to Y$ is a $\lambda$-*quotient map* if $\|T\|\,\mathrm{co}(T) \leqslant \lambda$. *Quotient maps* are 1-quotient maps according to the above terminology. A norm-one surjective operator $T\colon X \to Y$ is a quotient map if and only if $T$ is a (1+)-quotient map, that is, a $(1+\varepsilon)$-quotient map for every $\varepsilon > 0$.

The *Banach–Mazur distance* $\mathrm{d}(X, Y)$ between two isomorphic Banach spaces $X$ and $Y$ is defined by $\inf \|T\|\|T^{-1}\|$, where the infimum is taken over all isomorphisms $T$ from $X$ onto $Y$. As defined by Lindenstrauss and Rosenthal

[15], for $\lambda \geqslant 1$, a Banach space $X$ is said to be a $\mathcal{L}_{1,\lambda}$-*space* whenever for every finite-dimensional subspace $E$ of $X$, there is a finite-dimensional subspace $F$ of $X$ such that $F \supseteq E$ and $\mathrm{d}(F, l_1^{\dim F}) \leqslant \lambda$. We say that a Banach space $X$ is an $\mathcal{L}_{1,\lambda+}$-*space* if it is an $\mathcal{L}_{1,\lambda+\varepsilon}$-*space* for all $\varepsilon > 0$.

Following the notation from [8], we denote

$$\mathcal{F} = \{(n,i)\colon n = 0,1,\ldots, i = 0,1,\ldots,2^n-1\}$$

and, for $(n,i),(m,j) \in \mathcal{F}$, we write $(n,i) \geqslant (m,j)$ whenever

- $n \geqslant m$,
- $2^{n-m}j \leqslant i \leqslant 2^{n-m}(j+1)-1$.

Let $\Delta = \{0,1\}^{\mathbb{N}}$ be the Cantor set endowed with the metric

$$\mathrm{d}((a_n)_{n=1}^\infty,(b_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{1}{2^n}|a_n - b_n| \quad \big((a_n)_n,(b_n)_n \in \Delta\big).$$

By Miljutin's theorem [1, Lemma 4.4.7], $C[0,1]$ is isomorphic (but not isometric) to $C(\Delta)$. It is well-known that $C(\Delta)^*$ and $C[0,1]^*$ are linearly isometric, though.

**3. Proof of Theorem A.** The present section is devoted to the proof of Theorem A and is conveniently split into more digestible parts.

*Proof of Theorem A.* We split the proof into a number of steps.

Step 1. $\beta_{C(\Delta)^*}(X^*) \leqslant \alpha_{(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}}(X)$.

Since $Z = (\bigoplus_{n=1}^\infty \ell_\infty^{2^n})_{\ell_1}$ embeds isometrically into $(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$, it suffices to prove that $\alpha_Z(X) \geqslant \beta_{C(\Delta)^*}(X^*)$. For this, let us fix $c > \alpha_Z(X)$. Then there exists a contractive operator $R\colon Z \to X$ that is bounded below by $1/c$.

Let us consider a double-indexed family $(\Delta_{n,i})_{n=0,i=0}^{\infty,2^n-1}$ of clopen subsets of the Cantor set such that

(1) $\Delta_{0,0} = \Delta$, $\Delta_{n,i} = \Delta_{n+1,2i} \cup \Delta_{n+1,2i+1}$ $((n,i) \in \mathcal{F})$, and $\Delta_{n,i} \cap \Delta_{n,j} = \varnothing$ if $i \neq j$;
(2) the diameter of $\Delta_{n,i}$ is $1/2^n$ $(0 \leqslant i \leqslant 2^n-1)$.

We set $g_{n,i} = \mathbb{1}_{\Delta_{n,i}}$, which is a continuous function, $[g_{n,i}]_{i=0}^{2^n-1} \subseteq [g_{n+1,i}]_{i=0}^{2^{n+1}-1}$, $(g_{n,i})_{i=0}^{2^n-1}$ is isometrically equivalent to the unit vector basis of $\ell_\infty^{2^n}$ $(n \in \mathbb{N})$, and $\bigcup_{n=0}^\infty [g_{n,i}]_{i=0}^{2^n-1}$ is dense in $C(\Delta)$. We may then define an operator $T\colon Z \to C(\Delta)$ by the assignment $Te_{n,i} = g_{n,i}$. For each $n$, $T$ is an isometry when restricted to $[e_{n,i}\colon 0 \leqslant i \leqslant 2^n-1]$. Clearly, $\|T\| = 1$.

