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The Cremona problem in dimension 2

Wolfgang Bartenwerfer

Abstract. The Cremona conjecture, also called Jacobi problem, claims that a polynomial morphism $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ is invertible as a polynomial morphism if its Jacobian is constant and not zero. In this paper, we show that the conjecture is true for $n = 2$. The starting point of our proof is an important result of Shreeram Abhyankar. Then we use a computation in rigid geometry to achieve the result.

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Introduction. A polynomial map $(f, g) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is given by two polynomials f and g in two variables X and Y with complex scalars. We write f and g as sums of their homogenous components

$$f = f_m + \cdots + f_{m'} \quad \text{and} \quad g = g_n + \cdots + g_{n'} \,,$$

where f_μ respectively g_ν are linear combinations of the terms of total degree μ respectively ν. The forms f_m respectively g_n of highest degree are called the *leading forms*.

It was shown by Abhyankar that, for a given counterexample (f, g) to the Jacobian conjecture in dimension 2, one can assume that, after a suitable transformation of variables, the leading forms of f and g have the following shape

$$f_m = X^{m_1} Y^{m_2} \quad \text{and} \quad g_n = X^{n_1} Y^{n_2} \,;$$

cf. [1, Theorem 8.7] or [5, Corollary 10.2.22]. In this paper, we will show that this assumption leads to a contradiction. Thus the Jacobian conjecture is true in dimension 2.

1. Division algorithm. In this section, let $\mathbb{K}$ be an algebraically closed field of characteristic 0. The $\mathbb{K}$-algebra $L := \mathbb{K}[X,Y]_{XY}$ consists of all Laurent polynomials in two variables. It carries a canonical graduation of type $\mathbb{Z}$ given by the total degree function. Let H_n be the subspace of all homogenous Laurent polynomials of degree n including the zero polynomial. For $f = \sum_{\nu\in\mathbb{Z}} f_\nu \in L$ and $f \neq 0$, we set

$$\deg f := \sup\{\nu \in \mathbb{Z};\ f_\nu \neq 0\}\,.$$

On L, we have a filtration $(L_n;\ n \in \mathbb{N})$ where

$$L_n := \{f \in L;\ \deg f \leq -n\} = \bigoplus_{\nu \leq -n} H_\nu\,.$$

The completion with respect to this filtration is denoted by

$$A := \widehat{L} = \varprojlim L/L_n\,;$$

cf. [2, Chap. 3]. It consists of all series

$$\sum_{\mu=-\infty}^{m} f_\mu \quad \text{with} \quad f_\mu \in H_\mu \quad \text{for some} \quad m \in \mathbb{Z}\,;$$

cf. [5, Prop. 10.2.8]. The degree, the multiplication, and the filtration on A are declared as on L. The $\mathbb{K}$-algebra A represents the formal functions on a neighborhood of the twice punctured projective line at infinity which behave like meromorphic functions there. The algebra A has similar properties as the algebra $R^{\sim}$ defined in [5, Prop. 10.2.8]. In this section, we consider these functions without conditions of convergence; in Section 2, we will focus on that by means of rigid geometry.

Lemma 1.1. *An element $g \in A$ is a unit in A if and only if g is of the form*

$$g = c \cdot X^{n_1} Y^{n_2} \cdot (1 - v)$$

where $c \in \mathbb{K}^\times$, $n_i \in \mathbb{Z}$, $\deg(v) < 0$. Such a representation is unique. Such a unit g admits a k-th root for $0 \neq k \in \mathbb{Z}$ if and only if k divides both numbers n_1 and n_2.

