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# Groups with many abelian or self-normalizing subgroups

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**Abstract.** Groups in which every non-abelian subgroup is equal to its normalizer have been recently completely described. This paper investigates locally soluble groups with restrictions on subgroups which are neither abelian nor self-normalizing.

**Mathematics Subject Classification.** Primary 20F19; Secondary 20F22.

**Keywords.** Self-normalizing subgroup, Non-abelian subgroup, Minimal condition, Finite rank.

**1. Introduction.** The structure of groups in which the set of all non-normal subgroups is small in some sense has been investigated in many papers and in several different situations. In particular, Romalis and Sesekin [19–21] studied groups whose non-abelian subgroups are normal. A dual problem has been considered recently in [7] where a group $G$ is defined as an $\mathcal{H}$-*group* if any non-abelian subgroup $H$ of $G$ is self-normalizing (i.e. $H$ is equal to the normalizer $N_G(H)$ of $H$ in $G$). That paper deals with soluble $\mathcal{H}$-groups while finite non-soluble $\mathcal{H}$-groups were fully described later in [5]. Clearly, soluble $\mathcal{H}$-groups are metabelian and it turns out that any finite $\mathcal{H}$-group is either soluble or simple (see [5, Theorem 2.14]) and also that any infinite locally finite $\mathcal{H}$-group is soluble (see [5, Theorem 3.4]). The only finite non-abelian simple $\mathcal{H}$-groups are the alternating group Alt(5) of degree 5 or the projective special linear group $\mathrm{PSL}(2, 2^n)$, where $2^n - 1$ is a prime (see [5, Theorem 2.17]). A soluble non-nilpotent group $G$ is an $\mathcal{H}$-group if and only if $G = \langle x \rangle \ltimes G'$ where $x$ has order a power of a prime $p$, the derived subgroup $G'$ is a periodic abelian $p'$-group, and $x^p$ belongs to $C_G(G')$ (see [7, Theorem 1.7]). Of course, any nilpotent group does not have self-normalizing subgroups so that a nilpotent $\mathcal{H}$-group which is not abelian is a minimal non-abelian group and hence its structure is well-known.

In [7], it was also proved that any soluble group whose infinite non-abelian subgroups are self-normalizing is either a Chernikov group or an $\mathcal{H}$-group. Here in Sect. 2, we will consider minimal conditions related to $\mathcal{H}$-groups and the just quoted result will be generalized as follows.

**Theorem A1.** *Let $G$ be a locally soluble group satisfying the minimal condition on subgroups which are neither abelian nor self-normalizing. Then $G$ is either a Chernikov group or an $\mathcal{H}$-group.*

**Theorem A2.** *Let $G$ be a locally soluble group satisfying the minimal condition on non-$\mathcal{H}$-subgroups. Then $G$ is either a Chernikov group or an $\mathcal{H}$-group.*

Recall that a group $G$ is said to have *finite (Prüfer) rank* $r$ if every finitely generated subgroup of $G$ can be generated by $r$ elements and $r$ is the least such integer. In recent years, the influence on a locally soluble group of the behavior of the subgroups of infinite rank has been investigated (see for instance [3,4,6, 8,9,12,13]). This point of view will be adopted here in Sect. 3 were the following results will be obtained.

**Theorem B1.** *Let $G$ be a locally soluble group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. Then $G$ is an $\mathcal{H}$-group.*

**Theorem B2.** *Let $G$ be a locally (soluble-by-finite) group of infinite rank whose subgroups of infinite rank are $\mathcal{H}$-groups. Then $G$ is an $\mathcal{H}$-group.*

For notation and basic facts, we refer to [18].

**2. Minimal conditions.** The first three lemmas deal with a property of $\mathcal{H}$-groups that we need in what follows. Recall that a group $G$ is said to be *locally graded* if every finitely generated non-trivial subgroup of $G$ contains a proper subgroup of finite index. The class of locally graded groups is quite large and contains, in particular, all locally (soluble-by-finite) groups.

