On the finite index subgroups of Houghton's groups

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Abstract. Houghton's groups $H_2, H_3, \ldots$ are certain infinite permutation groups acting on a countably infinite set; they have been studied, among other things, for their finiteness properties. In this note we describe all of the finite index subgroups of each Houghton group, and their isomorphism types. Using the standard notation that $d(G)$ denotes the minimal size of a generating set for G, we then show, for each $n \in \{2, 3, \ldots\}$ and U of finite index in H_n, that $d(U) \in \{d(H_n), d(H_n) + 1\}$ and characterise when each of these cases occurs.

1. Introduction. Introduced in [11] by Houghton, the Houghton groups have since attracted attention for their finiteness properties [2,13], their growth [3,10], their many interesting combinatorial features [1,9,14,15] as well as other properties.

Definition. For $X \neq \emptyset$, let $\mathrm{Sym}(X)$ denote the group of all bijections on X. For $g \in \mathrm{Sym}(X)$, let $\mathrm{supp}(g) := \{x \in X : (x)g \neq x\}$, called the *support* of g. Then $\mathrm{FSym}(X) = \{g \in \mathrm{Sym}(X) : |\mathrm{supp}(g)| < \infty\}$ and $\mathrm{Alt}(X) \leq \mathrm{FSym}(X)$ consists of only the even permutations, meaning $[\mathrm{FSym}(X) : \mathrm{Alt}(X)] = 2$.

We give a brief overview of these groups for our purposes; more detailed introductions can be found, for example, in [1,4]. We will use right actions throughout. We define $\mathbb{N} := \{1, 2, \ldots\}$, let $G \leq_f H$ denote that G is a finite index subgroup of H, and for a group G and $g, h \in G$, let $[g, h] := g^{-1}h^{-1}gh$ and $g^h := h^{-1}gh$.

Definition. Let $n \in \{3, 4, \ldots\}$. Then the *nth Houghton group*, denoted H_n, is generated by $g_2, \ldots, g_n \in \mathrm{Sym}(X_n)$ where $X_n = \{1, \ldots, n\} \times \mathbb{N}$ and for $k \in \{2, \ldots, n\}$,

$$(i, m)g_k = \begin{cases} (1, m+1) & \text{if } i = 1 \text{ and } m \in \mathbb{N}, \\ (1, 1) & \text{if } i = k \text{ and } m = 1, \\ (k, m-1) & \text{if } i = k \text{ and } m \in \{2, 3, \ldots\}, \\ (i, m) & \text{otherwise.} \end{cases} \tag{1}$$

For each $n \in \{3, 4, \ldots\}$, we have that $\mathrm{FSym}(X_n) \leq H_n$. One way to see this is to first compute that $[g_2, g_3] = ((1, 1), (1, 2))$, and then observe that any 2-cycle with support in X_n can be conjugated, using the elements $g_2, \ldots, g_n$, to $((1, 1), (1, 2))$. Furthermore, as observed in [15], we have a short exact sequence of groups

$$1 \longrightarrow \mathrm{FSym}(X_n) \longrightarrow H_n \overset{\pi}{\longrightarrow} \mathbb{Z}^{n-1} \longrightarrow 1.$$

Here π is induced by defining $\pi(g_i) := e_{i-1}$ for $i = 2, \ldots, n$, where e_i denotes the vector in $\mathbb{Z}^{n-1}$ with ith entry 1 and other entries 0.

Definition. The second Houghton group is generated by the two cycle $((1, 1), (1, 2))$ together with the element g_2, defined analogously to (1) above. This is isomorphic to $\mathrm{FSym}(\mathbb{Z}) \rtimes \langle t \rangle$ where $t \in \mathrm{Sym}(\mathbb{Z})$ sends each $z \in \mathbb{Z}$ to $z + 1$.

There have been many papers with questions and results relating to the finite index subgroups of this family of groups, e.g. the questions on invariable generation in [15] and subsequent answers in [6], showing they all have solvable conjugacy problem in [4], and also all have the R_∞ property [5,8]. Some of these use the partial description of the finite index subgroups from [3]. We start by giving the first full description of them.

Theorem 1. *Let* $n \in \{3, 4, \ldots\}$ *and* $U \leq_f H_n$. *Then there exist* $c_2, \ldots, c_n \in \mathbb{N}$ *such that* $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$ *and either* U *is:*

(i) *equal to* $\langle \mathrm{FSym}(X_n), g_i^{c_i} : i = 2, \ldots, n \rangle$*; or*

(ii) *isomorphic to* $\langle \mathrm{Alt}(X_n), g_i^{c_i} : i = 2, \ldots, n \rangle$.