Proof. The proof can be left to the reader. For example, we have for the inverse

$$g^{-1} = c^{-1} \cdot X^{-n_1} Y^{-n_2} \cdot \left(\sum_{\nu=0}^{\infty} v^\nu\right).$$

For $c = 1$ and $k \in \mathbb{N}$, the k-th root is given by

$$g^{1/k} := X^{n_1/k} Y^{n_2/k} \cdot \left(\sum_{\nu=0}^{\infty} \binom{1/k}{\nu} (-v)^\nu\right)$$

if k divides n_1 and n_2. We consider this as the *canonical k-th root of g.* □

Corollary 1.2. *Let* $g = X^{n_1}Y^{n_2} \cdot (1+v) \in A$ *with* $\deg(v) < 0$ *and* $r \in \mathbb{Q}$ *be such that* $r \cdot n_1$ *and* $r \cdot n_2$ *belong to* $\mathbb{Z}$*, then*

$$g^r := X^{r\cdot n_1}Y^{r\cdot n_2} \cdot \left(\sum_{\nu=0}^{\infty} \binom{r}{\nu} v^\nu \right)$$

is well defined.

In the following, we denote by $\partial/\partial X$ respectively $\partial/\partial Y$ the partial derivatives of Laurent series. Obviously they give rise to $\mathbb{K}$-derivations on the $\mathbb{K}$-algebra A. They satisfy the usual rules for $\mathbb{K}$-derivations. Since the field $\mathbb{K}$ has characteristic 0, we have $\ker(\partial/\partial X\, , \partial/\partial Y) = \mathbb{K}$.

Definition 1.3. A couple (f,g) of elements of A is called a *Jacobian couple* if its Jacobian

$$\det \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{pmatrix} = d \in \mathbb{K}^\times$$

is constant and not 0.

As for polynomials in two variables, we also have the notion of a leading form for a Laurent series in A.

Proposition 1.4. *Consider a Jacobian couple* (f,g) *as introduced above, where* $m := \deg f$ *and* $n := \deg g$*. Assume that the leading form of* g *has the shape* $g_n = X^{n_1}Y^{n_2}$*.*

(a) Then we always have $m + n \geq 2$*.*

(b) If $m+n > 2$*, then* $f_m^n \cdot g_n^{-m}$ *is constant.*

Proof. (a) The homogenous components of the Jacobian of degree $m+n > 2$ vanish and for $m+n=2$ it is given by

$$\frac{\partial f_m}{\partial X} \cdot \frac{\partial g_n}{\partial Y} - \frac{\partial f_m}{\partial Y} \cdot \frac{\partial g_n}{\partial X}. \tag{1}$$

Since that the Jacobian is constant and the degree of a constant is 0, we see that $m+n \geq 2$.

(b) If $m+n > 2$, then the expression (1) is zero. We compute

$$\frac{\partial}{\partial X}\left(\frac{f_m^n}{g_n^m}\right) = \frac{f_m^{n-1} \cdot g_n^{m-1}}{g_n^{2m}} \cdot \left(\frac{\partial f_m}{\partial X} \cdot n \cdot g_n - \frac{\partial g_n}{\partial X} \cdot m \cdot f_m \right). \tag{2}$$

For homogenous polynomials, we have Euler's differential equation

$$n \cdot g_n = X\frac{\partial g_n}{\partial X} + Y\frac{\partial g_n}{\partial Y} \text{ and } m \cdot f_m = X\frac{\partial f_m}{\partial X} + Y\frac{\partial f_m}{\partial Y}.$$

Then the term in parentheses of equation (2) is equal to

$$\begin{aligned} &\frac{\partial f_m}{\partial X} \cdot \left(X\frac{\partial g_n}{\partial X} + Y\frac{\partial g_n}{\partial Y} \right) - \frac{\partial g_n}{\partial X} \cdot \left(X\frac{\partial f_m}{\partial X} + Y\frac{\partial f_m}{\partial Y} \right) \\ &= Y\left(\frac{\partial f_m}{\partial X} \cdot \frac{\partial g_n}{\partial Y} - \frac{\partial f_m}{\partial Y} \cdot \frac{\partial g_n}{\partial X} \right). \end{aligned}$$