*Claim 1.* If $W$ is a finite-dimensional Banach space and $S\colon W \to C(\Delta)$ is an operator, then for every $\varepsilon > 0$, there exists an operator $\widehat{S}\colon W \to Z$ so that $\|\widehat{S}\| \leqslant (1+\varepsilon)\|S\|$ and $\|S - T\widehat{S}\| \leqslant \varepsilon$.

*Proof of Claim 1.* Let us fix an Auerbach basis $(w_k, w_k^*)_{k=1}^N$ for $W$ ($\dim W = N$). So if $w = \sum_{k=1}^N a_k w_k \in W$, then for each $1 \leqslant j \leqslant N$, we get

$$|a_j| = \left|\left\langle w_j^*, \sum_{k=1}^N a_k w_k \right\rangle\right| \leqslant \|w_j^*\|\|w\| = \|w\|.$$

It follows that $\sum_{k=1}^{N}|a_k| \leqslant N\|w\|$. Let $\delta > 0$ be such that $\delta N \leqslant \varepsilon\|S\|$ and $\delta N \leqslant \varepsilon$. Then, there exist a positive integer $n$ and $(f_k)_{k=1}^{N}$ in $[g_{n,i}]_{i=0}^{2^n-1}$ so that $\|Sw_k - f_k\| < \delta$ $(k = 1, 2, \ldots, N)$. Let us write $f_k = \sum_{i=0}^{2^n-1} t_{k,i}g_{n,i}$ $(k = 1, 2, \ldots, N)$.

Let us define an operator $\widehat{S}\colon W \to Z$ by

$$\widehat{S}w_k = \sum_{i=0}^{2^n-1} t_{k,i}e_{n,i}.$$

We *claim* that $\widehat{S}$ is the required operator. Indeed, for $w = \sum_{k=1}^{N} a_k w_k \in W$, we have

$$\begin{aligned}\|\widehat{S}w\| &= \|\textstyle\sum_{k=1}^{N} a_k\widehat{S}w_k\| = \|\sum_{k=1}^{N} a_k T\widehat{S}w_k\|\\ &= \|\textstyle\sum_{k=1}^{N} a_k f_k\| \leqslant \|\sum_{k=1}^{N} a_k(f_k - Sw_k)\| + \|\sum_{k=1}^{N} a_k Sw_k\|\\ &\leqslant \textstyle\sum_{k=1}^{N}|a_k|\|f_k - Sw_k\| + \|S\|\|w\|\\ &\leqslant N\|w\|\delta + \|S\|\|w\|\\ &\leqslant (1+\varepsilon)\|S\|\|w\|.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|Sw - T\widehat{S}w\| &= \|\textstyle\sum_{k=1}^{N} a_k(Sw_k - \sum_{i=0}^{2^n-1} t_{k,i}g_{n,i})\|\\ &= \|\textstyle\sum_{k=1}^{N} a_k(Sw_k - f_k)\|\\ &\leqslant \delta N\|w\|\\ &\leqslant \varepsilon\|w\|.\end{aligned}$$

Let $\varepsilon > 0$. Since $C(\Delta)$ has the metric approximation property (see, e.g., [4] for the definition), there exists a net $(T_\alpha)_\alpha$ of finite-rank operators on $C(\Delta)$ such that

- $\limsup_\alpha \|T_\alpha\| \leqslant 1 + \varepsilon$,
- $\dim T_\alpha(C(\Delta)) \to \infty$,
- $T_\alpha \to I_{C(\Delta)}$ strongly.

For each $\alpha$, we may apply Claim 1 to the inclusion map $I_\alpha\colon T_\alpha[C(\Delta)] \to C(\Delta)$ in order to get an operator $\widehat{I_\alpha}\colon T_\alpha[C(\Delta)] \to Z$ such that

- $\|\widehat{I_\alpha}\| \leqslant 1 + \varepsilon$,
- $\|I_\alpha - T\widehat{I_\alpha}\| \leqslant (1 + \dim T_\alpha[C(\Delta)])^{-2}$.