If $U \leq_f H_2$, *then there exists* $c_2 \in \mathbb{N}$ *such that* $\pi(U) = \langle c_2 \rangle \leq \mathbb{Z}$ *and either (i) or (ii) above occurs or* U *is equal to* $\langle \mathrm{Alt}(X_2), ((1, 1), (1, 2)g_2) \rangle$.

Given $U, U' \leq_f H_n$ such that $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$ for $c_2, \ldots, c_n \in \mathbb{N}$ and $\pi(U') = \langle d_2 e_1, \ldots, d_n e_{n-1} \rangle$ for $d_2, \ldots, d_n \in \mathbb{N}$, one might wonder when $U \cong U'$. Clearly any permutation of the constants $c_2, \ldots, c_n$ produces an isomorphism. By considering $\mathrm{Aut}(U)$ and $\mathrm{Aut}(U')$, it seems that this is the only way for the groups to be isomorphic. Our methods do allow us to obtain the following.

Corollary 2. *Let* $n \in \{2, 3, \ldots\}$ *and* $c_2, \ldots, c_n \in \mathbb{N}$. *If* $U \leq_f H_n$ *and* $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$, *then either*

- *at least two of* $c_2, \ldots, c_n$ *are odd and* $U = \langle g_2^{c_2}, \ldots, g_n^{c_n}, \mathrm{FSym}(X_n) \rangle$*; or*
- U *is one of exactly* $2^{n-1} + 1$ *specific subgroups of* H_n.

We then extend the work in [7], where the groups $\langle \mathrm{Alt}(X_2), g_2^c \rangle$ were shown to be 2-generated for each $c \in \mathbb{N}$, by investigating the generation properties of each of these groups.

Notation. For a finitely generated group G, let $d(G) := \min\{|S| \, : \, \langle S \rangle = G\}$.

Every finite index subgroup of a finitely generated group is itself finitely generated. In the case of $\mathbb{Z}^n$, where $n \in \mathbb{N}$, we have that each $A \leq_f \mathbb{Z}^n$ satisfies $d(A) = d(\mathbb{Z}^n)$. There has been much work related to this notion, e.g. [12] and papers verifying the Nielsen–Schreier formula for families of groups other than free groups. Our second theorem states that there is a close connection between the minimal number of generators of a Houghton group and its finite index subgroups. Note that $d(H_2) = 2$ and $d(H_n) = n - 1$ for $n \in \{3, 4, \ldots\}$.

Theorem 3. *If $U \leq_f H_2$, then $d(U) = d(H_2)$. For $n \in \{3, 4 \ldots\}$ and $U \leq_f H_n$, we have that $d(U) \in \{d(H_n), d(H_n) + 1\}$. Furthermore, let $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$. Then $d(U) = d(H_n) + 1$ occurs exactly when both of the following conditions are met:*

  i) that $\mathrm{FSym}(X_n) \leq U$; and
  ii) either one or zero elements in $\{c_2, \ldots, c_n\}$ are odd.

Our proof involves providing a generating set. Theorem 1 allows us to replace U with either $\langle \mathrm{FSym}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$ or $\langle \mathrm{Alt}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$. Proposition 3.7 states that, in the second case, $d(U) = n - 1$. Lemma 3.5 and Lemma 3.6 combine to tell us that $\langle \mathrm{FSym}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle = \langle \mathrm{Alt}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$ when there are distinct $i, j \in \{2, \ldots, n\}$ that are both odd. Thus the only remaining possibility is that $U = \langle \mathrm{FSym}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle \neq \langle \mathrm{Alt}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$. Lemma 3.9 shows that in this case $d(U) = d(H_n) + 1$, by determining that the abelianization of U, with these conditions, is $C_2 \times \mathbb{Z}^{n-1}$. This complete categorisation provides us with subgroups of the Houghton groups with constant minimal number of generators on finite index subgroups.

Corollary 4. *Let $n \in \{3, 4, \ldots\}$ and define $G_2 := \langle \mathrm{Alt}(X_n), g_2^2, \ldots, g_n^2 \rangle \leq_f H_n$. If $U \leq_f G_2$, then $d(U) = d(G_2) = d(H_n)$.*

2. The structure of finite index subgroups of H_n. In this section, we prove Theorem 1. The following is well known.

Lemma 2.1. *Given $U \leq_f G$, there exists $N \leq_f U$ which is normal in G.*

Some structure regarding finite index subgroups of H_n is known; the following is particularly useful to us.

