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There are no topologically transitive operators in the noncommutative Schwartz space

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Abstract. The aim of this note is to prove that there are no topologically transitive operators in the noncommutative Schwartz space.

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1. Introduction. Let s and s' be the spaces of rapidly decreasing and slowly increasing sequences, respectively, equipped with their natural locally convex topologies. The so-called *noncommutative Schwartz space* is the space $\mathcal{S} := L(s', s)$ of all bounded linear operators acting from s' into s. This space becomes a Fréchet algebra in a natural way: if T_1 and T_2 are in $\mathcal{S}$, then the product T_1T_2 is defined by the formula $T_1T_2 = T_1 \circ \iota \circ T_2$, where ι is the natural embedding of s into s'. In fact, $\mathcal{S}$ embeds algebraically into the C^*-algebra $\mathcal{B}(\ell_2)$ of all bounded and linear operators on the Hilbert space ℓ_2.

The noncommutative Schwartz space is isomorphic (as a Fréchet *-algebra) to a number of other natural objects of analysis, e.g., $\mathcal{S} \simeq \mathcal{S}(\mathbb{R}^2)$—the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$ equipped with the Volterra multiplication $(f \cdot g)(x, y) := \int_{\mathbb{R}} f(x, z)g(z, y)dz$ and involution $f^*(x, y) := \overline{f(y, x)}$. It plays an important role, e.g., in K-theory—see [4,12], cyclic cohomology for crossed products—see [9,14], noncommutative geometry—see [3], operator spaces—see [7,8]. Another motivation comes from quantum mechanics where $\mathcal{S}$ is called the *space of physical states* and its dual is the so-called *space of observables*—see [6] for details.

Since $\mathcal{S}$ is a Fréchet algebra of operators, it is natural to ask about the dynamical properties of the elements of $\mathcal{S}$. Recall that an operator $T : s' \to s$ is *topologically transitive* if for every two non-empty and open sets $U \subset s'$,

$V \subset s$, there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$ and *hypercyclic* if there exists $x \in s'$ such that the set $\{T^n x : n \in \mathbb{N}\}$ is dense in s. It is clear that hypercyclicity of T implies that it is topologically transitive. Formally it could happen that the latter is a weaker property since s' is not a metric space.

The main goal of this note is to show that there are no topologically transitive operators in $\mathcal{S}$. The main difficulty of the paper is to understand the spectral properties of operators from $\mathcal{S}$, those are investigated in Sect. 2.

We refer the reader to [1,10,11] for unexplained details from linear dynamics and functional analysis, respectively.

2. Notation and terminology. Recall that

$$s = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} \colon \; |\xi|_t^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2t} < +\infty \text{ for all } t \geqslant 0 \right\}$$

and its topological dual

$$s' = \left\{ \eta = (\eta_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} \colon \; (|\eta|_t')^2 := \sum_{j=1}^{+\infty} |\eta_j|^2 j^{-2t} < +\infty \text{ for some } t \geqslant 0 \right\}$$

are the so-called spaces of *rapidly decreasing* and *slowly increasing sequences*, respectively.

Furthermore we consider the space $\mathcal{S} := L(s', s)$ of all linear and continuous operators from s' into s, equipped with the topology of uniform convergence on bounded sets. Consequently, the topology of $\mathcal{S}$ is given by the scale $(\|\cdot\|_t)_{t\geqslant 0}$ of norms, defined as

$$\|T\|_t := \sup\{|T\eta|_t \colon \; |\eta|_t' \leqslant 1\} \qquad (T \in \mathcal{S},\, t \geqslant 0).$$

If we denote $H_t := \ell_2((j^t)_{j\in\mathbb{N}})$, $t \in \mathbb{R}$, then $H_t' \cong H_{-t}$ and every $T \in \mathcal{S}$ is a Hilbert space operator in the sense that $T\colon H_t' \to H_t$ and

$$\|T\|_t = \|T\|_{H_t' \to H_t} \qquad (t \geqslant 0).$$

In other words, if we denote by $D_t := \operatorname{diag}(j^t)$, $t \in \mathbb{R}$, an infinite diagonal matrix, then D_t becomes simultaneously an isometry $D_t\colon H_t \to \ell_2$ and $D_t\colon \ell_2 \to H_t'$ and

$$\|T\|_t = \|D_t T D_t\|_{\mathcal{B}(\ell_2)} \qquad (T \in \mathcal{S},\, t \geqslant 0). \tag{1}$$

In particular, $\mathcal{S} = \operatorname{proj}_{t\geqslant 0} \mathcal{B}(H_t', H_t) = \operatorname{proj}_{k\in\mathbb{N}} \mathcal{B}(H_k', H_k)$. We will be using these properties interchangably.