Hence, for $f \in C(\Delta)$, we get

$$\begin{aligned}\|T\widehat{I_\alpha}T_\alpha f - f\| &\leqslant \|T\widehat{I_\alpha}T_\alpha f - I_\alpha T_\alpha f\| + \|T_\alpha f - f\|\\ &\leqslant \|T\widehat{I_\alpha} - I_\alpha\|\|T_\alpha\|\|f\| + \|T_\alpha f - f\| \to 0.\end{aligned}$$

Let $S$ be a $\sigma(\mathcal{B}(Z^*, C(\Delta)^*), Z^*\widehat{\otimes}_\pi C(\Delta))$-cluster point of the net $((\widehat{I_\alpha T_\alpha})^*)_\alpha$. We show that $ST^* = I_{C(\Delta)^*}$. Indeed, we choose a subnet $((\widehat{I_{\alpha'}T_{\alpha'}})^*)_{\alpha'}$ of $((\widehat{I_\alpha T_\alpha})^*)_\alpha$ so that $(\widehat{I_{\alpha'}T_{\alpha'}})^* \to S$ in the $\sigma(\mathcal{B}(Z^*, C(\Delta)^*), Z^*\widehat{\otimes}_\pi C(\Delta))$-topology. Then, for $f \in C(\Delta)$ and $\mu \in C(\Delta)^*$, we get $\langle(\widehat{I_{\alpha'}T_{\alpha'}})^*T^*\mu, f\rangle \to \langle ST^*\mu, f\rangle$. On the other hand, we have

$$\langle(\widehat{I_{\alpha'}T_{\alpha'}})^*T^*\mu, f\rangle = \langle\mu, TI_{\alpha'}T_{\alpha'}f\rangle \to \langle\mu, f\rangle.$$

Therefore, $\langle ST^*\mu, f\rangle = \langle \mu, f\rangle$.

*Claim 2.* There exists an operator $\widetilde{T}\colon C(\Delta)^* \to X^*$ so that $R^*\widetilde{T} = T^*$ and $\|\widetilde{T}\| \leqslant c(1+\varepsilon)$.

The proof of the claim is a variation of the Lindenstrauss' compactness argument (see [9, Proposition 1] and [14, Lemma 2]). Since certain amendments are required, we present the full reasoning.

*Proof of Claim 2.* We use the fact that $C(\Delta)^*$ is isometric to $L_1(\mu)$ for some infinite measure $\mu$, and as such, it is an $\mathcal{L}_{1,1+}$-space. Let $\Lambda$ be the collection of all finite-dimensional subspaces of $C(\Delta)^*$. Then, for each $\gamma \in \Lambda$, there exist $E_\gamma \in \Lambda$ with $\gamma \subseteq E_\gamma$ together with an isomorphism $U_\gamma\colon \ell_1^{\dim E_\gamma} \to E_\gamma$ so that $\|U_\gamma\|\|U_\gamma^{-1}\| \leqslant 1+\varepsilon$. Let $S_\gamma\colon Z \to E_\gamma^*$ be an operator such that $S_\gamma^* = T^*|_{E_\gamma}$ $(\gamma \in \Lambda)$. By the 1-injectivity of $\ell_\infty^{\dim E_\gamma}$, there is an operator $R_\gamma\colon X \to \ell_\infty^{\dim E_\gamma}$ so that $R_\gamma R = U_\gamma^* S_\gamma$ and $\|R_\gamma\| \leqslant \|U_\gamma^* S_\gamma\|\|R^{-1}\| \leqslant \|U_\gamma\|\|T\|\|R^{-1}\|$. Let $T_\gamma = R_\gamma^* U_\gamma^{-1}\colon E_\gamma \to X^*$. Then $R^*T_\gamma = T^*|_{E_\gamma}$ and $\|T_\gamma\| \leqslant c(1+\varepsilon)\|T\|$. For each $\gamma$, we define a non-linear, discontinuous function from $C(\Delta)^*$ to $X^*$ by

$$\widetilde{T_\gamma} f = \begin{cases} T_\gamma f, & f \in E_\gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\widetilde{T_\gamma})_\gamma$ is a net in the compact space

$$\prod_{f \in C(\Delta)^*} c(1+\varepsilon)\|T\|\|f\| B_{X^*},$$

and as such, it has a cluster point $\widetilde{T}$. Standard arguments show that $\widetilde{T}$ is linear, $R^*\widetilde{T} = T^*$, and $\|\widetilde{T}\| \leqslant c(1+\varepsilon)\|T\| = c(1+\varepsilon)$.

Finally, we get $SR^*\widetilde{T} = ST^* = I_{C(\Delta)^*}$ and hence

$$\beta_{C(\Delta)^*}(X^*) \leqslant \|\widetilde{T}\|\|SR^*\| \leqslant c(1+\varepsilon)^3.$$

Letting $\varepsilon \to 0$, we get $\beta_{C(\Delta)^*}(X^*) \leqslant c$. As $c$ is arbitrary, we get Step 1.
Step 2. $\beta_{L_1}(X^*) \leqslant \beta_{C[0,1]^*}(X^*)$.