Lemma 2.2 ([5, Prop. 2.5]). *Let X be a non-empty set and $\mathrm{Alt}(X) \leq G \leq \mathrm{Sym}(X)$. Then G has $\mathrm{Alt}(X)$ as a unique minimal normal subgroup.*

Let $n \in \{2, 3, \ldots\}$ and $U \leq_f H_n$. By Lemma 2.1, U contains a normal subgroup of H_n and so, by Lemma 2.2, $\mathrm{Alt}(X_n) \leq U$. Furthermore, $U \leq_f H_n$ and so $\pi(U) \leq_f \pi(H_n)$ where $\pi : H_n \to \mathbb{Z}^{n-1}$ sends g_i to e_{i-1} for $i = 2, \ldots, n$. Hence there exist minimal $k_2, \ldots, k_n \in \mathbb{N}$ such that $\pi(g_i^{k_i}) \in \pi(U)$ for each

$i \in \{2, \ldots, n\}$. But $\pi(g_i^{k_i}) = k_i e_{i-1}$, and so the preimage in H_n of $\pi(g_i^{k_i})$ is $\{\sigma g_i^{k_i} \; : \; \sigma \in \mathrm{FSym}(X_n)\}$. If $\mathrm{FSym}(X_n) \leq U$, then $g_2^{k_2}, \ldots, g_n^{k_n} \in U$. Otherwise $\mathrm{FSym}(X_n) \cap U = \mathrm{Alt}(X_n)$ and, for each $i \in \{2, \ldots, n\}$, either $g_i^{k_i} \in U$ or there exists $\omega_i \in \mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$ such that $\omega_i g_i^{k_i} \in U$; since $\mathrm{Alt}(X_n) \leq U$, we can specify that $\omega_i = ((1,1)\,(1,2))$. Then, using the minimality of $k_2, \ldots, k_n$,

$$U = \langle g_2^{k_2}, \ldots, g_n^{k_n}, \mathrm{FSym}(X_n) \rangle \text{ or } U = \langle \epsilon_2 g_2^{k_2}, \ldots, \epsilon_n g_n^{k_n}, \mathrm{Alt}(X_n) \rangle \quad (2)$$

where each ϵ_i is either trivial or $((1,1)\,(1,2))$. We now describe the isomorphism type for these subgroups. To do this, we introduce two families of finite index subgroups in H_n where $n \in \{2, 3, \ldots\}$.

Notation. For any given $n \in \{2, 3, \ldots\}$ and any $\mathbf{c} = (c_2, \ldots, c_n) \in \mathbb{N}^{n-1}$, let $F_{\mathbf{c}} := \langle \mathrm{FSym}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$ and $G_{\mathbf{c}} := \langle \mathrm{Alt}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$.

There are some $\mathbf{c} \in \mathbb{N}^n$ such that $G_{\mathbf{c}} = F_{\mathbf{c}}$, e.g. if $n \neq 2$ and $c_2 = \cdots = c_n = 1$.

Notation. Let $I_n := \{\mathbf{c} = (c_2, \ldots, c_n) \in \mathbb{N}^n \; : \; G_{\mathbf{c}} \neq F_{\mathbf{c}}\}$.

Lemma 3.5 and Lemma 3.6 together describe I_n. This description, together with (2), yields Corollary 2.

Notation. For $n \in \{2, 3, \ldots\}$ and $i \in \{1, \ldots, n\}$, let $R_i := \{(i, m) \; : \; m \in \mathbb{N}\} \subset X_n$.

Proposition 2.3. *Let $n \in \{3, 4, \ldots\}$, $\mathbf{c} = (c_2, \ldots, c_n) \in I_n$, each $\epsilon_2, \ldots, \epsilon_n$ be either trivial or $((i,1)\,(i,2))$, and $U = \langle \mathrm{Alt}(X_n), \epsilon_i g_i^{c_i} \; : \; i = 2, \ldots, n, \rangle \leq_f H_n$. Then U is isomorphic to $G_{\mathbf{c}}$.*

Proof. If there are $i \neq j$ such that $c_i = c_j = 1$, then $\mathrm{FSym}(X_n) \leq G_{\mathbf{c}}$ and so $G_{\mathbf{c}} = F_{\mathbf{c}}$, i.e. $\mathbf{c} \notin I_n$.

Now consider if $c_i = 1$ for some i and that $\epsilon_i = ((i,1)\,(i,2))$. Then $\epsilon_i g_i$ fixes $(i,1)$. Moreover, we have $(i,m)\epsilon_i g_i = (i, m-1)$ for all $m \geq 3$, $(i,2)\epsilon_i g_i = (1,1)$, $(1,m)\epsilon_i = (1,m+1)$ for all $m \geq 1$, and that $\epsilon_i g_i$ fixes all other points in X_n. This allows us to relabel X_n so that the point $(i,1)$ becomes part of another ray $j \in \{2, \ldots, n\}$.