This vanishes due to (1) since $m+n>2$. Thus we see that the left hand term of equation (2) is equal to 0. Analogously, one shows

$$\frac{\partial}{\partial Y}\left(\frac{f_m^n}{g_n^m}\right) = X\left(\frac{\partial f_m}{\partial Y}\cdot\frac{\partial g_n}{\partial X} - \frac{\partial f_m}{\partial X}\cdot\frac{\partial g_n}{\partial Y}\right) = 0\,. \tag{3}$$

So the total differential of $f_m^n \cdot g_n^{-m}$ vanishes. Thus, the function $f_m^n \cdot g_n^{-m}$ is constant. □

Corollary 1.5. *Let (f,g) be a Jacobian couple as in 1.4 satisfying $g_n = X^{n_1}Y^{n_2}$ with integers $n_1>0$, $n_2>0$. If $m+n>2$, then we have $f_m = c\cdot X^{m_1}Y^{m_2}$ with a constant $c\in\mathbb{K}^\times$. Moreover it holds*

$$\frac{m}{n} = \frac{m_1}{n_1} = \frac{m_2}{n_2}\,.$$

Therefore the following expression is well defined

$$g^{m/n} = X^{m_1}Y^{m_2}\left(1+\sum_{\nu=-\infty}^{n-1} g_\nu g_n^{-1}\right)^{m/n}\,.$$

In particular, we have $\left(g^{m/n}\right)^n = g^m$. Furthermore $|m_1-m_2|\neq|n_1-n_2|$ if $|m|\neq|n|$.

Proof. The first assertion follows from 1.4(b). For the second assertion, we use

$$n_1\cdot\frac{m}{n} = n_1\cdot\frac{m_1}{n_1} = m_1 \ \text{ and } \ n_2\cdot\frac{m}{n} = n_2\cdot\frac{m_2}{n_2} = m_2\,.$$

So we see $|m_1-m_2|\neq|n_1-n_2|$ if $|m|\neq|n|$. The formula for $g^{m/n}$ follows from 1.2. □

Now we turn to the division algorithm.

Proposition 1.6. *Let (f,g) be as in 1.5. Then there exists a rational number $r\in\mathbb{Q}$ such that $r\cdot n_1\in\mathbb{Z}$ and $r\cdot n_2\in\mathbb{Z}$ are integers, and a constant $c\in\mathbb{K}^\times$ with $\deg(f-c\cdot g^r)<m$. The couple (c,r) is uniquely determined; actually we have $r=m/n$ and $f_m = c\cdot X^{m_1}Y^{m_2}$.*

If, in addition, $\deg(f-c\cdot g^r)=2-n$, then $n_1\neq n_2$ and the leading form of $h:=f-c\cdot g^r$ is given by

$$h_{2-n} = \sum_{i+j=2-n} c_{i,j}X^iY^j\,,$$

where $c_{1-n_1,1-n_2}\neq 0$. Furthermore there is at most one index (i,j) with $(i,j)\neq(1-n_1,1-n_2)$ with $c_{i,j}\neq 0$. For this index, we have

$$\frac{i}{n_1} = \frac{j}{n_2} = \frac{2-n}{n}\,.$$

Proof. The first assertion follows from 1.5.

For the supplement, set $m' := \deg(f-c\cdot g^r) = 2-n$. The Jacobian d of the couple (h,g) is equal to the Jacobian of (f,g). Then we have

$$n_2\sum_{i+j=2-n} i\cdot c_{i,j}X^{i+n_1-1}Y^{j+n_2-1} - n_1\sum_{i+j=2-n} j\cdot c_{i,j}X^{i+n_1-1}Y^{j+n_2-1} = d.$$

For $(i,j) = (1-n_1, 1-n_2)$, it follows that

$$n_2(1-n_1)c_{1-n_1,1-n_2} - n_1(1-n_2)c_{1-n_1,1-n_2} = d\,.$$

Thus, we see $n_1 \neq n_2$ and $c_{1-n_1,1-n_2} \neq 0$. For all the other indices, we have

$$n_2 \cdot i \cdot c_{i,j} - n_1 \cdot j \cdot c_{i,j} = 0\,.$$

If $c_{i,j} \neq 0$, then

$$\frac{i}{n_1} = \frac{j}{n_2}\,.$$

Moreover we know $i + j = 2 - n$ and $n = n_1 + n_2$. This yields

$$\frac{i}{n_1} = \frac{j}{n_2} = \frac{2-n-i}{n_2}$$

and $i \cdot n_2 = (2-n-i)\cdot n_1$ and hence $i \cdot n = i \cdot (n_2 + n_1) = (2-n)\cdot n_1$ □