**Lemma 2.1.** *Let $G$ be a locally graded group whose finitely generated non-abelian subgroups are self-normalizing. Then $G$ is a locally (abelian-by-finite) $\mathcal{H}$-group.*

*Proof.* Let $E$ be any finitely generated subgroup of $G$ and assume, by contradiction, that $E$ is not abelian-by-finite. Since $G$ is locally graded, $E$ contains a proper normal subgroup $E_1$ of finite index. Then $E_1$ is finitely generated and non-abelian, so that $E_1 = N_G(E_1)$; hence $E_1 = E$. This contradiction proves that $E$ is abelian-by-finite. In particular, $G$ locally satisfies the maximal condition.

Let $H$ be any non-abelian subgroup of $G$ which is not finitely generated and let $g$ be any element of the normalizer $N_G(H)$. Consider any non-abelian finitely generated subgroup $X$ of $H$. Then $\langle X, g\rangle$ satisfies the maximal condition, so that the subgroup $Y = X^{\langle g\rangle}$ is a finitely generated non-abelian subgroup of $H^{\langle g\rangle} = H$ which is normalized by $g$. Since $N_G(Y) = Y$, it follows that $g$ belongs to $Y$ and so also to $H$. Thus $N_G(H) = H$. Therefore $G$ is an $\mathcal{H}$-group. □

**Lemma 2.2.** *Let $G$ be a locally graded group. If every finitely generated subgroup of $G$ is an $\mathcal{H}$-group, then $G$ is an $\mathcal{H}$-group.*

*Proof.* Let $H$ be any finitely generated subgroup of $G$ and let $g \in N_G(H) \setminus H$. Then the finitely generated subgroup $K = \langle H, g\rangle$ is an $\mathcal{H}$-group and $g$ belongs to $N_K(H) \setminus H$. Thus $H$ is abelian. Therefore all finitely generated subgroups of $G$ are either abelian or self-normalizing, hence application of Lemma 2.1 proves that $G$ is an $\mathcal{H}$-group. □

**Lemma 2.3.** *Let $G$ be a locally graded $\mathcal{H}$-group. Then $G$ is either finite or metabelian.*

*Proof.* The group $G$ is locally (abelian-by-finite) by Lemma 2.1. If $G$ is periodic, then $G$ is locally finite and so it is either finite or metabelian (see [5, Theorem 3.4]). Hence assume that $G$ is not periodic. In order to prove that $G$ is abelian, it can be assumed that $G$ is finitely generated. Then $G$ contains a normal abelian subgroup $A$ of finite index; in particular, $A$ is not periodic. If $g \in G$, then $A\langle g\rangle$ is a soluble non-periodic $\mathcal{H}$-group and hence it is abelian (see [5, Theorem 3.2] or [7, Lemma 1.2]). Therefore $A \leq Z(G)$, so that $G/Z(G)$ is finite. Thus $G'$ is finite (see [18, Part 1, Theorem 4.12]) and so $G$ is abelian (see [7, Lemma 1.2]). □

**Lemma 2.4.** *Let $G$ be a nilpotent group satisfying the minimal condition on subgroups which are neither abelian nor self-normalizing. Then $G$ is either abelian or a Chernikov group.*

*Proof.* Since a nilpotent group has no self-normalizing subgroups, $G$ satisfies the minimal condition on non-abelian subgroups and hence $G$ is either abelian or a Chernikov group (see [2]). □

**Lemma 2.5.** *Let $G$ be a group satisfying the minimal condition on subgroups which are neither abelian nor self-normalizing. Then either $G$ is abelian or $G/G'$ is periodic.*

