Circumscribed hyperbolic triangles

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Abstract. Some triangles in the hyperbolic plane have a circumscribed circle, and some do not. In this essay, we discuss hyperbolic polygons whose vertices lie on a circle, or a horocycle, or a hypercycle.

1. Introduction. Every triangle in the Euclidean plane has a circumscribed circle. By contrast, some triangles in the hyperbolic plane do not have a circumscribed circle: for example, if three points in the unit disc model of the hyperbolic plane have a circumscribed Euclidean circle of radius greater than 1, then they cannot lie on a hyperbolic circle. In fact, the existence or non-existence of the circumscribed circle of a triangle lies at the very heart of the differences between the two geometries for, as Farkas Bolyai (1775–1856) showed, when taken with Euclid's other axioms, *the parallel axiom holds if and only if every triangle has a circumscribed circle* (see [2, p. 128] and [5, p. 7]).

Although the absence of a circumscribing circle is often mentioned in an introduction to hyperbolic geometry, very few authors go further and include the striking fact that *a hyperbolic triangle with sides of lengths a, b, and c, where $a \leq b \leq c$, has a circumscribed circle if and only if*

$$\sinh c/2 < \sinh a/2 + \sinh b/2. \tag{1.1}$$

As (1.1), and its natural extension to horocycles and hypercycles, does not seem to be well known, we hope that the simple proof given below, and this expository, and partly historical, essay will make it more accessible.

We shall assume that readers are familiar with the extended complex plane $\mathbb{C}_\infty$ equipped with the chordal metric χ, where

$$\chi(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}},$$

and with the two most familiar models of the hyperbolic plane, namely

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}, \quad \mathbb{H} = \{x + iy \in \mathbb{C} \colon y > 0\},$$

with their associated hyperbolic metrics $2|dz|/(1 - |z|^2)$ and $|dz|/y$, respectively, of curvature -1. We shall use $\mathcal{H}$, with hyperbolic distance ρ, when either of these models could be used.

A *chordal circle* in $\mathbb{C}_\infty$ is a circle in the metric χ, and any three points in $\mathbb{C}_\infty$ lie on a unique chordal circle C (which is either a Euclidean circle, or a Euclidean straight line with ∞ attached). Now let z_1, z_2, and z_3 be three points in $\mathcal{H}$, and let C be the unique chordal circle through these points. Then C is a *hyperbolic circle* when it lies entirely in $\mathcal{H}$, a *horocycle* when it lies in $\mathcal{H}$ except for a single point at which it is tangent to the boundary of $\mathcal{H}$, and a *hypercycle* when it meets the exterior of $\mathcal{H}$. With these available, the inequality (1.1) is the first part of the following result which appears in [3, p. 118] following a long discussion of right-angled hyperbolic hexagons.

Theorem 1.1. *Let T be a hyperbolic triangle with sides of lengths a, b, and c, where $a \leq b \leq c$. Then the vertices of T lie on a circle, a horocycle, or a hypercycle, respectively, according as*

$$\sinh c/2 \begin{cases} < \sinh a/2 + \sinh b/2; \\ = \sinh a/2 + \sinh b/2; \\ > \sinh a/2 + \sinh b/2. \end{cases} \tag{1.2}$$

Theorem 1.1 was probably known long ago, but its origin seems to have been lost in the mist of history; indeed, many of the ideas that lead to Theorem 1.1 first appeared in the works of Farkas Bolyai, Johann Bolyai, and Lobachevski, and are still in use today. As remarked in [2], some of these ideas are now forgotten, but here we use them to give a simple and direct proof of Theorem 1.1. This result refers to the three sides of a triangle, but it can be used to derive a similar result for two sides and the included angle of a triangle.