It is well known that $L_1$ is isometric to a 1-complemented subspace of $C[0,1]^*$ (see, e.g., [1, p. 85]), which implies Step 2.
Step 3. $\alpha_{(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}}(X) \leqslant \beta_{L_1}(X^*)$.

Let $c > \beta_{L_1}(X^*)$. Then there exist operators $A\colon L_1 \to X^*, B\colon X^* \to L_1$ so that $BA = I_{L_1}, \|A\| = 1$, and $\|B\| < c$. Let $0 < \varepsilon < 1$ and $\varepsilon_n = \varepsilon/2^{2n+3}$ $(n = 0, 1, \ldots)$.

By [8, Lemma 3], we get $(f_{n,i})_{(n,i)\in\mathcal{F}}$ in $L_\infty$ and $(x_{n,i})_{(n,i)\in\mathcal{F}}$ in $X$ satisfying

(1) $\|f_{n,i}\|_1 = 1$ and $f_{n,i} \geqslant 0$ everywhere for all $(n,i) \in \mathcal{F}$;
(2) for each $n$ and $i \neq j$, $f_{n,i}(t)$ and $f_{n,j}(t)$ cannot be both non-zero for the same $t \in [0,1]$;
(3)

$$\langle Af_{n,i}, x_{m,j}\rangle = \begin{cases} 1, & (n,i) \geqslant (m,j), \\ 0, & \text{otherwise;} \end{cases}$$

(4) $\max_{0\leqslant i\leqslant 2^n-1}|t_i| \leqslant \|\sum_{i=0}^{2^n-1} t_i x_{n,i}\| \leqslant c(1+\varepsilon_n)\max_{0\leqslant i\leqslant 2^n-1}|t_i|$ ($n=0,1,\ldots;\ t_0,\ldots,t_{2^n-1}\in\mathbb{R}$).

We may now define recursively a sequence $(W_{n,i})_{(n,i)\in\mathcal{F}}$ of non-empty weak*-closed subsets of $B_{X^*}$ as follows:

- $W_{0,0} = \{x^*\in B_{X^*}\colon |\langle x^*, x_{0,0}\rangle - 1| \leqslant \varepsilon_0\}$,
- $W_{1,0} = W_{0,0}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{1,0}\rangle - 1|\leqslant \varepsilon_1, |\langle x^*, x_{1,1}\rangle|\leqslant\varepsilon_1\}$,
- $W_{1,1} = W_{0,0}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{1,1}\rangle - 1|\leqslant \varepsilon_1, |\langle x^*, x_{1,0}\rangle|\leqslant\varepsilon_1\}$,
- $W_{2,0} = W_{1,0}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{2,0}\rangle - 1|\leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle|\leqslant\varepsilon_2, j=1,2,3\}$,
- $W_{2,1} = W_{1,0}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{2,1}\rangle - 1|\leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle|\leqslant\varepsilon_2, j=0,2,3\}$,
- $W_{2,2} = W_{1,1}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{2,2}\rangle - 1|\leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle|\leqslant\varepsilon_2, j=0,1,3\}$,
- $W_{2,3} = W_{1,1}\cap\{x^*\in B_{X^*}\colon |\langle x^*, x_{2,3}\rangle - 1|\leqslant \varepsilon_2, |\langle x^*, x_{2,j}\rangle|\leqslant\varepsilon_2, j=0,1,2\}$,

and so on. By (3), each $W_{n,i}$ is non-empty. By the choice of $\varepsilon_n$, the sets $W_{n,i}, W_{n,j}$ are disjoint as long as $i\neq j$. Let

$$K = \bigcap_{n=0}^{\infty}\Big(\bigcup_{i=0}^{2^n-1} W_{n,i}\Big) \quad\text{and}\quad K_{n,i} = W_{n,i}\cap K \quad \big((n,i)\in\mathcal{F}\big).$$

By (3), $Af_{n,i}\in W_{m,j}$ if $(n,i)\geqslant(m,j)$, which implies that each $K_{n,i}$ is non-empty. By the construction of the sequence $(W_{n,i})$, we see that $K_{0,0} = K$, $K_{n+1,2i}\cup K_{n+1,2i+1} = K_{n,i}$, and $K_{n,i}\cap K_{n,j}=\varnothing$ if $i\neq j$.