Since $s \hookrightarrow s'$, we can define multiplication in $\mathcal{S}$ as

$$T_1 T_2 := T_1 \circ \iota \circ T_2 \qquad (T_1, T_2 \in \mathcal{S}),$$

where $\iota\colon s \to s'$, $\iota(\xi) := \xi$ is the formal embedding. Altogether it turns $\mathcal{S}$ into an m-convex Fréchet algebra. It comes endowed also with the involution (or the adjoint map) given as

$$\langle T^*\xi, \eta\rangle := \langle \xi, T\eta\rangle \qquad (\xi, \eta \in s',\, T \in \mathcal{S}).$$

It is worth noting that $s \hookrightarrow \ell_2$ and, by dualization, also $\ell_2 \hookrightarrow s'$ therefore $\mathcal{S}$ is algebraically contained in the C^*-algebra $\mathcal{B}(\ell_2)$ of all bounded and linear operators on the Hilbert space ℓ_2. Therefore multiplication in $\mathcal{S}$ is essentially the multiplication in $\mathcal{B}(\ell_2)$ with the additional property that the resulting operator belongs to $\mathcal{S}$. The same applies to involution.

The unitization of $\mathcal{S}$ will be denoted by $\mathcal{S}_1$. Clearly, the unit in $\mathcal{S}_1$ is the identity operator on ℓ_2 denoted by $\mathbb{1}$. The algebra $\mathcal{S}$ is called the *noncommutative Schwartz space* and the elements of $\mathcal{S}$ are called *smooth operators.* We refer the reader to [2,13] for more information on the properties of this algebra.

3. Spectral properties of operators in $\mathcal{S}$. We start by showing some spectral properties of smooth operators.

Proposition 3.1 ([5, Proposition 3.1 and Theorem 3.3]). *Every smooth operator is compact, i.e., $\mathcal{S} \hookrightarrow \mathcal{K}(\ell_2)$ and*

$$\sigma_{\mathcal{S}_1}(T) = \sigma_{\mathcal{B}(\ell_2)}(T) \qquad (T \in \mathcal{S}).$$

In particular, the spectrum of every smooth operator consists of zero and a (possibly) null sequence of eigenvalues.

Lemma 3.2. *For any $t \in \mathbb{R}$ and every smooth operator $T \in \mathcal{S}$, we have*

$$\sigma_{\mathcal{B}(\ell_2)}(D_t T D_{-t}) \subset \sigma_{\mathcal{S}_1}(T).$$

Proof. Let $t \in \mathbb{R}$ and $T \in \mathcal{S}$ be fixed. Suppose that $\lambda \in \rho_{\mathcal{S}_1}(T)$, i.e., there is $S \in \mathcal{S}$ such that

$$\left(S - \frac{1}{\lambda}\mathbb{1}\right)(T - \lambda\mathbb{1}) = (T - \lambda\mathbb{1})\left(S - \frac{1}{\lambda}\mathbb{1}\right) = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator on ℓ_2. Then

$$\ell_2 \xrightarrow{D_{-t}} H_t \hookrightarrow s' \xrightarrow{S} s \hookrightarrow H_t \xrightarrow{D_t} \ell_2$$

and

$$\left(D_t S D_{-t} - \frac{1}{\lambda}\mathbb{1}\right)(D_t T D_{-t} - \lambda\mathbb{1}) = (D_t T D_{-t} - \lambda\mathbb{1})\left(D_t S D_{-t} - \frac{1}{\lambda}\mathbb{1}\right) = \mathbb{1}.$$

Consequently, $\lambda \in \rho_{\mathcal{B}(\ell_2)}(D_t T D_{-t})$ and the proof is thereby complete. □

Corollary 3.3. *If a smooth operator $T \in \mathcal{S}$ satisfies*

$$\sigma_{\mathcal{S}_1}(T) \subset \mathbb{D}, \tag{2}$$

then the sequence $(T^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{S}$.