Corollary 1.2. *Let T be the hyperbolic triangle with sides of lengths a, b, and c, where $a \leq b \leq c$, and opposite angles α, β, and γ. Then the vertices of T lie on a circle, a horocycle, or a hypercycle, respectively, according as*

$$\sinh a/2 \sinh b/2 \begin{cases} < 1 + \cosh a/2 \cosh b/2 \cos \gamma; \\ = 1 + \cosh a/2 \cosh b/2 \cos \gamma; \\ > 1 + \cosh a/2 \cosh b/2 \cos \gamma. \end{cases}$$

Proof. The cosine rule in hyperbolic geometry gives

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma,$$

and as $\cosh x = 1 + 2\sinh^2(x/2)$ and $\sinh x = 2\sinh(x/2)\cosh(x/2)$, we see (after a little simplification) that $\left(\sinh a/2 + \sinh b/2\right)^2 - \sinh^2 c/2$ is

$$2\sinh a/2 \, \sinh b/2 \left[1 + \cosh a/2 \, \cosh b/2 \, \cos\gamma - \sinh a/2 \, \sinh b/2\right].$$

The result now follows from Theorem 1.1.   □

Theorem 1.1 and Corollary 1.2 also yield the following results.

Corollary 1.3. *A hyperbolic right-angled triangle with sides of lengths a, b, and c, and with $\gamma = \pi/2$, has a circumscribed circle, horocycle, or hypercycle, respectively, according as $\sinh a/2 \, \sinh b/2$ is less than, equal to, or greater than 1, respectively.*

Corollary 1.4. *Let T be an isosceles hyperbolic triangle with sides of lengths a, a, and c. If $a \geq c$ (for example, if T is an equilateral triangle), then T has a circumscribed circle. However, if $c > a$, then T has a circumscribed circle, horocycle, or hypercycle, respectively, according as $\sinh c/2$ is less than, equal to, or greater than $2\sinh a/2$, respectively.*

2. The background to Theorem 1.1. The hyperbolic distances $\rho_{\mathbb{D}}$ and $\rho_{\mathbb{H}}$ in the models $\mathbb{D}$ and $\mathbb{H}$, respectively, are given by the formulae

$$\rho_{\mathbb{D}}(z,w) = \log\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|}, \quad \rho_{\mathbb{H}}(z,w) = \log\frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}.$$

Despite these explicit formulae, experience shows that the most useful formula in the vast majority of circumstances is that for $\sinh\frac{1}{2}\rho(z,w)$. Indeed, if $\mathcal{H}$ is any model of the hyperbolic plane in which $\mathcal{H}$ is a chordal disc, with hyperbolic metric $\lambda_{\mathcal{H}}(z)\,|dz|$ of curvature -1, and associated hyperbolic distance $\rho_{\mathcal{H}}(z,w)$, then we have the following result.

Theorem 2.1. *For any disk model $\mathcal{H}$ of the hyperbolic plane,*

$$\sinh^2 \tfrac{1}{2}\rho_{\mathcal{H}}(z,w) = \tfrac{1}{4}|z - w|^2 \lambda_{\mathcal{H}}(z)\lambda_{\mathcal{H}}(w). \tag{2.1}$$

Note that (2.1) gives the formulae

$$\sinh\tfrac{1}{2}\rho_{\mathbb{D}}(z,w) = \frac{|z - w|}{\sqrt{(1 - |z|^2)(1 - |w|^2)}}, \tag{2.2}$$

$$\sinh\tfrac{1}{2}\rho_{\mathbb{H}}(z,w) = \frac{|z - w|}{2\sqrt{\mathrm{Im}(z)\,\mathrm{Im}(w)}}, \tag{2.3}$$

in the models $\mathbb{D}$ and $\mathbb{H}$. The proof of Theorem 2.1 is easy. First, if f is a Möbius map of one disc model, say $\mathcal{H}$, onto another, say $\mathcal{D}$, then

$$\rho_{\mathcal{H}}(z,w) = \rho_{\mathcal{D}}\big(f(z), f(w)\big);$$
$$\lambda_{\mathcal{H}}(z) = f'(z)\,\lambda_{\mathcal{H}}\big(f(z)\big).$$

Since

$$\big(f(z) - f(w)\big)^2 = (z - w)^2\, f'(z)\, f'(w),$$

we see that both sides of (2.1) are invariant under Möbius maps. It follows that (2.1) need only be verified in any chosen model of the hyperbolic plane, and

since the formulae (2.2) and (2.3) are well known to be true, (2.1) follows for any model. Theorem 2.1 will play a significant role in our proof of Theorem 1.1.