Corollary 1.7. *Keep the assumptions of 1.6. Then we have $n_1 \neq n_2$ and there exist a natural number $s \in \mathbb{N}$, constants $c_\sigma \in \mathbb{K}$, and rational numbers $r_\sigma \in \mathbb{Q}$ satisfying*

$$r_1 > r_2 > \cdots > r_s = \frac{2-n}{n}$$

such that

$$G = \sum_{\sigma=1}^{s} c_\sigma \cdot g^{r_\sigma}$$

belongs to A and the leading term of $(f-G)$ fulfills

$$(f-G)_{2-n} = c_{1-n_1,1-n_2} X^{1-n_1} Y^{1-n_2}\,.$$

Proof. Apply 1.6 inductively. Note that $\deg(f-G)$ is always an integer and that $(f-G\,,\,g)$ is a Jacobian couple. Therefore the procedure stops after finitely many steps until we arrive at the situation $\deg(f-G) = 2-n$ since there is at each step at most one term which has to be cancelled. In the case $\deg(f-G) = 2-n$, we apply the additional claim of 1.6. Then we obtain for the leading form

$$(f-G)_{2-n} = c_{1-n_1,1-n_2} X^{1-n_1} Y^{1-n_2} + c_{i,j} X^i Y^j\,,$$

where $i/n_1 = j/n_2 = (2-n)/n$ as follows from 1.6. Then we subtract

$$c_{i,j} \cdot g^{(2-n)/n} := c_{i,j} \cdot \left((X^{n_1} Y^{n_2}) \cdot \left(1 + \sum_{\nu=-\infty}^{n-1} g_\nu g_n^{-1} \right) \right)^{(2-n)/n}$$

which cancels the term $c_{i,j} X^i Y^j$. Thus the assertion is proved. □

2. Convergence of the division algorithm. In the following, we make use of some elementary results in rigid geometry; for a general reference, we cite [3] or [4]. We consider an algebraically closed field $\mathbb{K}$ which is complete with respect to a non-Archimedean valuation and which has residue characteristic 0. We assume that $\mathbb{K}$ contains the field K of characteristic 0 as a subfield, where K is the algebraically closed field over which the Jacobian problem is posed. Such a field can be constructed in the following way: Consider the field of fractions K' of $K[[T]]$ and define $\mathbb{K}$ as the topological algebraic closure of K'. The canonical valuation on $K[[T]]$ extends to a valuation of $\mathbb{K}$. Note that we write valuations in the multiplicative way. So we obtain on $\mathbb{K}^2$ a canonical structure of rigid space in the sense of Tate. On each subset $V \subset \mathbb{K}^2$, we have the spectral norm of functions f

$$|f|_V := \sup \{ |f(x)| \, ; \, x \in V \} .$$

In particular, we have the notion of an affinoid domain $V \subset \mathbb{K}^2$; for example, bounded domains described by finitely many inequalities

$$V := \{x \in \mathbb{K}^2 \, ; \, 1 \leq |f_i(x)| \, , \, |g_j(x)| \leq 1 \text{ for } i = 1, \dots, r \, , \, j = 1, \dots, s\}$$

with polynomials $f_i, g_j \in \mathbb{K}[X, Y]$ are affinoid domains. Affinoid functions on such a domain are functions which can be uniformly approximated by rational functions without poles in V. Such functions are bounded and take their maximal absolute value in V. Thus the spectral norm $|f|_V$ is always a non-negative real number which actually lies in the value group of $\mathbb{K}$. We are mainly interested in domains of the following shape