Let us define an operator $T\colon X\to C(K)$ by $\langle Tx, x^*\rangle = \langle x^*, x\rangle$ ($x\in X, x^*\in K$). Then $|\langle Tx_{n,i}, x^*\rangle - 1|\leqslant\varepsilon_n$ if $x^*\in K_{n,i}$, and $|\langle Tx_{n,i}, x^*\rangle|\leqslant\varepsilon_n$ if $x^*\in\bigcup_{j\neq i}K_{n,j}$. Set $g_{n,i} = \mathbb{1}_{K_{n,i}}$, which is continuous as $K_{n,i}$ is clopen. Then $\|Tx_{n,i} - g_{n,i}\|\leqslant\varepsilon_n$. Moreover, $[g_{n,i}]_{i=0}^{2^n-1}\subseteq[g_{n+1,i}]_{i=0}^{2^{n+1}-1}$, $(g_{n,i})_{i=0}^{2^n-1}$ is isometrically equivalent to the unit vector basis of $\ell_\infty^{2^n}$ for all $n$, and

$$[g_{n,i}\colon (n,i)\in\mathcal{F}] = \overline{\bigcup_{n=0}^{\infty}[g_{n,i}]_{i=0}^{2^n-1}}$$

is isometric to $C(\Delta)$. Let $Z$ be a subspace of $C(\Delta)$ isometric to $(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$ and let $(z_{n,j})_{n=1,j=0}^{\infty,n-1}$ be a basis of $Z$ isometrically equivalent to the unit vector basis of $(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$. Fix $n\geqslant 1$. Then there exist $m>n$ and unit vectors $h_{n,j}\in[g_{m,i}]_{i=0}^{2^m-1}$ so that $\|z_{n,j}-h_{n,j}\|\leqslant\varepsilon/2^{n+3}$ ($j=0,1,\ldots,n-1$). We write $h_{n,j} = \sum_{i=0}^{2^m-1}a_{i,j}g_{m,i}$ and define $y_{n,j} = \sum_{i=0}^{2^m-1}a_{i,j}x_{m,i}\in X$.

*Claim 3.* For all $(t_{n,j})_{n=1,j=0}^{\infty,n-1}\in(\bigoplus_{n=1}^\infty \ell_\infty^n)_{\ell_1}$, we have

$$\Big(1-\frac{\varepsilon}{2}\Big)\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}| \leqslant \Big\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}t_{n,j}y_{n,j}\Big\| \leqslant c(1+\varepsilon)^2\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.$$

Indeed, by (4), we get

$$\begin{aligned}\left\|\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\right\| &= \left\|\sum_{i=0}^{2^m-1}\left(\sum_{j=0}^{n-1} a_{i,j} t_{n,j}\right) x_{m,i}\right\| \\ &\leqslant c(1+\varepsilon_m) \max_{0\leqslant i\leqslant 2^m-1}\left|\sum_{j=0}^{n-1} a_{i,j} t_{n,j}\right| \\ &= c(1+\varepsilon_m)\| \sum_{j=0}^{n-1} t_{n,j} h_{n,j}\| \\ &\leqslant c(1+\varepsilon_m)\left(\left\|\sum_{j=0}^{n-1} t_{n,j} z_{n,j}\right\| + \sum_{j=0}^{n-1} t_{n,j}(h_{n,j}-z_{n,j})\|\right) \\ &\leqslant c(1+\varepsilon_m)\Big(\max_{0\leqslant j\leqslant n-1} |t_{n,j}| + n\varepsilon/2^{n+3}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|\Big) \\ &\leqslant c(1+\varepsilon)^2 \max_{0\leqslant j\leqslant n-1}|t_{n,j}|.\end{aligned}$$

Consequently,

$$\left\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\right\| \leqslant \sum_{n=1}^{\infty}\left\|\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\right\| \leqslant c(1+\varepsilon)^2\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.$$

On the other hand, by the choice of $m$ and $h_{n,j}$, we arrive at

$$\begin{aligned}\|Ty_{n,j}-z_{n,j}\| &\leqslant \|Ty_{n,j}-h_{n,j}\| + \|h_{n,j}-z_{n,j}\| \\ &= \left\|\sum_{i=0}^{2^m-1} a_{i,j}(Tx_{m,i}-g_{m,i})\right\| + \varepsilon/2^{n+3} \\ &\leqslant \varepsilon_m 2^m \max_{0\leqslant i\leqslant 2^m-1}|a_{i,j}| + \varepsilon/2^{n+3} \\ &\leqslant \varepsilon/2^{n+3} + \varepsilon/2^{n+3} = \varepsilon/2^{n+2}.\end{aligned}$$