3. Pencils of geodesics and their orthogonal trajectories. The idea of a pencil of geodesics occurs in the very early publications on hyperbolic geometry, and the three types of pencils in a model $\mathcal{H}$ of the hyperbolic plane are as follows:

- an *elliptic pencil* is the family of geodesics that pass through a given point in $\mathcal{H}$;
- a *parabolic pencil* is the family of geodesics that have a given endpoint on $\partial\mathcal{H}$;
- a *hyperbolic pencil* is the family of geodesics that are orthogonal to a given geodesic in $\mathcal{H}$.

As usual, we say that two geodesics α and β in $\mathcal{H}$ (i) *meet* (*intersect*, or are *concurrent*) if they have a non-empty intersection in $\mathcal{H}$, (ii) are *parallel* if they have a common end-point on $\partial\mathcal{H}$, and (iii) are *ultra-parallel* if they have a common orthogonal geodesic. Thus any two geodesics in an elliptic pencil are concurrent, any two geodesics in a parabolic pencil are parallel, and any two geodesics in a hyperbolic pencil are ultra-parallel.

Pencils of geodesics are intimately connected to circles, horocycles and hypercycles, for each pencil $\mathcal{P}$ has a family $\mathcal{O}(\mathcal{P})$ of orthogonal trajectories, each of which is orthogonal to every geodesic in $\mathcal{P}$. Explicitly,

- if $\mathcal{P}$ is the elliptic pencil of geodesics through w, then $\mathcal{O}(\mathcal{P})$ is the set of all circles with centre w;
- if $\mathcal{P}$ is the parabolic pencil of geodesics that end at ζ, then $\mathcal{O}(\mathcal{P})$ is the set of all horocycles that are tangent to $\partial\mathcal{H}$ at ζ;
- if $\mathcal{P}$ is the hyperbolic pencil of geodesics that are orthogonal to the geodesic α, then $\mathcal{O}(\mathcal{P})$ is the family of all hypercycles that have the same endpoints as α.

We shall use the term *cycle* to indicate either a circle, a horocycle, or a hypercycle, and although there are various alternative ways to introduce these three families of cycles, all that matters here is that *each cycle is an orthogonal trajectory of some pencil of geodesics.*

For any points z and w in $\mathcal{H}$, we let $[z,w]$ be the Euclidean, and $\langle z,w\rangle$ the hyperbolic, geodesic segments, respectively, each with endpoints z and w. The *orthogonal bisector* of $\langle z,w\rangle$ is the geodesic, denoted by $B(z,w)$, which consists of those points that are (hyperbolically) equidistant from z and w, and these geodesics play a crucial role in our discussion. Instead of starting with a triangle and asking whether or not the vertices lie on a cycle, we may start with a cycle and consider its chords. Explicitly, a *chord* of a cycle C is a geodesic segment $\langle z,w\rangle$, where z and w are on C, and we have the following result (which goes back to Lobachevski, though not with this terminology).

Lemma 3.1. *Let C be a cycle and let $\mathcal{B}$ be the set of orthogonal bisectors of the chords of C. If C is a circle, then $\mathcal{B}$ is an elliptic pencil; if C is a horocycle, then $\mathcal{B}$ is a parabolic pencil; if C is a hypercycle, then $\mathcal{B}$ is a hyperbolic pencil.*

Proof. If we use the standard invariance arguments, we need only consider the case when C is either the circle $\{z \colon |z| = r\}$ in $\mathbb{D}$, or the horocycle $\{z \colon y = 1\}$ in $\mathbb{H}$, or the hypercycle $\{tw \colon t > 0\}$, where $w \in \mathbb{H}$. In each of these cases, we can take w_1 and w_2 on C, and then find the orthogonal bisector of $\langle w_1, w_2 \rangle$ by requiring that $\sinh \frac{1}{2} \rho(z, w_1) = \sinh \frac{1}{2} \rho(z, w_2)$, and using (2.2) or (2.3). In the case when C is the given hypercycle, the orthogonal bisector of the chord $\langle w, tw \rangle$, where $t > 1$, is part of the Euclidean circle $|z| = \sqrt{t}|w|$, and this is orthogonal to the (vertical) geodesic given by $x = 0$. $\qquad\square$

For alternative proofs, see [4, p. 100] and [5, p. 53]. It is clear that if $\mathcal{B}$ is an elliptic pencil, then the original curve C is a circle whose centre is at the common point of the geodesics in the elliptic pencil. Given this, it is perhaps worth remarking that Lobachevski actually defined a horocycle to be a curve with the property that the set of orthogonal bisectors of its chords forms a parabolic pencil (informally a circle whose centre is at infinity).