$$W_{\varepsilon,\rho} := \{(x, y) \in \mathbb{K}^2 \, ; \, \varepsilon \leq |x| \leq \rho \, , \, \varepsilon \leq |y| \leq \rho\}$$

for values $\varepsilon \leq \rho$ belonging to the value group of $\mathbb{K}$. The affinoid functions on $W_{\varepsilon,\rho}$ are exactly the Laurent series which converge on $W_{\varepsilon,\rho}$. Of particular interest will be the following domains

$$U_{\varepsilon,\rho} := \{(x, y) \in \mathbb{K}^2 \, ; \, \varepsilon \leq |x| = |y| \leq \rho\} .$$

These subsets are also affinoid and they are open subsets in the rigid analytic sense.

Lemma 2.1. *Keep the above notations. Let ε, ρ be elements of the value group $|\mathbb{K}^\times|$ with $\rho \geq \varepsilon$.*

(a) If v is an affinoid function on $U := U_{\varepsilon,\rho}$ with $|v|_U < 1$, then the series

$$h := \sum_{\nu=0}^{\infty} \binom{r}{\nu} v^\nu ,$$

for any $r \in \mathbb{Q}$, converges uniformly on $U_{\varepsilon,\rho}$ and gives rise to an affinoid function there. In particular, $(1+v)^r$ is well-defined and affinoid on $U_{\varepsilon,\rho}$.

(b) Let $g = g_n + \cdots + g_0 \in \mathbb{K}[X, Y]$ be a polynomial with homogenous components g_ν of degree ν. Assume $g_n = X^{n_1} Y^{n_2}$. Then there exists an ε in $|\mathbb{K}^\times|$ such that $|g_\nu(x, y)| < |g_n(x, y)|$ for all $(x, y) \in U_{\varepsilon,\rho}$ and all $\nu = 0, \dots, n-1$ and $\rho \geq \varepsilon$. Especially, for any $r \in \mathbb{Q}$ with $n_1 \cdot r \in \mathbb{Z}$ and

$n_2 \cdot r \in \mathbb{Z}$, the function g^r is well-defined and affinoid on $U_{\varepsilon,\rho}$ for all ρ with $\rho \geq \varepsilon$.

Proof. (a) Since the residue field of $\mathbb{K}$ has characteristic 0, the absolute value $|\binom{r}{\nu}| = 1$ is equal to 1. Therefore the series converges on $U_{\varepsilon,\rho}$ for all $\rho \geq \varepsilon$.
(b) For all monomials $X^{\nu_1}Y^{\nu_2}$ of g_ν with $\nu < n$, we have $|x^{\nu_1}y^{\nu_2}| \leq |x^{n_1}y^{n_2}|$ if $(x,y) \in U_{\varepsilon,\rho}$ and $\varepsilon > 1$. If we now choose $\varepsilon \geq |c_{\nu_1,\nu_2}|$ for all the coefficients c_{ν_1,ν_2} of g_ν for all $\nu = 0,\ldots,n-1$, then the assertion follows by (a).

For the last assertion, note that $(X^{n_1}Y^{n_2})^r = X^{m_1}Y^{m_2}$, where $n_1 \cdot r = m_1$ and $n_2 \cdot r = m_2$ with $m_1, m_2 \in \mathbb{Z}$. Then it follows from (a). □

Proposition 2.2. *Let (f,g) be a Jacobian couple of polynomials with homogenous decompositions*

$$f = X^{m_1}Y^{m_2} + \sum_{\mu=0}^{m-1} f_\mu \quad \textit{and} \quad g = X^{n_1}Y^{n_2} + \sum_{\nu=0}^{n-1} g_\nu$$

in $\mathbb{K}[X,Y]$ with $n_1 > 0$, $n_2 > 0$, where $m := m_1 + m_2$ and $n := n_1 + n_2$.

