Abelian sections of the symmetric groups with respect to their index

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Abstract. We show the existence of an absolute constant $\alpha > 0$ such that, for every $k \geq 3$, $G := \mathrm{Sym}(k)$, and for every $H \leqslant G$ of index at least 3, one has $|H/H'| \leq |G : H|^{\alpha/\log\log|G:H|}$. This inequality is the best possible for the symmetric groups, and we conjecture that it is the best possible for every family of arbitrarily large finite groups.

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1. Introduction. Abelian quotients of permutation groups attracted attention for the first time in [4], where the authors show that an abelian section of $\mathrm{Sym}(k)$ has order at most $3^{k/3}$ for every $k \geq 3$. Better bounds hold for primitive groups [1], and for transitive groups [5]. In these notes, a different aspect concerning the subgroups of the symmetric groups is revealed: as the index increases, the abelian quotients grow as slowly as possible.

Theorem 1. *There exists an absolute constant $\alpha > 0$ such that for every $k \geq 1$ and every $H \leqslant G := \mathrm{Sym}(k)$ of index at least 3, one has*

$$|H/H'| \leq |G : H|^{\alpha/\log\log|G:H|}.$$

This bound is sharp in a number of situations, which make the proof by induction somewhat challenging. For example, equality is satisfied for infinitely many k by elementary abelian groups of order $p^{\lfloor k/p \rfloor}$ having all of their orbits of cardinality p. More is said in Sect. 5, where we show some evidences towards the fact that Theorem 1 is the best possible for every family of arbitrarily large finite groups.

2. Preliminaries. Unless explicitly stated otherwise, all the logarithms are to base 2, and $\exp(x) := 2^x$. We will often use inequalities for the factorial function: to avoid useless calculations, we recall three estimates, which provide increasing accuracy.

Lemma 2 (Factorial estimates). *Let $k \geq 1$. Then*

(i)
$$k^{k/2} \; \leq \; k! \; \leq \; k^k.$$

(ii)
$$\frac{k^k}{e^{k-1}} \; \leq \; k! \; \leq \; \frac{k^{k+1}}{e^{k-1}}.$$

(iii)
$$\left(\frac{k}{e}\right)^k \sqrt{2\pi k} \; \leq \; k! \; \leq \; \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \cdot e^{1/12k}.$$

Proof. The right side of (i) is obvious and the left side is equivalent to $\sqrt{k} \leq (k!)^{1/k}$. These are the geometric means of $\{1, k\}$ and $\{1, 2, \ldots, k\}$ respectively. Since the product $j(k-j)$ is maximal where j is close to $k/2$, the first mean is at most the second, as desired. For (ii), let us notice that

$$\frac{k^k}{k!} = \prod_{j=1}^{k-1} \frac{(j+1)^j}{j^j}, \qquad \text{and} \qquad \frac{k!}{k^{k+1}} = \prod_{j=1}^{k-1} \frac{j^{j+1}}{(j+1)^{j+1}}.$$

To obtain the left side, we use $\frac{j+1}{j} \leq e^{1/j}$, so that

$$\prod_{j=1}^{k-1} \frac{(j+1)^j}{j^j} \leq \prod_{j=1}^{k-1} e = e^{k-1}.$$

To obtain the right side, we use $\frac{j}{j+1} \leq e^{-1/(j+1)}$, so that

$$\prod_{j=1}^{k-1} \frac{j^{j+1}}{(j+1)^{j+1}} \leq \prod_{j=1}^{k-1} e^{-1} = \frac{1}{e^{k-1}}.$$

Finally, (iii) is the sharp estimate of Robbins [7]. $\qquad\qquad\square$

As it is easy to see, taking logarithms in Lemma 2 (i), we see that

$$\tfrac{1}{2} k \log k \; \leq \; \log(k!) \; \leq \; k \log k,$$

and that

$$\tfrac{1}{2} \log k \; \leq \; \log \log(k!) \; \leq \; 2 \log k$$

for all $k \geq 3$.