Proof. Let the smooth operator $T \in \mathcal{S}$ satisfy (2). Since $D_t^{-1} = D_{-t}$, we obtain that for every $n \in \mathbb{N}$ and every $t \geqslant 0$,

$$\begin{aligned}
\|T^n\|_t &= \|D_t T^n D_t\|_{\mathcal{B}(\ell_2)} \\
&= \|\underbrace{D_t T D_{-t} D_t T D_{-t} \cdots D_t T D_{-t}}_{n-1} D_t T D_t\|_{\mathcal{B}(\ell_2)} \\
&\leqslant \|(D_t T D_{-t})^{n-1}\|_{\mathcal{B}(\ell_2)} \|T\|_t.
\end{aligned}$$

From Lemma 3.2, the spectral radius formula, and compactness of the spectrum, it now follows that there is $\varepsilon > 0$ such that

$$\nu(D_tTD_{-t}) = \lim_{n\to\infty} \|(D_tTD_{-t})^n\|_{\mathcal{B}(\ell_2)}^{1/n} \leqslant 1-\varepsilon.$$

Hence there is $N \in \mathbb{N}$ such that for every $n \geqslant N$, we have

$$\|(D_tTD_{-t})^n\|_{\mathcal{B}(\ell_2)} \leqslant 1.$$

If we now define $C_t := \max\{\|T\|_t, \|T^2\|_t, \ldots, \|T^N\|_t, 1\} \cdot \|T\|_t$, then

$$\sup_{n\in\mathbb{N}} \|T^n\|_t \leqslant C_t < \infty.$$

Consequently, $(T^n)_{n\in\mathbb{N}}$ is a bounded sequence in the noncommutative Schwartz space. □

4. Main result.

Theorem 3.1. *There are no topologically transitive operators in $\mathcal{S}$. In particular, the operators in $\mathcal{S}$ are not hypercyclic.*

Proof. Let $T \in \mathcal{S}$ be arbitrary. There are two possible cases: either $\sigma_{\mathcal{S}_1}(T) = \{0\}$ or there exists $0 \neq \lambda \in \sigma_{\mathcal{S}_1}(T)$.

If $\sigma_{\mathcal{S}_1}(T) = \{0\}$, then from Corollary 3.3, the sequence $(T^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{S}$ and therefore it is equicontinuous. In particular, for every zero neighbourhood $U \subset s'$, there is a zero neighbourhood $V \subset s$ such that

$$T^n(U) \subset \frac{1}{2}V \qquad (n \in \mathbb{N}).$$

We choose now $\xi \in s \setminus V$ and suppose that there is $n \in \mathbb{N}$ and $\eta \in s$ such that

$$\eta \in T^n(U) \cap \left(\xi + \frac{1}{2}V\right).$$

This implies that for some $\zeta \in \frac{1}{2}V$, we have

$$\xi = \eta - \zeta \in \frac{1}{2}V + \frac{1}{2}V = V.$$

This contradicts the choice of $\xi \in s$ and shows that in this case T is not topologically transitive.

If there exists $0 \neq \lambda \in \sigma_{\mathcal{S}_1}(T)$, then from Proposition 3.1, it follows that λ is an eigenvalue and we let f be a holomorphic function on a neighbourhood of $\sigma_{\mathcal{S}_1}(T)$ such that $f(\lambda) = 1$ and $f(z) = 0$ for $z \in \sigma_{\mathcal{S}_1}(T) \setminus \{\lambda\}$. Using the holomorphic functional calculus (which is available in $\mathcal{S}$ by [12, Lemma 1.3]), we can now consider the operator $f(T) \in \mathcal{S}$. Let $M = \mathrm{Im}(f(T))$. It is clear that M is a non-trivial and finite dimensional subspace of s (every non-zero element of M is an eigenvector of the compact operator $f(T)$). The properties of the functional calculus imply that the diagram

$$\begin{tikzcd} s' \arrow[r, "T"] \arrow[d, "f(T)"'] & s \arrow[d, "f(T)_{|s}"] \\ M \arrow[r, "T_{|M}"] & M \end{tikzcd}$$

commutes and one can easily verify that topological transitivity of T would imply topological transitivity of $T_{|M}$. Since M is finite dimensional this implies that T is not topologically transitive. $\square$

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