A key consequence of Lemma 3.1 is that we can decide whether a given cycle C is a circle, a horocycle, or a hypercycle simply by considering the orthogonal bisectors of just *two* of its chords (for any two bisectors determine a unique pencil). In particular, when considering whether or not the three vertices of a triangle lie on a circle, a horocycle, or a hypercycle, we need only consider the orthogonal bisectors of *two* of its sides (and this was a key point in Lobachevski's arguments). Of course, a consideration of the orthogonal bisectors naturally leads us to consider one half of the chord length or, equivalently, one half of the length of a side of the triangle.

4. The proof of Theorem 1.1. Throughout, we shall suppose that the hyperbolic triangle T has vertices z_1, z_2, and z_3 which are labelled so that

$$\rho(z_1, z_3) = \max\{\rho(z_2, z_1), \rho(z_2, z_3)\}.$$

Then Theorem 1.1 can be restated in the form that the z_j lie on a circle, a horocycle, or a hypercycle, respectively, according as

$$\sinh \tfrac{1}{2}\rho(z_1, z_3) \begin{cases} < \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3); \\ = \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3); \\ > \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3). \end{cases}$$

We have three cases to consider, namely when the vertices z_j of T lie on a circle, a horocycle, or a hypercycle.

Case 1: *The z_j lie on a hyperbolic circle.*
As the required result is invariant under hyperbolic isometries, and as the z_j lie on a circle, we may use the model $\mathbb{D}$ and assume that the three vertices z_j of T lie on some hyperbolic circle given by $|z| = r$, where $0 < r < 1$. Then, from (2.2), we have

$$\sinh \tfrac{1}{2}\rho(z_i, z_j) = \frac{|z_i - z_j|}{1 - r^2},$$

and the inequality $|z_1 - z_3| < |z_1 - z_2| + |z_2 - z_3|$ now shows that

$$\sinh \tfrac{1}{2}\rho(z_1, z_3) < \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3).$$

Case 2: *The z_j lie on a horocycle.*

As the required result is invariant under hyperbolic isometries, and as the z_j lie on a horocycle, we may use the model $\mathbb{H}$ and assume that the three vertices z_j of T lie on the horocycle H given by $\{z \colon y = 1\}$. Then, from (2.3), we see that

$$\sinh \tfrac{1}{2}\rho(z_i, z_j) = \tfrac{1}{2}|z_i - z_j|,$$

and this implies that z_2 lies between z_1 and z_3 on H, and that

$$\sinh \tfrac{1}{2}\rho(z_1, z_3) = \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3).$$

Case 3: *The z_j lie on a hypercycle.*

As the required result is invariant under hyperbolic isometries, and as the z_j lie on a hypercycle, we may use the model $\mathbb{H}$ and assume that the three vertices z_j of T lie on the hypercycle K given by $\{r \exp(i\theta) \colon r > 0\}$, where $0 < \theta < \pi/2$. We then have the situation illustrated in Fig. 1, where, directly from (2.2), we have

$$\sinh \tfrac{1}{2}\rho(z, w) = \frac{\sinh \tfrac{1}{2}\rho(i|z|, i|w|)}{\sin \theta}.$$

It follows from this that z_2 lies between z_1 and z_3 on K, and since

$$\sinh \tfrac{1}{2}\rho(i|z_1|, i|z_3|) = \sinh \tfrac{1}{2}\big(\rho(i|z_1|, i|z_2| + \rho(i|z_2|, i|z_3|)\big)$$
$$> \sinh \tfrac{1}{2}\rho(i|z_1|, i|z_2|) + \sinh \tfrac{1}{2}\rho(i|z_2|, i|z_3|),$$

we see that $\sinh \tfrac{1}{2}\rho(z_1, z_3) > \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3)$. The proof of Theorem 1.1 is now complete.