Lemma 3. *Let G be an arbitrary group, $N \lhd G$, and $G/N \cong Q$. Then*
$$|Q/Q'| \; \leq \; |G/G'| \; \leq \; |Q/Q'| \cdot |N/N'|.$$

Proof. We have

$$|G/G'| = |G : NG'| \cdot |NG' : G'| = |(G/N) : (G/N)'| \cdot |N : N \cap G'|.$$

Since $N' \subseteq N \cap G'$, the proof follows. $\qquad\square$

To prove Theorem 1, we seek for an absolute constant $C > 0$ such that, for every sufficiently large k and every $H \leqslant \mathrm{Sym}(k)$ of index at least 3, one has

$$\log |H/H'| \ \leq \ C \cdot \frac{\log |\mathrm{Sym}(k) : H|}{\log \log |\mathrm{Sym}(k) : H|}. \tag{2.1}$$

By $f(n) \ll g(n)$ and $g(n) \gg f(n)$ we mean the same thing, namely, that there is $C > 0$ such that $f(n) \leq C \cdot g(n)$ for all $n \geq C$. Moreover, we will frequently use the fact that the function $x/\log x$ is an increasing function when $x \geq 3$.

Remark 4 (Small subgroups). Choose a large k, and let $G := \mathrm{Sym}(k)$. We notice that (2.1) is true for subgroups H such that $\log |H| \leq \frac{1}{4}k \log k$. In fact, via Lemma 2 (i), we have

$$\log |G : H| \geq \log(k!) - \tfrac{1}{4}k \log k \geq \tfrac{1}{4}k \log k.$$

From the main theorem of [4], we have $|H/H'| \leq 3^{k/3}$, and so

$$\log |H/H'| \leq 2k \leq 16 \cdot \frac{\frac{1}{4}k \log k}{\log(\frac{1}{4}k \log k)} \leq 16 \cdot \frac{\log |G : H|}{\log \log |G : H|}.$$

From the main theorem of [6], a primitive subgroup of G which does not contain $\mathrm{Alt}(k)$ has size at most 4^k. Since $\mathrm{Alt}(k)$ is perfect, primitive groups do not affect the proof of Theorem 1.

3. Transitive subgroups. In short, looking inside $\mathrm{Sym}(k)$, we use two sharp inequalities with respect to k: one for the index of a maximal transitive group, and one for the abelianization of a transitive group, which is provided by [5]. The structure of the groups we are left with is trapped by some smaller symmetric or alternating group, and this allows to argue that their abelian quotients are small.

Lemma 5. *Let $W := \mathrm{Sym}(k/b)^b \rtimes \mathrm{Sym}(b) \leqslant G$ for some $2 \leq b \leq k/2$ which divides k. Then*

$$\log |G : W| \ \geq \ \frac{k \log b}{3}$$

if k is sufficiently large.

Proof. We will prove the same inequality with the natural logarithm in place of log. Taking the logarithm in Lemma 2 (ii), we obtain

$$k \ln k - (k-1) \leq \ln(k!) \leq (k+1) \ln k - (k-1).$$

Using these inequalities three times, we can write

$$\ln|G:W| = \ln(k!) - \ln|W| = \ln(k!) - \ln(b!) - b\ln((k/b)!) \geq$$
$$k\ln k - (k-1) - (b+1)\ln b + (b-1) - b(k/b+1)\ln(k/b) + b(k/b-1) =$$
$$k\ln k - k - b\ln b - \ln b - k\ln(k/b) - b\ln(k/b) + k =$$
$$k\ln k - k - b\ln b - \ln b - k\ln k + k\ln b - b\ln k + b\ln b + k =$$
$$(k-1)\ln b - b\ln k.$$

It remains to prove that this is at least $\frac{1}{3}k\ln b$, and arranging the terms , we see that this is equivalent to

$$\frac{\frac{2}{3}k - 1}{\ln k} \geq \frac{b}{\ln b}.$$

Now we observe that, for large enough k , we have

$$\frac{\frac{2}{3}k - 1}{\ln k} \geq \frac{(k/2)}{\ln(k/2)}, \tag{3.1}$$

because $\left(\frac{2}{3}k - 1\right)/(k/2)$ converges to $4/3$, while $\ln(k)/\ln(k/2)$ converges to 1. Finally, for every $2 \leq b \leq k/2$, the right side of (3.1) is at least $b/\ln b$, because $x/\ln x$ is an increasing function. $\qquad\square$