If $c_i \neq 1$ and $\epsilon_i = ((i,1)\,(i,2))$, then we can relabel R_i as R_i' by swapping the labels on the points $(i,1)$ and $(i,2)$ so that $\epsilon_i g_i^{c_i}$ acts on $R_i' \cup R_1$ in the same way as $g_i^{c_i}$ acts on $R_i \cup R_1$. $\square$

Lemma 2.4. *Let $k \in \{2, 3, \ldots\}$, $\epsilon = ((1,1)\,(1,2))$, and $U = \langle \mathrm{Alt}(X_n), \epsilon g_2^k \rangle \leq_f H_2$. Then U is isomorphic to $\langle \mathrm{Alt}(X_2), g_2^k \rangle$.*

Proof. In a similar way to the preceding proof, we can relabel R_2 as R_2' (by swapping the labels on the points $(2,1)$ and $(2,2)$) so that ϵg_2^k acts on $R_1 \cup R_2'$ in the same way that g_2^k acts on $R_1 \cup R_2$. $\square$

Remark 2.5. The preceding lemma may hold for $k = 1$. Such an isomorphism would not be induced by a permutation of X_2, however: any element g in $\langle \mathrm{Alt}(X_2), ((1,1)\,(1,2))t \rangle$ with $\pi(g) = 1$ cannot have an infinite orbit equal to X_2.

3. Generation of the groups $F_\mathbf{c}$ and $G_\mathbf{c}$. We start with the case of H_2, first dealing with the 'exceptional' case.

Lemma 3.1. *The group* $\langle \mathrm{Alt}(X_2), ((1,1)\ (1,2))g_2 \rangle$ *is 2-generated.*

Proof. Note that $\langle \mathrm{Alt}(X_2), ((1,1)\ (1,2))g_2 \rangle \cong \langle \mathrm{Alt}(\mathbb{Z}), (1\ 2)t \rangle \leq \mathrm{FSym}(\mathbb{Z}) \rtimes \langle t \rangle$ where $t : z \to z + 1$ for all $z \in \mathbb{Z}$. Our aim will be to show that $S = \{(0\ 1\ 2), (0\ 1)t\}$ generates $\langle \mathrm{Alt}(\mathbb{Z}), (1\ 2)t \rangle$. To do this, we will use $\langle S \rangle$ to construct all 3-cycles $(0\ a\ b)$ with $0 < a < b$. With these elements we can then also produce every 3-cycle of the form $(a\ b\ c)$ where $0 < a < b < c$, by conjugating $(0\ a\ b)$ by $(0\ c\ 2c)$. Then every 3-cycle in $\mathrm{Alt}(\mathbb{N} \cup \{0\})$, or its inverse, is accounted for by such elements. Conjugation by some power of $(0\ 1)t$ yields every 3-cycle in $\mathrm{Alt}(\mathbb{Z})$, and we recall that $\mathrm{Alt}(\mathbb{Z})$ is generated by the set of all 3-cycles with support in $\mathbb{Z}$.

In addition to our first simplification, note that it is sufficient to show that $(0, 1, k+1) \in \langle (0\ 1\ 2), (0\ 1)t \rangle$ for every $k \in \mathbb{N}$. This is because any 3-cycle $(0\ a\ b)$ with $0 < a < b$ will be conjugate, by $(0\ 1)t$, to an element of the form $(0, 1, k+1)$. We start with $\sigma_1 := (0, k, k+1)$. If $k = 1$, then we are done. Otherwise, conjugate σ_1^{-1} by $(0, k-1, k)$ to obtain $\sigma_2 = (0, k-1, k+1)$. If $k > 2$, conjugate σ_2^{-1} by $(0, k-2, k-1)$ to obtain σ_3. Continuing in this way yields the result. $\square$

The following results conclude the H_2 case. We include the proof here as we will adapt it for $F_\mathbf{c}$ and $G_\mathbf{c}$. The notation $\Omega_k := \{1, \ldots, k\}$ is helpful for these results.

Lemma 3.2 ([5, Lem. 3.6]). *Let* $k \in \{3, 4, \ldots\}$ *and* $t : z \to z + 1$ *for all* $z \in \mathbb{Z}$. *Then* $\langle \mathrm{Alt}(\mathbb{Z}), t^k \rangle$ *is generated by* $\mathrm{Alt}(\Omega_{2k}) \cup \{t^k\}$.