If we apply the division algorithm of 1.6 and 1.7 to f and set $v := \sum_{\nu=0}^{n-1} g_\nu g_n^{-1}$, then there exists an $\varepsilon \in |\mathbb{K}^\times|$ such that the formal series G defined in 1.7 converges on every affinoid domain $U_{\varepsilon,\rho}$ for all $\rho \geq \varepsilon$ and gives rise to an affinoid function there.

After a possible enlarging of ε, the function $(f-G)$ has the form

$$(f-G)_{U_{\varepsilon,\rho}} = eX^{1-n_1}Y^{1-n_2}(1+u)$$

with $e \in \mathbb{K}^\times$, $\deg(u) < 0$, is affinoid on each $U_{\varepsilon,\rho}$, and satisfies $|u|_{U_{\varepsilon,\rho}} < 1$.

Proof. The claim follows from Lemma 2.1. □

In the following, we will compute the cardinality of the fibers of $(f-G, g)$ on $U_{\varepsilon,\rho}$.

Proposition 2.3. *Let (f,g) be a Jacobian couple as in 2.2. Thus we have the map*

$$(f-G,g) := \left(eX^{1-n_1}Y^{1-n_2}(1+u),\, X^{n_1}Y^{n_2}(1+v)\right) : U_{\varepsilon,\rho} \longrightarrow \mathbb{K}^2 .$$

Set $k := \gcd(n_1,n_2)$. Then, for any domain $V := U_{\varepsilon',\rho'} \subset U_{\varepsilon,\rho}$, the fibers of the morphism $(f-G\,,\,g^{1/k})\,|_V$ consist of exactly $|n_1-n_2|/k$ points. The fibers of $(f\,,\,g)\,|_V$ consist of exactly $|n_1-n_2|$ points.

Proof. We abbreviate

$$\Psi := (\psi_1,\psi_2) := \left(f-G\;,\; g^{1/k}\right)|_V .$$

Since $|u|_V < 1$ and $|v|_V < 1$, the map Ψ gives rise to a map

$$|\Psi| := (|\psi_1|,|\psi_2|) : |V| := \{(|x|,|y|) \in |\mathbb{K}^\times|^2 \,;\, (x,y) \in V\} \longrightarrow |\mathbb{K}^\times| \times |\mathbb{K}^\times|,$$
$$(|x|,|y|) \longmapsto \left(|e|\cdot|x|^{1-n_1}|y|^{1-n_2},\, |x|^{n_1/k}|y|^{n_2/k}\right).$$

Due to the construction, all numbers $k \cdot r_\sigma$ are integers by 1.6 since $r_\sigma \cdot n_1 \in \mathbb{Z}$ and $r_\sigma \cdot n_2 \in \mathbb{Z}$. Obviously, this map is injective. So, for every $(x_0,y_0) \in V$, the map Ψ induces a mapping

$$\Psi_{(x_0,y_0)} : \{(x,y) \in V\,;\, |x| = |x_0|\,,\, |y| = |y_0|\} \\ \longrightarrow \big\{(x,y) \in \mathbb{K}^2\,;\, |x| = |\psi_1(x_0)|\,,\, |y| = |\psi_2(y_0)|\big\}\,.$$

Next we will compute the cardinality of the fibers of $\Psi_{(x_0,y_0)}$. After adjusting the radii $|x_0|$ and $|y_0|$ to 1 and the constant e to 1, we are concerned with a morphism of type $\Phi : W \longrightarrow W$ with

