Algebraic surfaces determine analyticity of functions

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Abstract. Let $f: X \rightarrow \mathbb{R}$ be a function defined on a nonsingular real algebraic set X of dimension at least 3. We prove that f is an analytic (resp. a Nash) function whenever the restriction $f|_S$ is an analytic (resp. a Nash) function for every nonsingular algebraic surface $S \subset X$ whose each connected component is homeomorphic to the unit 2-sphere. Furthermore, the surfaces S can be replaced by compact nonsingular algebraic curves in X , provided that $\dim X \geq 2$ and f is of class $\mathcal{C}^\infty$.

Mathematics Subject Classification. 14P05, 14P20, 58A07, 32C05, 32C25.

Keywords. Real algebraic set, Analytic function, Nash function.

By [3, Theorem 1], a real-valued function on a real analytic manifold of dimension at least 3 is analytic whenever all its restrictions to analytic submanifolds homeomorphic to the unit 2-sphere are analytic. In the present note, we prove a variant of this result for functions defined on nonsingular real algebraic sets. Henceforth, we abbreviate *real analytic* to *analytic*. Besides analytic functions, we also consider Nash functions. We refer to [1] for the general theory of the latter class of functions.

Unless explicitly stated otherwise, all subsets of $\mathbb{R}^n$ are endowed with the Euclidean topology, induced by the standard norm.

Recall that $M \subset \mathbb{R}^m$ is a *Nash manifold* if it is a semialgebraic subset and an analytic submanifold (in particular, M is a closed subset of some open subset of $\mathbb{R}^m$). A function $f: M \rightarrow \mathbb{R}$ is called a *Nash function* if it is analytic and its graph is semialgebraic. Equivalently, f is a Nash function if and only if it is analytic and there is a relation

$$\sum_{i=0}^k \varphi_i(x) f(x)^{k-i} = 0 \text{ for all } x \in M,$$

where $k \geq 1$ and the $\varphi_i: \mathbb{R}^m \rightarrow \mathbb{R}$ are polynomial functions, with φ_0 not identically 0 on any connected component of M . For this reason, Nash functions are called *algebraic functions* in the older literature, see for example [5].

Let $X \subset \mathbb{R}^m$ be an irreducible nonsingular algebraic set of dimension $n \geq 0$. We define $d(X)$ to be the supremum of the number of points in the intersection $X \cap L$, where L runs through the family of all affine $(m - n)$ -planes in $\mathbb{R}^m$ that are transverse to X . Clearly, $d(X)$ is a positive integer, see [11, Theorem 11.5.3]. Given an integer k with $1 \leq k \leq n - 1$, we denote by $\mathcal{F}_k(X)$ the collection of all nonsingular algebraic subsets $Z \subset X$ having at most $d(X)$ connected components, each of which is homeomorphic to the unit k -sphere $\mathbb{S}^k$. In what follows, we only make use of $\mathcal{F}_k(X)$ with $k = 1$ and $k = 2$.

Theorem 1. *Let $f: X \rightarrow \mathbb{R}$ be a function defined on an irreducible nonsingular algebraic set $X \subset \mathbb{R}^m$ of dimension $n \geq 3$. Assume that the restriction $f|_S$ is an analytic (resp. a Nash) function for every algebraic surface $S \in \mathcal{F}_2(X)$. Then f is an analytic (resp. a Nash) function.*

The proof of Theorem 1 requires some preparation. Along the way, we establish results which are of independent interest. We emphasize that the function f in Theorem 1 is not assumed to be continuous.

Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{R}^n$. For any integer k with $1 \leq k \leq n - 1$, we denote by $\mathcal{E}_k(\mathbb{B}^n)$ the collection of all Euclidean k -spheres in $\mathbb{B}^n$ passing through the origin, that is, all algebraic sets $\Sigma^k \subset \mathbb{R}^n$ of the form

$$\Sigma^k = \{x \in \mathbb{R}^n: \|x - c\| = \|c\|\} \cap V,$$

where $c \in \mathbb{R}^n$, $0 < \|c\| < \frac{1}{2}$, and $V \subset \mathbb{R}^n$ is a vector subspace of dimension $k + 1$. In our results only $\mathcal{E}_k(\mathbb{B}^n)$ with $k = 1$ and $k = 2$ are relevant.

Theorem 2. *Let $f: \mathbb{B}^n \rightarrow \mathbb{R}$ be a function defined on the open unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. Assume that $n \geq 3$ and the restriction $f|_{\Sigma^2}$ is an analytic (resp. a Nash) function for every Euclidean 2-sphere $\Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n)$. Then f is an analytic (resp. a Nash) function.*

The analytic case in Theorem 2 is already settled in [3, Theorem 2]. It plays the key role in the proof of the Nash case.

The inversion

$$\mu: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad \mu(x) = \frac{x}{\|x\|^2},$$

is a biregular isomorphism. It maps $\mathbb{B}^n \setminus \{0\}$ onto the complement $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ of the closed unit ball $\overline{\mathbb{B}^n}$ and gives a one-to-one correspondence between the Euclidean k -spheres in $\mathcal{E}_k(\mathbb{B}^n)$ and the affine k -planes contained in $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$.