Lemma 3.3 ([5, Lem. 3.7]). *Let* $k \in \mathbb{N}$ *and* $t : z \to z + 1$ *for all* $z \in \mathbb{Z}$. *Then* $G_k := \langle \mathrm{Alt}(\mathbb{Z}), t^k \rangle$ *and* $F_k := \langle \mathrm{FSym}(\mathbb{Z}), t^k \rangle$ *are 2-generated.*

Proof. We start with G_k. We will show that we can find, for each $k \in \{3, 4, \ldots\}$, an $\alpha_k \in \mathrm{Alt}(\mathbb{Z})$ such that $\langle t^k, \alpha_k \rangle$ contains $\mathrm{Alt}(\mathbb{Z})$. Then, for $d \in \{1, 2, 3\}$, $\langle t^d, \alpha_6 \rangle$ contains $\langle t^6, \alpha_6 \rangle$ and so contains $\mathrm{Alt}(\mathbb{Z})$. We can therefore fix some $k \in \{3, 4, \ldots\}$, meaning all 3-cycles in $\mathrm{Alt}(\Omega_{2k})$ are conjugate.

Let $r = \binom{2k}{3}$ and let $\omega_1, \ldots, \omega_r$ be a choice of distinct 3-cycles in $\mathrm{Alt}(\Omega_{2k})$ with $\omega_i \neq \omega_j^{-1}$ for every $i, j \in \{1, \ldots, r\}$. Thus $\langle \omega_1, \ldots, \omega_r \rangle = \mathrm{Alt}(\Omega_{2k})$. Set $\sigma_0 = (1\ 3)$ and $\sigma_{r+1} = (2\ 3)$ and note that $(\sigma_{r+1}^{-1} \sigma_0^{-1} \sigma_{r+1}) \sigma_0 = (1\ 2\ 3)$. Now choose $\sigma_1, \ldots, \sigma_r \in \mathrm{Alt}(\Omega_{2k})$ so that for each $m \in \{1, \ldots, r\}$, we have $\sigma_m^{-1} (1\ 2\ 3) \sigma_m = \omega_m$.

Let $\alpha_k := \prod_{i=0}^{r+1} t^{-2ik} \sigma_i t^{2ik}$ and, for $m \in \{1, \ldots, r+1\}$, let $\beta_m := t^{2mk} \alpha_k t^{-2mk}$. Then $\beta_{r+1}^{-1} \alpha_k^{-1} \beta_{r+1} \alpha_k = \sigma_{r+1}^{-1} \sigma_0^{-1} \sigma_{r+1} \sigma_0 = (1\ 2\ 3)$ and, for $m \in \{1, \ldots, r\}$, we have that $\beta_m^{-1}(1\ 2\ 3)\beta_m = \omega_m$. Hence $\langle \alpha_k, t^k \rangle = G_k$.

We can now adapt the element α_k to an element γ_k so, for each $k \in \{3, 4, \ldots\}$, we have that $\langle \gamma_k, t^k \rangle = F_k$. One way to do so is to set $\gamma_k := (\sigma_0')(\prod_{i=1}^{r+1} t^{-2ik} \sigma_i t^{2ik})$ where $\sigma_0' = (1\ 3)(4\ 5)$. This means that $[\sigma_{r+1}, \sigma_0']$ still equals $(1\ 2\ 3)$, and so $\mathrm{Alt}(\mathbb{Z}) \leq \langle \gamma_k, t^k \rangle$, but that $\mathrm{Alt}(\mathbb{Z}) \cup \gamma_k(\mathrm{Alt}(\mathbb{Z})) = \mathrm{FSym}(\mathbb{Z}) \leq \langle \gamma_k, t^k \rangle$ as well. $\square$

We now work with a fixed $n \in \{3, 4, \ldots\}$. The following well-known commutator identities will be helpful for a few of our remaining proofs.

$$[a, bc] = a^{-1}(bc)^{-1}a(bc) = a^{-1}c^{-1}acc^{-1}a^{-1}b^{-1}abc = [a, c][a, b]^c \quad (3)$$

$$[ab, c] = (ab)^{-1}c^{-1}(ab)c = b^{-1}(a^{-1}c^{-1}ac)bb^{-1}c^{-1}bc = [a, c]^b[b, c] \quad (4)$$

Remark 3.4. Let $a, b, c \in H_n$ and sgn denote the sign function on $\mathrm{FSym}(X_n)$. From the identities (3) and (4) above, we have that $\mathrm{sgn}([a, bc]) = \mathrm{sgn}([a, b])\mathrm{sgn}([a, c])$ and $\mathrm{sgn}([ab, c]) = \mathrm{sgn}([a, c])\mathrm{sgn}([b, c])$.