This implies that

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} y_{n,j}\right\| &\geqslant \left\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} Ty_{n,j}\right\| \\ &\geqslant \left\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j} z_{n,j}\right\| - \left\|\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} t_{n,j}(Ty_{n,j}-z_{n,j})\right\| \\ &\geqslant \sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}| - \sum_{n=1}^{\infty} n \max_{0\leqslant j\leqslant n-1}|t_{n,j}|\frac{\varepsilon}{2^{n+2}} \\ &\geqslant \left(1-\frac{\varepsilon}{2}\right)\sum_{n=1}^{\infty}\max_{0\leqslant j\leqslant n-1}|t_{n,j}|.\end{aligned}$$

Finally, by Claim 3, we get

$$\alpha_{(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}}(X) \leqslant c(1+\varepsilon)^2 / \Big(1 - \frac{\varepsilon}{2}\Big).$$

Letting $\varepsilon \to 0$ yields $\alpha_{(\bigoplus_{n=1}^{\infty} \ell_\infty^n)_{\ell_1}}(X) \leqslant c$. Since $c$ was arbitrary, the proof of Step 3 is completed.

Step 4. $\beta_{L_1}(X^*) \leqslant \theta_{C(\Delta)}(X)$.

This step follows from (1.1) together with Step 2. We are now ready to establish the final step of the proof.

Step 5. Suppose that $X$ is separable. Then $\theta_{C(\Delta)}(X) \leqslant \beta_{L_1}(X^*)$.

Let $c > \beta_{L_1}(X^*)$. Then there exist operators $A\colon L_1 \to X^*, B\colon X^* \to L_1$ so that $BA = I_{L_1}, \|A\| = 1$, and $\|B\| < c$.

Let $(f_{n,i})_{(n,i)\in\mathcal{F}}$ be a family of functions in $L_\infty$, $(x_{n,i})_{(n,i)\in\mathcal{F}}$ in $X$, and $(W_{n,i})_{(n,i)\in\mathcal{F}}$ associated to $\varepsilon_n = 1/2^{2n+2}$ $(n = 0, 1, \ldots)$ as described in Step 3. Since $X$ is separable, we may assume that the d-diameter of $W_{n,i} \leqslant 2^{-n}$ for each $i$, where d is a metric giving the relative $\sigma(X^*, X)$-topology on $B_{X^*}$. Let

$$K = \bigcap_{n=0}^{\infty} \left( \bigcup_{i=0}^{2^n-1} W_{n,i} \right) \quad \text{and} \quad K_{n,i} = W_{n,i} \cap K \ \ \big((n,i) \in \mathcal{F}\big).$$

Then $K$ is a compact, totally disconnected metric space without isolated points, hence homeomorphic to $\Delta$. Moreover, $K_{0,0} = K, K_{n+1,2i} \cup K_{n+1,2i+1} = K_{n,i}$, and $K_{n,i} \cap K_{n,j} = \varnothing$ if $i \neq j$. Hence $K = \bigcup_{i=0}^{2^n-1} K_{n,i}$ for all $n$. As seen in Step 3, the operator $T\colon X \to C(K)$, defined by $\langle Tx, x^* \rangle = \langle x^*, x \rangle$ $(x \in X, x^* \in K)$, satisfies $\|Tx_{n,i} - g_{n,i}\| \leqslant \varepsilon_n$, where $g_{n,i} = \mathbb{1}_{K_{n,i}} \in C(K)$.

An argument analogous to Step 1 yields that, if $W$ is a finite-dimensional Banach space and $S\colon W \to C(K)$ is an operator, then, for every $\varepsilon > 0$, there exists an operator $\widehat{S}\colon W \to X$ so that $\|\widehat{S}\| \leqslant c(1+\varepsilon)\|S\|$ and $\|S - T\widehat{S}\| \leqslant \varepsilon$.

Fix $\varepsilon > 0$. By an argument analogous to the one from Step 1, we get an operator $S\colon X^* \to C(K)^*$ with $\|S\| \leqslant c(1+\varepsilon)^2$ so that $ST^* = I_{C(K)^*}$. This means that

$$\theta_{C(\Delta)}(X) = \theta_{C(K)}(X) \leqslant c(1+\varepsilon)^2.$$

Letting $\varepsilon \to 0$, we arrive at $\theta_{C(\Delta)}(X) \leqslant c$. As $c$ is arbitrary, the proof is complete. □

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