Theorem 6 ([5, Theorem 1]). *If $G \leqslant \mathrm{Sym}(k)$ is a transitive permutation group, then*

$$|G/G'| \leq 4^{k/\sqrt{\log k}}.$$

In reality, the proof of Theorem 1 requires a slightly better result than Theorem 6 itself. Given a finite group R and a prime $p \geq 2$, let $a_p(R)$ be the number of the abelian composition factors of R of order p. We define

$$a(R) := \sum_{p\,\mathrm{prime}} a_p(R) \cdot \log p.$$

Informally, this is the logarithm of the "abelian portion" of R. We also introduce some more notation about wreath products. Let $W := R^b \rtimes \mathrm{Sym}(b)$. We denote by $\rho_{\mathrm{Sym}(b)} : W \to \mathrm{Sym}(b)$ the projection over the top group, and for every $j = 1, \ldots, b$, we denote by $\prod_{j=1}^b R_j$ the base subgroup. For every j, we have

$$N_W(R_j) = R_j \times (R^{b-1} \rtimes \mathrm{Sym}(b-1)).$$

This allows to consider the projections $\rho_j : N_W(R_j) \to R_j$.

Proposition 7. *Let $G \leqslant R\wr_k \mathrm{Sym}(k)$ be such that $\rho(G)$ is transitive and $\rho_j(N_G(R_j)) = R_j$ for every $j = 1, \ldots, k$. Then*

$$\log|G/G'| \leq (1 + a(R))\frac{2k}{\sqrt{\log k}}.$$

Proof. As an intermediate result towards Theorem 6, [5, Lemma 3.1] says that the inequality

$$\log|G/G'| \leq \frac{(2/\sqrt{\pi}) \cdot a(R)k}{\sqrt{\log k}} + \log|\rho(G)/(\rho(G))'|$$

is true under our hypothesis. Since $\rho(G) \leqslant \mathrm{Sym}(k)$ is transitive, putting back Theorem 6, and noting that $2/\sqrt{\pi} < 2$, we obtain the claimed inequality.   □

Proof (Proof of Theorem 1 for transitive H). We will always suppose that k is larger than any constant. We have already settled primitive groups at the end of Sect. 2, so let $H \leqslant G$ be transitive but not primitive, and contained in a maximal transitive group $W := \mathrm{Sym}(a)^b \rtimes \mathrm{Sym}(b)$ as in Lemma 5. In particular, we choose W in such a way that a is the smallest possible (equivalently, the blocks of imprimitivity have minimal size). Then $\log|G : H| \geq \log|G : W| \geq \frac{1}{3}k \log b$. If $\log b \geq \sqrt{\log k}$, from Theorem 6, we have

$$\log|H/H'| \leq \frac{2k}{\sqrt{\log k}} \ll \frac{\frac{1}{3}k\sqrt{\log k}}{\log(\frac{1}{3}k\sqrt{\log k})} \leq \frac{\frac{1}{3}k \log b}{\log(\frac{1}{3}k \log b)} \leq \frac{\log|G : H|}{\log\log|G : H|},$$

and then (2.1) is true. Thus, it remains to control *all* the cases

$$1 \leq \log b \leq \sqrt{\log k}. \tag{3.2}$$

For such a fixed b, we take a closer look at $H \leqslant W$. Let us recall the notation we introduced just before the statement of Proposition 7. First, the projection over the top group $\rho_{\mathrm{Sym}(b)}(H) \leqslant \mathrm{Sym}(b)$ is transitive (otherwise $H \leqslant \mathrm{Sym}(k)$ itself would be not transitive). This implies that the projections ρ_j of $N_H(\mathrm{Sym}(a))$ in $\mathrm{Sym}(a)$ are all isomorphic for every $1 \leq j \leq b$. Let us denote by $H_{proj} \leqslant \mathrm{Sym}(a)$ one of these projections. Since the blocks of imprimitivity have minimal size by the construction of W, we get that $H_{proj} \leqslant \mathrm{Sym}(a)$ is primitive. If H_{proj} does not contain $\mathrm{Alt}(a)$, then from the main theorem of [6], we have $|H_{proj}| \leq 4^a$. By the imprimitive embedding theorem , we have $H \leqslant (H_{proj})^b \rtimes \mathrm{Sym}(b)$, and so $|H| \leq 4^{ab} \cdot b!$. Using also Lemma 2 (i), and (3.2), it follows that

$$\log|G : H| \geq \tfrac{1}{2}k \log k - 2k - b \log b \gg k \log k - (\sqrt{\log k})2^{\sqrt{\log k}} \gg k \log k.$$

As we have seen in Remark 4, this implies that H is too small.