*Proof.* Let $x$ be any element of $G$ such that the coset $xG'$ has infinite order. Since $G$ has the minimal condition on subgroups which are neither abelian nor self-normalizing, from the consideration of the infinite descending chain of $G$-invariant subgroups

$$\left\langle x^2, G'\right\rangle > \left\langle x^4, G'\right\rangle > \cdots > \left\langle x^{2^n}, G'\right\rangle > \cdots,$$

it follows that there exists $h \in \mathbb{N}$ such that $\langle x^{2^h}, G'\rangle$ is abelian. Similarly also $\langle x^{3^k}, G'\rangle$ is abelian for some $k \in \mathbb{N}$. Therefore Fitting's theorem yields that

$$\langle x, G'\rangle = \langle x^{2^h}, G'\rangle\langle x^{3^k}, G'\rangle$$

is nilpotent. It follows that if $x_1G', \ldots, x_tG'$ are elements of infinite order of $G/G'$, the subgroup $\langle x_1, G'\rangle \cdots \langle x_t, G'\rangle$ is nilpotent by Fitting's theorem and hence even abelian by Lemma 2.4. Therefore, if $G/G'$ is not periodic and $\mathcal{I}$ is the set of all elements $x$ of $G$ such that the coset $xG'$ has infinite order, the group $G = \langle\langle x, G'\rangle : x \in \mathcal{I}\rangle$ itself is abelian. □

**Lemma 2.6.** *Let $G = \langle x, A \rangle$ where $A$ is a non-periodic finitely generated abelian normal subgroup whose index in $G$ is a power of a prime. If $G$ satisfies the minimal condition on subgroups which are neither abelian nor self-normalizing, then $G$ is abelian.*

*Proof.* Consider first the case in which $\langle x \rangle \cap A = \{1\}$. Assume further that $A$ is torsion-free, and let the index of $A$ in $G$ be a power of the prime $p$. If $\langle x \rangle A^{p^n} = \langle x \rangle A^{p^{n+1}}$ for some positive integer $n$, then

$$A^{p^n} = \langle x \rangle A^{p^{n+1}} \cap A^{p^n} = A^{p^{n+1}} (\langle x \rangle \cap A^{p^n}) = A^{p^{n+1}}$$

and this is not possible since $A$ is a free abelian group of finite rank. Thus $\langle x \rangle A^{p^n} \neq \langle x \rangle A^{p^{n+1}}$ for every positive integer $n$. On the other hand, $G/A^{p^n}$ is a finite $p$-group, so that it is nilpotent and this implies that $\langle x \rangle A^{p^n}$ is a subnormal subgroup of $G$ for any positive integer $n$. Therefore the consideration of the descending chain

$$\langle x \rangle A^{p} > \langle x \rangle A^{p^2} > \cdots > \langle x \rangle A^{p^n} > \cdots$$

and the minimal condition on subgroups which are neither abelian nor self-normalizing gives that $\langle x \rangle A^{p^n}$ is abelian for some positive integer $n$. Hence $G = \left(\langle x \rangle A^{p^n}\right) A$ is nilpotent. Therefore application of Lemma 2.4 gives that $G$ is abelian when $A$ is torsion-free. If $A$ is not torsion-free and $T$ is the subgroup consisting of all elements of finite order of $A$, the previous argument gives that $G/T$ is abelian. Therefore $G'$ is periodic and, since $G$ is not periodic, it follows from Lemma 2.5 that $G$ is abelian.

Assume now that $\langle x \rangle \cap A \neq \{1\}$; notice that $\langle x \rangle \cap A \leq Z(G)$. The first part of this proof yields that either $A/\langle x \rangle \cap A$ is periodic or $G/\langle x \rangle \cap A$ is abelian. If $A/\langle x \rangle \cap A$ is periodic, then $G/\langle x \rangle \cap A$ is finite; thus $G'$ is finite (see [18, Part 1, Theorem 4.12]) and so, since $G$ is not periodic, it follows from Lemma 2.5 that $G$ must be abelian. On the other hand, if $G/\langle x \rangle \cap A$ is abelian, we have that $G$ is nilpotent and so even abelian by Lemma 2.4. The lemma is proved. □

The next result also solves the non-periodic case of Theorem A1.

**Lemma 2.7.** *Let $G$ be a locally graded group satisfying the minimal condition on subgroups which are neither abelian nor self-normalizing. If $G$ is not periodic, then $G$ is abelian.*

*Proof.* In order to prove that $G$ is abelian, without loss of generality, it can be supposed that $G$ is finitely generated.