5. Right-angled triangles. We have seen that if T is a hyperbolic right-angled triangle whose shorter sides have lengths $2\ell_1$ and $2\ell_2$, then T has

   (i) a circumscribed circle if and only if $\sinh \ell_1 \sinh \ell_2 < 1$;
  (ii) a circumscribed horocycle if and only if $\sinh \ell_1 \sinh \ell_2 = 1$;
 (iii) a circumscribed hypercycle if and only if $\sinh \ell_1 \sinh \ell_2 > 1$.

In this section, we examine the classical geometry that underlies this result and, in particular, we shall show that it follows directly from the classical angle of parallelism formula and the Lambert quadrilateral that was studied by (and named after) J.H. Lambert (1728–1777).
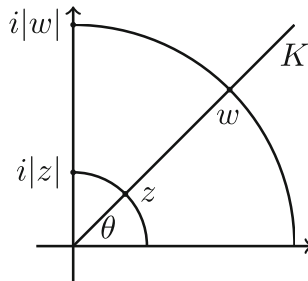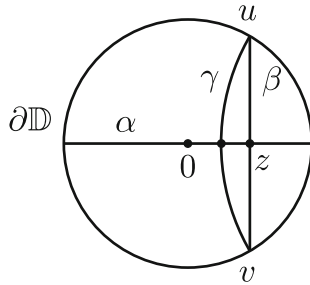

FIGURE 1. The hypercycle K

FIGURE 2. The geodesic γ is the bisector $B(0, z)$
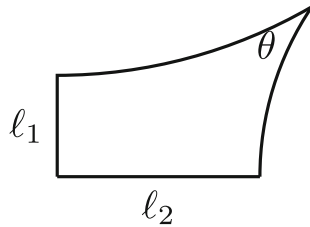

FIGURE 3. The Lambert quadrilateral: $\sinh \ell_1 \sinh \ell_2 = \cos \theta$

First, we give a visual representation of the orthogonal bisector $B(0, z)$ of a hyperbolic segment $\langle 0, z \rangle$ in the model $\mathbb{D}$ of the hyperbolic plane: see Fig. 2.

Lemma 5.1. *Let z be a non-zero point in the unit disc $\mathbb{D}$, let α be the hyperbolic geodesic through 0 and z, and let β be the Euclidean chord through z and orthogonal to α. Then the hyperbolic geodesic γ with the same endpoints as β is the orthogonal bisector $B(0, z)$.*

Proof. The hyperbolic geodesic γ is an arc of a Euclidean circle C that is orthogonal to the unit cirle $\partial \mathbb{D}$, and it is well known that this implies that 0 and z are inverse points with respect to C. However, the inversion across C coincides with the hyperbolic reflection across the geodesic γ; thus γ is the set of points equidistant from 0 and z. $\square$

A *Lambert quadrilateral* is a hyperbolic quadrilateral with three right-angles, and one other angle, say θ, where $0 \leq \theta < \pi/2$ (see Fig. 3); if $\theta = 0$, then the corresponding vertex lies on the boundary of $\mathcal{H}$. It is well known ([1, p. 156]) that for the Lambert quadrilateral in Fig. 3, we have

$$\sinh \ell_1 \sinh \ell_2 = \cos \theta.$$

In particular, if $\theta = 0$, then $\sinh \ell_1 \sinh \ell_2 = 1$.

We now return to the case of a right-angled hyperbolic triangle. In Fig. 4, we have a right-angled triangle T with vertices 0, z_1, and z_2 (and whose hypothenuse is not drawn), and the orthogonal bisectors of the sides $\langle 0, z_1 \rangle$ and $\langle 0, z_2 \rangle$ pass through w_1 and w_2, respectively, and meet at the point A on
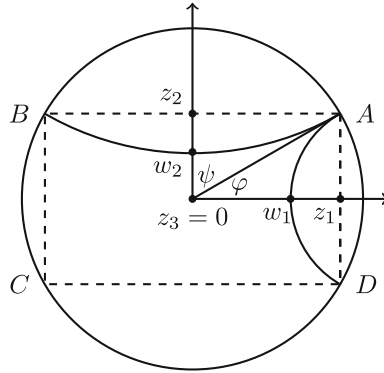
Figure 4. Midpoints and Euclidean chords

the unit circle. The hyperbolic quadrilateral with vertices 0, w_1, A, and w_2 is a Lambert quadrilateral, so that $\sinh \rho(0, w_1) \sinh \rho(0, w_2) = 1$. This is the second case in Theorem 1.1, and a continuity argument then leads to the first and third cases in that result.