We are left with the cases where either $H_{proj} = \mathrm{Alt}(a)$ or $H_{proj} = \mathrm{Sym}(a)$. If $H_{proj} = \mathrm{Alt}(a)$, then from Proposition 7 (notice that $a(R) = 0$ in this case), we have

$$\log|H/H'| \leq \frac{2b}{\sqrt{\log b}}.$$

Using again (3.2), we obtain

$$\log|H/H'| \leq \frac{2b}{\sqrt{\log b}} \ll \frac{2^{\sqrt{\log k}}}{(\log k)^{1/4}} \ll \frac{k}{\log k} \ll \frac{\log|G : W|}{\log\log|G : W|} \leq \frac{\log|G : H|}{\log\log|G : H|}.$$

If $H_{proj} = \mathrm{Sym}(a)$, then from Proposition 7 (notice that $a(R) = 1$ in this case), we have

$$\log|H/H'| \leq \frac{4b}{\sqrt{\log b}}.$$

Using a last time (3.2), we obtain $\log|H/H'| \ll \frac{\log|G:H|}{\log\log|G:H|}$ as before, and the proof of the transitive case is complete.   □

4. Intransitive subgroups. Let H be contained in $\mathrm{Sym}(a) \times \mathrm{Sym}(b)$ for some $a + b = k$ and $1 \le b \le a \le k - 1$. Consider the projection $\rho : H \to \mathrm{Sym}(b)$. The factorized subgroup $Ker(\rho) \times \rho(H) \le \mathrm{Sym}(a) \times \mathrm{Sym}(b)$ has the same size as H, while it has no smaller abelianization from Lemma 3. Thus, we can suppose $H = A \times B$ for some $A \le \mathrm{Sym}(a)$ and $B \le \mathrm{Sym}(b)$. We can also suppose that a, as k, is larger than any constant. We have

$$\log |H/H'| = \log |A/A'| + \log |B/B'|,$$

and now we go by induction distinguishing two cases with respect to b.

4.1. Small b. First we suppose $b \le M$, where $M > 0$ is a large positive integer to be fixed later. If $a!/|A| < a$, then $|H/H'| \le M!$, and (2.1) follows easily (with some C depending on M). If $a!/|A| \ge a$, then by induction, we have $\log |A/A'| \le C \frac{\log(a!/|A|)}{\log \log(a!/|A|)}$. So, it is enough to prove that for some $C(M)$ and every $a!/|A| \ge a$, one has

$$C \frac{\log(a!/|A|)}{\log \log(a!/|A|)} + M! \;\le\; C \frac{\log\left(\frac{a!}{|A|} \cdot \frac{(a+1)\cdots(a+b)}{b!}\right)}{\log \log\left(\frac{a!}{|A|} \cdot \frac{(a+1)\cdots(a+b)}{b!}\right)} \tag{4.1}$$

for all a large enough. Set $x := a!/|A|$. We notice that, for all $a \ge 3$,

$$\frac{(a+1)\cdots(a+b)}{b!} = \binom{a+b}{a} \ge a \ge \frac{\log(a!)}{2 \log \log(a!)} \ge \frac{\log x}{2 \log \log x}.$$

We can assume $x \ge a$. Now we compute the following limit.