Assume first that $G$ is soluble. Suppose by contradiction that $G$ is not abelian and let $G$ be a counterexample with minimal derived length. Application of Lemma 2.5 yields that $G/G'$ is finite, hence $G'$ is likewise a non-periodic finitely generated group with the minimal condition on subgroups which are neither abelian nor self-normalizing, and so $G'$ is abelian by the minimality of derived length of $G$. Let $G/G' = \langle x_1G', \dots, x_tG' \rangle$ with every $x_iG'$ of prime-power order. Lemma 2.6 yields that every $\langle x_i, G' \rangle$ is abelian, so that $G = \langle x_1, G' \rangle \cdots \langle x_t, G' \rangle$ is nilpotent and hence even abelian by

Lemma 2.4. This contradiction proves the statement for non-periodic soluble groups with the minimal condition on subgroups which are neither abelian nor self-normalizing.

In the general case, let $G$ be locally graded. Since $G$ is finitely generated, it contains a descending chain $G_1 > \cdots > G_n > \cdots$ of proper normal subgroups of finite index; in particular, each $G_i$ is not periodic. Since $G$ satisfies the minimal condition on subgroups which are neither abelian nor self-normalizing, there exists a positive integer $n$ such that $G_n$ is abelian. If $g \in G$, then $\langle g \rangle G_n$ is a soluble non-periodic group with the minimal condition on subgroups which are neither abelian nor self-normalizing and hence it is abelian by the first part of the proof. Therefore $G_n \leq Z(G)$ and so $G/Z(G)$ is finite; thus $G'$ is finite (see [18, Part 1, Theorem 4.12]). Hence $G \neq G'$ and Lemma 2.5 yields that either $G$ is abelian or $G/G'$ is periodic. Since $G$ is not periodic, it follows that $G$ is abelian. □

*Proof of Theorem A1.* Lemma 2.7 allows us to suppose that $G$ is periodic. Assume that $G$ is not a Chernikov group, and let $H$ be an arbitrary finite non-abelian subgroup of $G$. Let $g \in N_G(H)$, and consider the finite subgroup $K = \langle H, g \rangle$. Then $G$ contains an abelian subgroup $A$ which does not satisfy the minimal condition such that $A^K = A$ (see [22]). Hence the socle $S$ of $A$ is infinite and so it is possible to find finite $K$-invariant subgroups $A_1, A_2, \ldots$ of $S$ such that

$$\langle A_n : n \in \mathbb{N} \rangle = \operatorname*{Dr}_{n \in \mathbb{N}} A_n$$

(see [11, Lemma 3.8]). Clearly, the subgroup $\langle A_n : n \in \mathbb{N} \rangle \cap K$ is contained in a direct product of finitely many $A_n$'s, so that, by replacing $\langle A_n : n \in \mathbb{N} \rangle$ with a suitable subgroup of finite index, it can be assumed that

$$\langle A_n : n \in \mathbb{N} \rangle \cap K = \{1\}.$$

For any $n \in \mathbb{N}$, consider the $K$-invariant subgroup

$$B_n = \operatorname*{Dr}_{i \geq n} A_i.$$

Then $B_1 H > \cdots > B_n H > \cdots$ is an infinite descending chain of non-abelian subgroups of $G$ and so there exists a positive integer $k$ such that $N_G(B_k H) = B_k H$. Since $g \in N_G(B_k H)$, it follows that

$$g \in B_k H \cap K = H(B_k \cap K) = H.$$

Thus $H = N_G(H)$. Therefore all finite non-abelian subgroups of $G$ are self-normalizing and so $G$ is an $\mathcal{H}$-group by Lemma 2.1. □

Next we consider groups with the minimal condition on subgroups which do not have the property $\mathcal{H}$. The proof of Theorem A2 will be essentially the same as Theorem A1, but we need a corresponding version of Lemma 2.4.