Alternatively, we can use the classical angle of parallelism formula. In the situation given in Fig. 4, this formula [1, p. 145] gives

$$\sinh \rho(0, w_1) \tan \varphi = 1 = \sinh \rho(0, w_2) \tan \psi.$$

Since $\varphi + \psi = \pi/2$, we have $\tan \varphi \tan \psi = 1$ which gives the same conclusion. It is also clear from Fig. 4 that we can phrase this result in Euclidean terms as follows: let the vertices of T be z_1, z_2, and z_3, where z_1 is positive, z_2 is purely imaginary, and $z_3 = 0$. Then z_1, z_2, and z_3 are concyclic, or lie on a horocycle, or lie on a hypercycle, according as $|z_1|^2 + |z_2|^2$ is less than 1, equal to 1, or greater than 1, respectively.

Similar geometric ideas can be used to illustrate, explain, and prove the other results given above; however, the geometric proofs are less efficient in as far as it is often necessary to consider different cases that correspond to different geometric configurations. Briefly, for a triangle, these proofs depend on the geometry of the polygon formed by the three bisectors of its sides and also, if two bisectors are ultra-parallel, the common orthogonal geodesic to these sides. Such situations involve a study of hyperbolic quadrilaterals, pentagons, and hexagons, and such an analysis can be found in [3, pp. 86–87].

6. Finite sets of points on a cycle. The arguments used above can also be used to establish the following *necessary* (but not sufficient) conditions on any finite sequence $z_1, \ldots, z_n$ of points to lie on a circle, a horocycle, or a hypercycle.

Theorem 6.1. *Suppose that the points $z_1, \ldots, z_n$, where $n \geq 3$, are points in the hyperbolic plane that lie, in this order, along some chordal circle C. If C is a circle, then*

$$\sinh \tfrac{1}{2}\rho(z_1, z_n) < \sinh \tfrac{1}{2}\rho(z_1, z_2) + \cdots + \sinh \tfrac{1}{2}\rho(z_{n-1}, z_n).$$
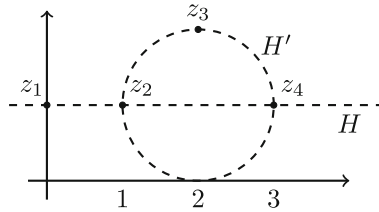
FIGURE 5. Four points not on any cycle

If C is a horocycle, then

$$\sinh \tfrac{1}{2}\rho(z_1, z_n) = \sinh \tfrac{1}{2}\rho(z_1, z_2) + \cdots + \sinh \tfrac{1}{2}\rho(z_{n-1}, z_n).$$

If C is a hypercycle, then

$$\sinh \tfrac{1}{2}\rho(z_1, z_n) > \sinh \tfrac{1}{2}\rho(z_1, z_2) + \cdots + \sinh \tfrac{1}{2}\rho(z_{n-1}, z_n).$$

It is important to note that while three points in the hyperbolic plane necessarily lie on some cycle, in general, four or more points will not. Thus Theorem 6.1 necessarily includes the hypothesis that $z_1, \ldots, z_n$ do lie on some cycle. Moreover, it is possible for z_1, z_2, z_3, and z_4 (for example) to satisfy

$$\sinh \tfrac{1}{2}\rho(z_1, z_4) = \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_3) + \sinh \tfrac{1}{2}\rho(z_3, z_4), \quad (6.1)$$

but *not lie on any cycle*. Consider, for example, the points z_j in Fig. 5. By Theorem 1.1, we have

$$\sinh \tfrac{1}{2}\rho(z_1, z_4) = \sinh \tfrac{1}{2}\rho(z_1, z_2) + \sinh \tfrac{1}{2}\rho(z_2, z_4),$$
$$\sinh \tfrac{1}{2}\rho(z_2, z_4) = \sinh \tfrac{1}{2}\rho(z_2, z_3) + \sinh \tfrac{1}{2}\rho(z_3, z_4),$$

thus (6.1) holds.

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