Recall that $\mathbf{c} \in \mathbb{N}^{n-1}$ is in I_n if and only if $G_{\mathbf{c}} \neq F_{\mathbf{c}}$.

Lemma 3.5. *Let* $\mathbf{c} = (c_2, \ldots, c_n) \in \mathbb{N}^n$. *We have that* $\mathbf{c} \in I_n$ *if and only if* $[g_i^{c_i}, g_j^{c_j}] \in \mathrm{Alt}(X_n)$ *for all* $i, j \in \{2, \ldots, n\}$.

Proof. If there exist $i, j \in \{2, \ldots, n\}$ such that $[g_i^{c_i}, g_j^{c_j}] \notin \mathrm{Alt}(X_n)$, then $G_{\mathbf{c}}$ contains an odd permutation and so $G_{\mathbf{c}} = F_{\mathbf{c}}$. Recall that $G_{\mathbf{c}} = \langle \mathrm{Alt}(X_n), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$ and that we have a homomorphism $\pi : H_n \to \mathbb{Z}^{n-1}$ induced by sending g_i to e_{i-1} for $i = 2, \ldots, n$. Then $\pi(G_{\mathbf{c}}) \leq_f \pi(H_n) = \mathbb{Z}^{n-1}$, and so $\pi(G_{\mathbf{c}})$ is free abelian of rank $n - 1$. Define $S := \{g_2^{c_2}, \ldots, g_n^{c_n}\}$ so that $\langle S \cup \mathrm{Alt}(X_n) \rangle = G_{\mathbf{c}}$ and $\pi(S)$ is a linearly independent set. Let $t_i := \pi(g_i^{c_i})$ for $i = 2, \ldots, n$. Thus

$$\pi(\langle S \rangle) = \langle t_2, \ldots, t_n \mid R \rangle \text{ where } R = \{[t_i, t_j] : i \neq j\}. \quad (5)$$

Now let $\alpha \in \langle S \rangle \cap \mathrm{FSym}(X_n)$. We will show that $\alpha \in \mathrm{Alt}(X_n)$, meaning that $\mathbf{c} \in I_n$. First, express α as a word w in $S^{\pm 1}$. Thus $\alpha = w(g_2^{c_2}, \ldots, g_n^{c_n})$ and $\pi(\alpha) = w(t_2, \ldots, t_n)$ for the same word w. But also $\pi(\alpha) = \underline{0}$ which, from (5), means that $w(t_2, \ldots, t_n)$ is in the normal closure of R. Thus $w(t_2, \ldots, t_n)$ is a product of conjugates of powers of commutators in terms of $t_2, \ldots, t_n$. Then $\alpha = w(g_2^{c_2}, \ldots, g_n^{c_n})$ must also be a product of conjugates of powers of commutators in terms of $g_2^{c_2}, \ldots, g_n^{c_n}$. Our assumption that $[g_i^{c_i}, g_j^{c_j}] \in \mathrm{Alt}(X_n)$ for all $i, j \in \{2, \ldots, n\}$ together with Remark 3.4 means that $\alpha \in \mathrm{Alt}(X_n)$. □

The following gives a clearer description of the set I_n.

Lemma 3.6. *Let* $i, j \in \{2, \ldots, n\}$ *and* $c_i, c_j \in \mathbb{N}$. *Then* $[g_i^{c_i}, g_j^{c_j}] \notin \mathrm{Alt}(X_n)$ *if and only if* c_i *and* c_j *are both odd.*

Proof. Note that $[g_i, g_j] \in \mathrm{FSym}(X_n) \backslash \mathrm{Alt}(X_n)$ whereas $[g_i^2, g_j^2], [g_i, g_j^2], [g_i^2, g_j] \in \mathrm{Alt}(X_n)$. Now, by repeatedly applying Remark 3.4, we can reduce c_i and c_j to be either 1 or 2, depending on whether they are odd or even respectively. □

Combining Lemma 3.5 and Lemma 3.6, we see that $\mathbf{c} = (c_2, \ldots, c_n) \in \mathbb{N}^{n-1}$ is in I_n if and only if at least two of $c_2, \ldots, c_n$ are odd.

Proposition 3.7. *Given* $\mathbf{c} = (c_2, \ldots, c_n) \in \mathbb{N}^{n-1}$, *we have that* $d(G_{\mathbf{c}}) = d(H_n)$.