**Lemma 2.8.** *Let $G$ be a nilpotent group satisfying the minimal conditions on non-$\mathcal{H}$-subgroups. Then $G$ is either abelian or a Chernikov group.*

*Proof.* Let $G_1 > \cdots > G_n > \cdots$ an infinite descending chain of non-abelian subgroups. Then there exists a positive integer $n$ such that $G_n$ is an $\mathcal{H}$-group. Since $G$ is nilpotent, it follows that $G_n$ is a minimal non-abelian group and hence it is finite. This contradiction proves that $G$ satisfies the minimal condition on non-abelian subgroups, thus $G$ is either abelian or a Chernikov group (see [2]). □

**Lemma 2.9.** *Let $G$ be a locally graded group satisfying the minimal conditions on non-$\mathcal{H}$-subgroups. If $G$ is not periodic, then $G$ is abelian.*

*Proof.* Since any locally graded non periodic $\mathcal{H}$-group is abelian, the proof can be obtained in a similar way as in the proofs of Lemmas 2.5, 2.6, and 2.7, replacing in the argument Lemma 2.4 with Lemma 2.8. □

*Proof of Theorem A2.* Lemma 2.9 allows us to suppose that $G$ is periodic. Assume that $G$ is not a Chernikov group. Consider any finite subgroup $H$ of $G$. In order to prove the statement, by Lemma 2.2, it is enough to prove that $H$ is an $\mathcal{H}$-group. The group $G$ contains an abelian subgroup $A$ which does not satisfy the minimal condition such that $A^H = A$ (see [22]). As in the proof of Theorem A1, $H$-invariant subgroups $B_1, B_2, \ldots$ of $A$ can be found such that $B_1H > \cdots > B_nH > \cdots$ is an infinite descending chain. Hence there exists a positive integer $k$ such that $B_kH$ is an $\mathcal{H}$-group. Thus $H$ is likewise an $\mathcal{H}$-group and the theorem is proved. □

Finally we conclude this section with a result concerning groups in which non-$\mathcal{H}$-subgroups fall into finitely many conjugacy classes.

**Proposition 2.10.** *Let $G$ be an infinite locally graded group with finitely many conjugacy classes of non-$\mathcal{H}$-subgroups. Then $G$ is an $\mathcal{H}$-group.*

*Proof.* Since any locally graded $\mathcal{H}$-group is either finite or soluble by Lemma 2.3 and hence also abelian-by-finite (see [5, Lemma 2.3]), the group $G$ is locally (abelian-by-finite) (see [15, Proposition 3.3]). Then the elements of any chain of non-$\mathcal{H}$-subgroups are pairwise not conjugate (see [1, Lemma 4.6.3]) and hence, since $G$ has finitely many conjugacy classes on non-$\mathcal{H}$-subgroups, any chain of non-$\mathcal{H}$-subgroups subgroups of $G$ is finite. In particular, $G$ satisfies both the minimal and the maximal condition on non-$\mathcal{H}$-subgroups. Assume that $G$ is not an $\mathcal{H}$-group, so that $G$ is periodic by Lemma 2.9. Let $H$ be any minimal element of the set of all non-$\mathcal{H}$-subgroups of $G$. Then all proper subgroups of $H$ are $\mathcal{H}$-subgroups and hence it follows from Lemma 2.2 that $H$ is finitely generated. Thus $H$ is finite. Since $G$ is infinite and locally finite, it follows that there are infinitely many elements $g_1, g_2, \ldots$ of $G$ which give a strictly infinite ascending chain of subgroups

$$H < \langle H, g_1\rangle < \cdots < \langle H, g_1, \ldots, g_n\rangle < \cdots .$$

Then $\langle H, g_1, ..., g_n\rangle$ is an $\mathcal{H}$-group for some $n \in \mathbb{N}$, a contradiction which proves the statement. □

## 3. Restrictions on subgroups of infinite rank.

**Lemma 3.1.** *Let $G$ be a group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. If $G'$ has finite rank, then $G$ is abelian.*

*Proof.* Let $X$ be any finitely generated subgroup of $G$. Then $XG'/G'$ is an abelian finitely generated group and hence, as $G'$ has finite rank, the subgroup $XG'$ has likewise finite rank. Thus $G/XG'$ has infinite rank and so there exists a proper subgroup of infinite rank $Y$ of $G$ containing $XG'$; clearly $Y$ is a normal subgroup and so it is abelian. Hence $X$ is itself abelian and so the lemma is proved. □