Lemma 8.

$$\lim_{x \to +\infty} \left(\frac{\log\left(\frac{x \log x}{2 \log \log x}\right)}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)} - \frac{\log x}{\log \log x} \right) = 1.$$

Proof. We have

$$\frac{\log\left(\frac{x \log x}{2 \log \log x}\right)}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)} - \frac{\log x}{\log \log x}$$

$$= \left(\frac{\log x}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)} - \frac{\log x}{\log \log x} \right) + \frac{\log \log x}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)} - \frac{\log(2 \log \log x)}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)}.$$

Then, for $x \to +\infty$, it is easy to see that the second term converges to 1, while the third term converges to zero. For the first term, we have that this is equal to

$$\frac{\log x}{\log \log\left(\frac{x \log x}{2 \log \log x}\right)} \left(1 - \frac{\log \log\left(\frac{x \log x}{2 \log \log x}\right)}{\log \log x} \right) \to 0.$$

$\square$

From Lemma 8, we have that

$$\frac{\log\left(\frac{a!}{|A|}\cdot\frac{(a+1)\cdots(a+b)}{b!}\right)}{\log\log\left(\frac{a!}{|A|}\cdot\frac{(a+1)\cdots(a+b)}{b!}\right)} - \frac{\log(a!/|A|)}{\log\log(a!/|A|)} \geq 1/2$$

is true for every a large enough. Thus, for $C := 2M!$, we obtain that (4.1) is true for every a large enough, as desired.

4.2. Large b. Let us suppose $a \geq b > M$, where $M > 0$ is again a large positive integer to be fixed later. By induction, we have

$$\log|A/A'| + \log|B/B'| \leq \frac{\log(a!/|A|)}{\log\log(a!/|A|)} + \frac{\log(b!/|B|)}{\log\log(b!/|B|)}.$$

Let $x := a!/|A|$ and $y := b!/|B|$. Since $|G:H| = \frac{(a+b)!}{|A||B|} = xy\binom{a+b}{a}$, we need to prove that

$$\frac{\log x}{\log\log x} + \frac{\log y}{\log\log y} \leq \frac{\log(xy\binom{a+b}{a})}{\log\log(xy\binom{a+b}{a})} \tag{4.2}$$

is true for every $a \geq b > 100$, $a \leq x \leq a!$, $b \leq y \leq b!$.

Lemma 9. *Let X, Y, K be positive integers larger than* 2. *If*

$$\frac{\log X}{\log\log X} + \frac{\log Y}{\log\log Y} \leq \frac{\log(XY \cdot K)}{\log\log(XY \cdot K)},$$

then

$$\frac{\log x}{\log\log x} + \frac{\log y}{\log\log y} \leq \frac{\log(xy \cdot K)}{\log\log(xy \cdot K)}$$

for every $3 \leq x \leq X$ *and* $3 \leq y \leq Y$.

Proof. We will argue replacing x, y, X, Y, K with their logarithms (to the base 2). Fix $K \geq \log 3$, and set

$$f(x,y) := \frac{x+y+K}{\log(x+y+K)} - \frac{x}{\log x} - \frac{y}{\log y}.$$

We will prove that $f(x,y)$ is non-increasing in x and y. To do this, we can replace $\log_2$ with $\ln$ in the definition of f. When considered in $(1,+\infty) \times (1,+\infty)$, $f(x,y)$ is an analytic function. Computing the partial derivative with respect to x, we obtain

$$\frac{\partial f}{\partial x} \leq -\frac{1}{\ln x} + \frac{1}{(\ln x)^2} + \frac{1}{\ln(x+y+K)} \leq 0.$$

Since the expression of f is symmetric with respect to x and y, we have $\frac{\partial f}{\partial y} \leq 0$ as before, and the proof follows. $\square$

From the previous lemma, it is enough to check (4.2) when $x = a!$ and $y = b!$. The next inequality is really about the inverse function of the gamma function, and concludes the proof of Theorem 1.

Proposition 10. *There exists an absolute constant $M > 0$ such that, whenever $M \le b \le a$, then*

$$\frac{\log(a!)}{\log\log(a!)} + \frac{\log(b!)}{\log\log(b!)} \le \frac{\log((a+b)!)}{\log\log((a+b)!)}.$$

Proof. Taking the natural logarithm in Lemma 2 (iii), we obtain

$$k(\ln k - 1) + \frac{\ln(2\pi k)}{2} \le \ln(k!) \le k(\ln k - 1) + \frac{\ln(2\pi k)}{2} + \frac{1}{12k}.$$