$$W := \{(x,y) \in \mathbb{K} \times \mathbb{K}\,;\, |x| = 1, |y| = 1\}\,,$$

sending $(x,y) \in W$ to $(x^{1-n_1} \cdot y^{1-n_2} \cdot (1+u(x,y))\,,\, x^{n_1/k} \cdot y^{n_2/k} \cdot (1+v(x,y))^{1/k}$. The degree of this map can be calculated via its reduction. The algebra of affinoid functions on W which are bounded by 1 is given by $\mathbb{K}^\circ\langle X, Y, X^{-1}, Y^{-1}\rangle$, where $\mathbb{K}^\circ$ denotes the valuation ring of $\mathbb{K}$. Denote by $\tilde{\mathbb{K}}$ the residue field of the valued field $\mathbb{K}$ and by $\tilde{\mathbb{K}}^\times$ its multiplicative group. The reduction of W is given by the spectrum of the $\tilde{\mathbb{K}}$-algebra $\tilde{W} = \tilde{\mathbb{K}}[\tilde{x}\,,\, \tilde{x}^{-1}\,,\, \tilde{y}\,,\, \tilde{y}^{-1}]$, where $\tilde{x}$ resp. $\tilde{y}$ is the reduction of x resp. y. Since $|u| < 1$ and $|v| < 1$, the map of the reductions coincides with the mapping

$$\tilde{\Phi} : \tilde{\mathbb{K}}^\times \times \tilde{\mathbb{K}}^\times \longrightarrow \tilde{\mathbb{K}}^\times \times \tilde{\mathbb{K}}^\times\,,\; (\tilde{x}, \tilde{y}) \longmapsto \left(\tilde{x}^{1-n_1}\tilde{y}^{1-n_2}\,,\, \tilde{x}^{n_1/k}\tilde{y}^{n_2/k}\right)\,.$$

The degree of this map is $|n_1 - n_2|/k$ as claimed; cf. Lemma 2.4 below. This degree is the degree of Φ since a finite generating system of the reduced module via $\tilde{\Phi}$ lifts to a generating system via Φ due to the lemma of Nakayama [3, 1.2.4/6]. Linear independence is also preserved as one easily checks.

It remains to compute the cardinality of the fibers of $(f\,,\, g)\,|_V$. Recall from 1.7 that $G(x,y)$ is a function of $g(x,y)^{1/k}$. Therefore, we have that the fiber of $(f - G, g^{1/k})|_V$ of a point $(x,y) \in V$ with image $(z_1, z_2) := (f - G, g^{1/k})(x,y)$ coincides with the fiber of $(f, g^{1/k})|_V)$ over the point $(z_1 + c_1, z_2)$ where $c_1 := G(x,y)$ depends only on $z_2 = g^{1/k}(x,y)$. Thus, we see that the cardinality of the fiber of Ψ coincides with that of $(f, g^{1/k})|_V$. Therefore the cardinality of the fiber of $(f,g)|_V$ is equal to $|n_1 - n_2|$ since Φ is finite and étale. □

Lemma 2.4. *Let k be a field and let $m_1, m_2 \in \mathbb{Z}$ be non-zero and $m_1 \neq m_2$. Let x, y be variables and $r \in \mathbb{Z}$. Then the extension of k-algebras*

$$k[x^{1-rm_1}y^{1-rm_2}\,, x^{rm_1-1}y^{rm_2-1}\,,\, x^{m_1}y^{m_2}\,,\, x^{-m_1}y^{-m_2}] \longrightarrow k[x, x^{-1}, y, y^{-1}]$$

is finite flat of degree $|m_1 - m_2|$.

Proof. Obviously we have

$$\begin{aligned} &k[x^{1-rm_1}y^{1-rm_2}\,, x^{rm_1-1}y^{rm_2-1}\,,\, x^{m_1}y^{m_2}\,,\, x^{-m_1}y^{-m_2}] \\ &= k[xy\,, x^{-1}y^{-1}\,,\, x^{m_1}y^{m_2}\,,\, x^{-m_1}y^{-m_2}] \\ &= k[xy\,, x^{-1}y^{-1}\,,\, y^{m_2-m_1}\,,\, y^{m_1-m_2}]\,. \end{aligned}$$