**Lemma 3.2.** *Let $G$ be a group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. If $G$ is abelian-by-finite, then $G$ is an $\mathcal{H}$-group.*

*Proof.* Let $A$ be an abelian normal subgroup of finite index of $G$. Then $A$ has infinite rank and there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of finitely generated $G$-invariant subgroups of $A$ such that

$$X = \langle X_n : n \in \mathbb{N}\rangle = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, X_n,$$

$r(X_n) < r(X_{n+1})$ for every $n \in \mathbb{N}$ (see [8, Lemma 6]). Let $H$ be any finitely generated non-abelian subgroup of $G$, and let $g \in N_G(H)$. Then $K = \langle H, g\rangle$ is abelian-by-finite and finitely generated, hence $K \cap X$ is likewise finitely generated and so, by replacing $X$ with a suitable subgroup (which is a direct product of infinitely many of the $X_n$'s), it can be assumed that $X \cap K = \{1\}$. Then $Y = \langle X_n : n \geq 2\rangle$ is a proper $G$-invariant subgroup of infinite rank of $X$, hence $HY$ is a proper non-abelian subgroup of infinite rank of $G$ and so $HY = N_G(HY)$. Since $g \in N_G(HY)$, it follows that

$$g \in HY \cap K = H(Y \cap K) = H.$$

Thus $H = N_G(H)$. Therefore every finitely generated non-abelian subgroup of $G$ is self-normalizing and hence $G$ is an $\mathcal{H}$-group by Lemma 2.1. □

As in many problems concerning groups of infinite rank, the existence of a proper normal subgroup of infinite rank plays a crucial role. We have in fact the following result on which the proof of Theorem B1 will depend.

**Proposition 3.3.** *Let $G$ be a locally graded group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. If $G$ contains a proper normal subgroup of infinite rank, then $G$ is an $\mathcal{H}$-group.*

*Proof.* By Lemma 3.2, it is enough to prove that $G$ is abelian-by-finite. Let $N$ be a proper normal subgroup of infinite rank. Then $N$ is abelian, and so $G/N$ is locally graded (see [17]). Clearly $G/N$ is an $\mathcal{H}$-group and hence Lemma 2.3 yields that either $G/N$ is finite or $G'' \leq N$. If $G/N$ is finite, $G$ is abelian-by-finite and so we have done. Assume that $G'' \leq N$, so $G \neq G'$. Lemma 3.1 allows us to suppose that $G'$ has infinite rank, so that $G'$ is abelian. Clearly it can be assumed that $G$ is not abelian, hence $G$ cannot be nilpotent (see [14, Theorem A]) and so it follows from Fitting's theorem that $G/G'$ cannot be the product of two proper subgroups. On the other hand, if $a, b \in G$ are such that $[a, b] \neq 1$, then $\langle a, b, G' \rangle$ is a non-abelian normal subgroup of infinite rank and hence $G = \langle a, b, G' \rangle$; therefore $G/G'$ is finitely generated. Hence $G/G'$ is a cyclic group of prime-power order and so $G$ is abelian-by-finite. The proof is completed. □

**Corollary 3.4.** *Let $G$ be a locally (soluble-by-finite) group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. If $G$ has no simple homomorphic images of infinite rank, then $G$ is an $\mathcal{H}$-group.*

*Proof.* Assume, by contradiction, that $G$ is not an $\mathcal{H}$-group. Then any proper normal subgroup of $G$ has finite rank by Proposition 3.3; in particular, it follows from Lemma 3.1 that $G = G'$. Let $N$ be any proper normal subgroup of $G$ and assume that $N$ is not abelian. Then $N$ has finite rank and $G/N$ is a locally (soluble-by-finite) group of infinite rank whose subgroups of infinite rank are self-normalizing; hence $G/N$ is soluble (see [6]) which is a contradiction because $G$ is perfect. Therefore any proper normal subgroup of $G$ is abelian. Since any proper normal subgroup of $G$ has finite rank and the product of finitely many normal subgroups of finite rank has likewise finite rank, it follows that the join $L$ of all proper normal subgroups of $G$ is abelian. Hence $G \neq L$ and so $L$ has finite rank. Thus $G/L$ has infinite rank and so, since clearly $G/L$ is simple, we obtain the contradiction which concludes the proof. □