Then

$$\frac{\log(a!)}{\log\log(a!)} + \frac{\log(b!)}{\log\log(b!)} = \frac{\ln(a!)}{\ln\left(\frac{\ln(a!)}{\ln 2}\right)} + \frac{\ln(b!)}{\ln\left(\frac{\ln(b!)}{\ln 2}\right)}$$

$$\le \frac{a(\ln a - 1) + \frac{\ln(2\pi a)}{2} + \frac{1}{12a}}{\ln\left(\frac{1}{\ln 2}(a(\ln a - 1) + \frac{\ln(2\pi a)}{2} + \frac{1}{12a})\right)} + \frac{b(\ln b - 1) + \frac{\ln(2\pi b)}{2} + \frac{1}{12b}}{\ln\left(\frac{1}{\ln 2}(b(\ln b - 1) + \frac{\ln(2\pi b)}{2} + \frac{1}{12b})\right)}$$

$$\le \frac{(a+b)(\ln(a+b) - 1) + \frac{\ln(2\pi(a+b))}{2}}{\ln\left(\frac{1}{\ln 2}((a+b)(\ln(a+b) - 1) + \frac{\ln(2\pi(a+b))}{2})\right)}$$

$$\le \frac{\ln((a+b)!)}{\ln\left(\frac{\ln((a+b)!)}{\ln 2}\right)} = \frac{\log((a+b)!)}{\log\log((a+b)!)}.$$

Indeed, the inequality in the middle is true for sufficiently large a and b because comparing the leading terms in the asymptotic expansions of both sides, we obtain

$$a\left(1 - \frac{1}{\ln a}\right) + b\left(1 - \frac{1}{\ln b}\right) \le (a+b)\left(1 - \frac{1}{\ln(a+b)}\right).$$

$\square$

5. Arbitrary finite groups. From Lemma 2 (i), we have

$$k = \frac{k\log k}{\log k} \ge \frac{\log(k!)}{2\log\log(k!)}.$$

Thus, for every k which is a multiple of 3, $G := \mathrm{Sym}(k)$, and $H \le G$ an elementary abelian 3-group of size $3^{k/3}$, we obtain

$$|H/H'| = 2^{\frac{k\log 3}{3}} \ge \exp\left(\frac{(\log 3/6)\log|G|}{\log\log|G|}\right) \ge \exp\left(\frac{(\log 3/6)\log|G:H|}{\log\log|G:H|}\right).$$

(5.1)

This shows that Theorem 1 is the best possible for the symmetric groups. When G is an arbitrary finite group, we have the following.

Proposition 11. *Every finite group G of size at least 3 has an abelian section of size at least $|G|^{1/6(\log\log|G|)^2}$.*

Proof. It is well known that every group of size p^n has derived length at most $\log n$. Using pigeonhole on the derived series, we see that such a group has an abelian section of size at least $p^{n/\log n}$. Thus, every nilpotent group H of

size $p_1^{n_1} \cdots p_k^{n_k}$ has an abelian section of size at least $p_1^{n_1/\log n_1} \cdots p_k^{n_k/\log n_k}$. Since $\log n_i \leq \log \log |H|$ for every $i = 1, \ldots, k$, it follows that this size is at least $|H|^{1/\log \log |H|}$. Now, a result of Pyber [8, Corollary 2.3 (a)] shows that every finite group G has a solvable subgroup of size at least $|G|^{1/2 \log \log |G|}$. By another result of Heineken [2, Corollary], a finite solvable group S has a nilpotent subgroup of size at least $|S|^{1/3}$. Then, putting all together, an arbirary finite group G has an abelian section of size at least

$$|H|^{1/\log \log |H|} \geq |S|^{1/3 \log \log |S|} \geq |G|^{1/6 (\log \log |G|)(\log \log |S|)} \geq |G|^{1/3 (\log \log |G|)^2},$$

where we used that $x^{1/2 \log \log x}$ is an increasing function when $x \geq 7$. An analysis of groups of small order concludes the proof.   □

Arguing as in (5.1), Proposition 11 shows that Theorem 1 is not far from the best possible for every family of arbitrarily large finite groups. It is an intriguing question whether the 2 in the exponent in Proposition 11 can be removed: in fact, we conjecture that this can be done. Finally, it is worth to notice that a positive answer to a question of Pyber [3, Problem 14.76] would imply such an improvement of Proposition 11, showing again that Theorem 1 is the best possible for every family of arbitrarily large finite groups.

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