Proof. We start with the case that $\mathbf{c} \in I_n$. By possibly relabelling the branches of X_n, set $c_3 := \max\{c_2, \ldots, c_n\}$. Thus $c_3 \neq 1$, as otherwise $c_2 = c_3 = 1$

and $\mathbf{c} \notin I_n$. Let $k := 2c_3$ and $\Omega^* := \{(1,1), \ldots, (1,2k)\}$. Thus $2k \geq 8$ and all 3-cycles in $\mathrm{Alt}(\Omega^*)$ are conjugate in $\mathrm{Alt}(\Omega^*)$. We claim that $\mathrm{Alt}(X_n) \leq \langle \mathrm{Alt}(\Omega^*), g_2^{c_2}, \ldots, g_n^{c_n} \rangle$. By Lemma 3.2, $\langle \mathrm{Alt}(\Omega^*), g_3^{c_3} \rangle$ is a subgroup of H_n which generates $\langle \mathrm{Alt}(R_1 \cup R_3), g_3^{c_3} \rangle \cong \langle \mathrm{Alt}(\mathbb{Z}), t^{c_3} \rangle$. Most importantly, $\langle \mathrm{Alt}(\Omega^*), g_3^{c_3} \rangle$ contains $\mathrm{Alt}(R_1)$. Now, given any $\sigma \in \mathrm{FSym}(X_n)$, there exists a word of the form $g_2^{d_2} \ldots g_n^{d_n}$ for some $d_i \in c_i \mathbb{N}$ which conjugates σ to σ', where $\mathrm{supp}(\sigma') \subset R_1$. Hence $\langle \mathrm{Alt}(\Omega^*), g_2^{c_2}, \ldots, g_n^{c_n} \rangle = G_\mathbf{c}$. Our next aim is to adapt the proof of Lemma 3.3. In particular, we will show that there exists a $\sigma \in \mathrm{Alt}(X_n)$ such that $\langle \sigma g_2^{c_2}, g_3^{c_3}, \ldots, g_n^{c_n} \rangle = G_\mathbf{c}$. For any $\omega \in \mathrm{FSym}(X_n)$, let $\sigma_\omega := [\omega g_2^{c_2}, g_3^{2k}]$, which we can calculate using (4) as

$$[\omega, g_3^{2k}]^{g_2^{c_2}}[g_2^{c_2}, g_3^{2k}] = g_2^{-c_2}(\omega^{-1} g_3^{-2k} \omega g_3^{2k}) g_2^{c_2}[g_2^{c_2}, g_3^{2k}]. \tag{6}$$

Observe that $\mathrm{supp}([g_2^{c_2}, g_3^{2k}]) \subseteq \{(1,1), \ldots, (1, c_2 + 2k)\}$ and, since $k = 2c_3 > 2c_2$, that $(1, 4k+1), (1, 4k+3) \notin \mathrm{supp}([g_2^{c_2}, g_3^k])$. Let $\delta_0 := ((1, 2k - c_2 + 1), (1, 2k - c_2 + 3))$ and note, using (6), that $\mathrm{supp}(\sigma_{\delta_0}) \cap \{(1, m) : m = 4k + 2 \text{ or } m \geq 4k + 4\} = \emptyset$ and σ_{δ_0} swaps $(1, 4k+1)$ and $(1, 4k+3)$. We now restrict to those $\omega \in \mathrm{Alt}(X_n)$ with

$$\mathrm{supp}(\omega \delta_0) \subset \bigcup_{j \in \mathbb{N}} (\Omega^*) g_3^{-4jk}.$$

The following implications of this restriction are all important in relation to (6):

- $\mathrm{supp}(\sigma_\omega) \subset R_1 \cup R_3$;
- $\mathrm{supp}(\omega^{-1}) \cap \mathrm{supp}(g_3^{-2k} \omega g_3^{2k}) = \emptyset$;
- $(x)g_2^{-c_2} \omega^{-1} g_2^{c_2} = (x)\omega^{-1}$ for all $x \in R_3$; and
- $(x)g_2^{-c_2}(g_3^{-2k} \omega g_3^{2k})g_2^{c_2} = (x)g_3^{-2k} \omega g_3^{2k}$ for all $x \in R_3$.