*Proof of Theorem B1.* Any simple locally soluble group has prime order (see [18, Part 1, Corollary 1 on page 154]), hence the statement follows immediately from Corollary 3.4. □

**Corollary 3.5.** *Let $G$ be a group of infinite rank whose non-abelian subgroups of infinite rank are self-normalizing. If $G \neq G'$, then $G$ is an $\mathcal{H}$-group.*

*Proof.* If $G'$ has finite rank, then $G$ is abelian by Lemma 3.1. If $G'$ has infinite rank, then $G'$ is abelian and the result follows from Theorem B1. □

Finally we consider groups in which all proper subgroups of infinite rank are $\mathcal{H}$-groups.

**Lemma 3.6.** *Let $G$ be a locally graded group of infinite rank whose proper subgroups of infinite rank are $\mathcal{H}$-groups. If $G'$ has finite rank, then $G$ is an $\mathcal{H}$-group.*

*Proof.* Let $X$ be any finitely generated subgroup of $G$. Then $XG'/G'$ is abelian and finitely generated and so, since $G'$ has finite rank, $XG'$ has finite rank. Hence $G/XG'$ is an abelian group of infinite rank and so there exists a subgroup of infinite rank $Y$ such that $XG' < Y < G$. Thus $Y$ is an $\mathcal{H}$-group and so $X$ is likewise an $\mathcal{H}$-group. Therefore $G$ itself is an $\mathcal{H}$-group by Lemma 2.2. □

**Lemma 3.7.** *Let $G$ be a non-periodic locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are $\mathcal{H}$-groups. Then $G$ is abelian.*

*Proof.* Assume, by contradiction, that $G$ is not abelian. Hence $G$ is not an $\mathcal{H}$-group, so that Lemma 3.6 yields that $G'$ is a subgroup of infinite rank and hence $G/G'$ is finitely generated (see [10, Lemma 2.8] and Lemma 2.2). Since every soluble $\mathcal{H}$-group is either abelian or periodic, the derived subgroup $G'$ is periodic (see [10, Theorem A]). Hence $G/G'$ is not periodic and so we may write $G/G' = \langle x_1G', \ldots, x_tG'\rangle$ with any $x_iG'$ of infinite order. If $i \in \{1, \ldots, t\}$, $p$ and $q$ are distinct primes, the subgroups $\langle x_i^p, G'\rangle$ and $\langle x_i^q, G'\rangle$ are proper normal non-periodic subgroups of $G$ of infinite rank, so that they are both $\mathcal{H}$-groups and hence even abelian. Thus

$$G = \langle x_1^p, G'\rangle \langle x_1^q, G'\rangle \cdots \langle x_t^p, G'\rangle \langle x_t^q, G'\rangle$$

is nilpotent. Therefore, since $G/G'$ is finitely generated, $G$ itself is finitely generated (see [18, Part 1, Theorem 2.26]) and hence $G$ has finite rank. A contradiction which concludes the proof. □

*Proof of Theorem B2.* Lemma 2.3 yields that all subgroups of infinite rank of $G$ are soluble, and so $G$ itself is soluble (see [9, Theorem 9]). Moreover, Lemma 3.7 allows us to assume that $G$ is periodic. Let $H$ be any finite subgroup of $G$, $g \in N_G(H) \setminus H$, and $K = \langle H, g\rangle$. Then $K$ is finite and there exist abelian $K$-invariant subgroups of infinite rank $A_1$ and $A_2$ such that $A_2 < A_1$ and $K \cap A_1 = \{1\}$ (see [16]). Then $A_2K$ is a proper subgroup of infinite rank of $G$, so that it is an $\mathcal{H}$-group. Since $g \in N_{A_2K}(H) \setminus H$, it follows that $H$ is abelian. Therefore all finite non-abelian subgroups of $G$ are self-normalizing and hence $G$ in an $\mathcal{H}$-group by Lemma 2.1. □

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