Moreover, we have that the extension

$$k[xy\,, x^{-1}y^{-1}\,,\, y^{m_2-m_1}\,,\, y^{m_1-m_2}] \longrightarrow k[xy\,, x^{-1}y^{-1}\,,\, y\,,\, y^{-1}] = k[x\,, x^{-1}\,,\, y\,,\, y^{-1}]$$

is finite flat of degree $|m_1 - m_2|$. □

3. The contradiction. Now we have all the preparations to deduce the main result of our article.

Proposition 3.1. *There does not exist a Jacobian couple* (f,g) *of polynomials* $f,g \in \mathbb{C}[X,Y]$ *with homogenous decompositions*

$$f = \sum_{\mu=0}^{m} f_\mu \in \mathbb{K}[X,Y] \ \text{ and } \ g = \sum_{\nu=0}^{n} f_\nu \in \mathbb{K}[X,Y]$$

where $f_m = X^{m_1}Y^{m_2}$ *and* $g_n = X^{n_1}Y^{n_2}$ *with* $m_1m_2 \neq 0\,,\, n_1n_2 \neq 0$.

Proof. First of all we perform a field extension $\mathbb{C} \hookrightarrow \mathbb{K}$ as introduced in Section 2. It is clear that it suffices to show the assertion for the field $\mathbb{K}$. Assume that (f,g) is such a couple in the $\mathbb{K}$-algebra A.

Assume first $m = n$. If $m+n = 2$, then we would have $m = m_1 = 1$ and $n = n_2 = 1$ without loss of generality. Obviously, that case cannot occur as a counterexample. If $m+n > 2$, then we have $m_1 = n_1$ and $m_2 = n_2$ due to 1.5. So we can replace f by $h := f - g$. Due to 2.2, the leading form of the polynomial h also has the shape $h_r = a \cdot X^{r_1}Y^{r_2}$ and $r := r_1 + r_2 < n$. Thus we can assume $m \neq n$. Moreover, we have $m_1 \neq n_1$ and $m_2 \neq n_2$ and $|m_1 - m_2| \neq |n_1 - n_2|$ due to 1.5. Thus we see that we can start just from the beginning with $m \neq n$.

Now we apply Proposition 2.3. So there exists a function in A

$$G := \sum_{\sigma=1}^{s} c_\sigma \cdot g^{r_\sigma}$$

as in 1.5 such that $h := (f - G)$ is of degree $(2-n)$ and h has a leading form of the shape

$$h_{2-n} := c_{1-n_1,1-n_2} X^{1-n_1}Y^{1-n_2}\,.$$

Note that all the monomials of h have negative degree. Furthermore, there exists a domain $U := U_{\varepsilon,\rho}$ such that for any subdomain $V := U_{\varepsilon',\rho'} \subset U$ the restriction $(h,g)|_V$ has fibers with cardinality $n' = |n_1 - n_2|$, which coincides with the degree of the map $(f,g)|_V$; cf. 2.3.

If we interchange f und g, then, after a possible shrinking of U, the degree of $(f,g)|_V$ would be $m' = |m_1 - m_2|$. This has to be equal to n', but we have $m' \neq n'$ due to 1.5. Contradiction! □

Summarizing the arguments we obtain the main result. Indeed, by Abhyankar's result, a counterexample would give rise to a Jacobian couple of the given shape, which cannot exist due to Proposition 3.1.

Theorem 3.2. *Let* $(f,g) : \mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial morphism. If the Jacobian of* (f,g) *is constant and unequal zero, then* (f,g) *is an isomorphism.*

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References

[1] Abhyankar, S.S.: Some thoughts on the Jacobian conjecture. I. J. Algebra **319**(2), 493–548 (2008)

[2] Bourbaki, N.: Commutative Algebra. Chapters 1–7. Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer, Berlin (1998)

[3] Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry. Grundlehren der mathematischen Wissenschaften, 261. Springer, Berlin (1984)

[4] Bosch, S.: Lectures on Formal and Rigid Geometry. Lecture Notes in Mathematics, 2105. Springer, Cham (2014)

[5] van den Essen, A.: Polynomial Automorphisms and the Jacobian Conjecture. Progress in Mathematics, 190. Birkhäuser, Basel (2000)

Wolfgang Bartenwerfer
Gropiusweg 3
44801 Bochum
Germany
e-mail: w.bartenwerfer@gmx.de

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