Now, by mimicking the proof of Lemma 3.3, we can choose one such ω, which we will denote by α, so that $\mathrm{Alt}(R_1) \leq \langle g_3^k, \sigma_\alpha \rangle$. Let $r = \binom{2k}{3}$ and $\omega_1, \ldots, \omega_r$ be a set of 3-cycles such that $\langle \omega_1, \ldots, \omega_r \rangle = \mathrm{Alt}(\Omega^*)$. Choose $\sigma_1, \ldots, \sigma_r \in \mathrm{Alt}(\Omega^*)$ such that $\sigma_i^{-1}((1,1)\,(1,2)\,(1,3))\sigma_i = \omega_i$ for $i = 1, \ldots, r$. Also set $\sigma_{r+1} := ((1,2)\,(1,3))$ and $\sigma_0 := ((1,1)\,(1,3))$. Define $\delta_i := g_3^{4ki}\sigma_i g_3^{-4ki}$ for $i = 1, \ldots, r+1$ and $\alpha := \delta_0 \ldots \delta_{r+1}$. Define $\beta_m := g_3^{-4mk}\sigma_\alpha g_3^{4mk}$ for $m = -1, \ldots, r+1$. Then, as in Lemma 3.3, we have that $[\beta_{r+1}, \beta_{-1}] = [\sigma_{d+1}, \sigma_0]$ and $\beta_i[\sigma_{d+1}, \sigma_0]\beta_i^{-1} = \omega_i$ for $i = 1, \ldots, r$.

We end our proof with the case where $\mathbf{c} \notin I_n$. From Lemma 3.5 and Lemma 3.6, there then exist $i, j \in \{2, \ldots, n\}$ such that $[g_i^{c_i}, g_j^{c_j}] \in \mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$. Using the element $\sigma_\alpha \in \mathrm{Alt}(X_n)$ defined above, we claim that $\langle \sigma_\alpha g_2^{c_2}, g_3^{c_3}, \ldots, g_n^{c_n} \rangle = F_\mathbf{c}$. We note that if $[g_i^{c_i}, g_j^{c_j}] \in \mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$, then, by Remark 3.4, so is $[\sigma g_i^{c_i}, g_j^{c_j}]$ for any $\sigma \in \mathrm{FSym}(X_n)$. Hence $\langle \sigma_\alpha g_2^{c_2}, g_3^{c_3}, \ldots, g_n^{c_n} \rangle$ contains both $\mathrm{Alt}(X_n)$ and an odd permutation, and so $d(G_\mathbf{c}) = d(H_n)$. $\square$

We now look at the final case of determining $d(F_\mathbf{c})$ where $\mathbf{c} \in I_n$.

Lemma 3.8. *Let $\mathbf{c} \in I_n$. Then the commutator subgroup of $F_\mathbf{c}$ equals $\mathrm{Alt}(X_n)$.*

Proof. Note that $[\mathrm{Alt}(X_n), \mathrm{Alt}(X_n)] = \mathrm{Alt}(X_n)$. Let $\sigma \prod_{i \in I} g_i^{p_i}$ and $\omega \prod_{j \in J} g_j^{q_j}$ be in $F_\mathbf{c}$, where $I, J \subseteq \{2, \ldots, n\}$, $\{p_i : i \in I\} \subset \mathbb{Z}$, $\{q_j : j \in J\} \subset \mathbb{Z}$, and

$\sigma, \omega \in \mathrm{FSym}(X_n)$. Then their commutator, using (3), will be a product of conjugates of elements of the form $[\sigma \prod_{i \in I} g_i^{p_i}, \omega]$ or $[\sigma \prod_{i \in I} g_i^{p_i}, g_k^{q_k}]$ for some $k \in J$. Applying (4) to these elements will produce a product of conjugates of elements of the form

$$[\sigma, \omega], [g_l^{p_l}, \omega], [\sigma, g_k^{q_k}], \text{ or } [g_l^{p_l}, g_k^{q_k}]$$

for some $l \in I$ and $k \in J$. The first 3 of these are clearly in $\mathrm{Alt}(X_n)$. The 4th is also in $\mathrm{Alt}(X_n)$ from our assumption that $\mathbf{c} \in I_n$ together with Lemma 3.5. Hence, from Remark 3.4, a commutator of any two elements in $F_{\mathbf{c}}$ is in $\mathrm{Alt}(X_n)$. □

Lemma 3.9. *Let* $\mathbf{c} = (c_2, \ldots, c_n) \in I_n$. *Then* $d(F_{\mathbf{c}}) = d(H_n) + 1$.

Proof. Recall that $d(G_{\mathbf{c}}) = n - 1$. Including an element from $\mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$ to a minimal generating set for $G_{\mathbf{c}}$ therefore yields a generating set of $F_{\mathbf{c}}$ of size n. So we need only show that, with the given hypothesis, no generating set of $F_{\mathbf{c}}$ of size $n - 1$ exists. From Lemma 3.8, the abelianization of $F_{\mathbf{c}}$ is $\mathbb{Z}^{n-1} \times C_2$. Thus $F_{\mathbf{c}}$ cannot be generated by $n - 1$ elements. □

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