Proof of Theorem 2. As indicated above, we may assume that f is an analytic function and the restriction $f|_{\Sigma^2}$ is a Nash function for every Euclidean 2-sphere $\Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n)$. Hence the function

$$g := f \circ (\mu|_{\mathbb{R}^n \setminus \overline{\mathbb{B}^n}}): \mathbb{R}^n \setminus \overline{\mathbb{B}^n} \rightarrow \mathbb{R}$$

is analytic and its restriction to any affine 2-plane contained in $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ is a Nash function. Evidently, the restriction of g to any affine line contained in

$\mathbb{R}^n \setminus \overline{\mathbb{B}}^n$ is a Nash function. It follows from [2, Theorem 2.4] that g is a Nash function on $\mathbb{R}^n \setminus \overline{\mathbb{B}}^n$, and therefore so is f on $\mathbb{B}^n$. $\square$

Proposition 3. *Let $X \subset \mathbb{R}^m$ be an irreducible nonsingular real algebraic set of dimension $n \geq 1$, and let p be a point in X . Then there exists a linear map $\lambda: \mathbb{R}^m \rightarrow \mathbb{R}^n$ for which the following hold:*

- (i) *The restriction $\lambda|_X: X \rightarrow \mathbb{R}^n$ is a proper map with finite fibers (some fibers may be empty).*
- (ii) *The map $\lambda|_X$ is transverse to $\lambda(p)$.*

Proof. Let $Y := X - p$ be the translate of X . For each linear map $\alpha: \mathbb{R}^m \rightarrow \mathbb{R}^n$, the restriction $\alpha|_Y: Y \rightarrow \mathbb{R}^n$ induces a homomorphism of the coordinate rings (=rings of polynomial functions)

$$(\alpha|_Y)^*: A(\mathbb{R}^n) \rightarrow A(Y).$$

Let $L(m, n)$ be the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, which we identify with the space $M(n, m)$ of all n -by- m matrices with real entries. By a suitable version of the Noether normalization theorem, see [6, Theorem 13.3] and its proof, there is a nonempty Zariski open subset $\Omega \subset L(m, n)$ such that for each $\beta \in \Omega$, the homomorphism $(\beta|_Y)^*$ is injective and the ring $A(Y)$ is integral over $A(\mathbb{R}^n) \cong \text{Im}(\beta|_Y)^*$ (equivalently, $A(Y)$ is a finitely generated $A(\mathbb{R}^n)$ -module). It follows that β is surjective and the restriction $\beta|_Y$ is a proper map with finite fibers. Now we choose a linear map $\gamma \in \Omega$ such that the derivative of $\varphi := \gamma|_Y: Y \rightarrow \mathbb{R}^n$ at the origin $0 \in Y$ is an isomorphism. After a coordinate change, we may assume that γ is the canonical projection $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \rightarrow \mathbb{R}^n$.

For any constant $\varepsilon > 0$, we set

$$M_\varepsilon := \{t = (t_{ij}) \in M(n, m) : |t_{ij}| < \varepsilon \text{ for } 1 \leq i \leq n, 1 \leq j \leq m\}$$

and consider the map $\Phi: Y \times M_\varepsilon \rightarrow \mathbb{R}^n$ defined by

$$\Phi(x, t) = (x_1 + \sum_{j=1}^m t_{1j}x_j, \dots, x_n + \sum_{j=1}^m t_{nj}x_j)$$

where $x = (x_1, \dots, x_m) \in Y$ and $t = (t_{ij}) \in M_\varepsilon$. If ε is sufficiently small, then Φ is a submersion since for each point $x \neq 0$, the restriction of Φ to $\{x\} \times M_\varepsilon$ is a submersion, and φ is a submersion at the origin $0 \in Y$. Hence, according to the standard consequence of Sard's theorem [7, p. 79, Theorem 2.7], the map $\Phi_t: Y \rightarrow \mathbb{R}^n$, $\Phi_t(x) = \Phi(x, t)$, is transverse to the origin $0 \in \mathbb{R}^n$ for some $t \in M_\varepsilon$. There is a linear map $\lambda \in L(m, n)$ with $\lambda|_Y = \Phi_t$. If ε is small, then λ belongs to Ω and has the required properties. $\square$

Proof of Theorem 1. Let p be a point in X and let $\lambda: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear map as in Proposition 3. We can choose a constant $r > 0$ such that

$$(\lambda|_X)^{-1}(B(\lambda(p), r)) = U_1 \cup \dots \cup U_l,$$

where $B(\lambda(p), r) \subset \mathbb{R}^n$ is the open ball centered at $\lambda(p)$ with radius r , the U_i are pairwise disjoint open subsets of X , $\lambda|_{U_i}: U_i \rightarrow B(\lambda(p), r)$ are Nash

isomorphisms, and $p \in U_1$. Clearly, $l \leq d(X)$. Define the map $\pi: X \rightarrow \mathbb{R}^n$ by

$$\pi(x) = \frac{1}{r}(\lambda(x) - \lambda(p)) \text{ for } x \in X.$$

Then

$$\pi^{-1}(\mathbb{B}^n) = U_1 \cup \cdots \cup U_l$$

and the restriction $\pi|_{U_i}: U_i \rightarrow \mathbb{B}^n$ is a Nash isomorphism for $i = 1, \dots, l$.

If $\Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n)$, then $S(\Sigma^2) := \pi^{-1}(\Sigma^2) \in \mathcal{F}_2(X)$. Assume that for every $\Sigma^2 \in \mathcal{E}_2(\mathbb{B}^n)$, the restriction $f|_{S(\Sigma^2)}$ is an analytic (resp. a Nash) function. Then, by Theorem 2, the composite $f \circ (\pi|_{U_1})^{-1}: \mathbb{B}^n \rightarrow \mathbb{R}$ is an analytic (resp. a Nash) function. It follows that f is an analytic (resp. a Nash) function, the point $p \in X$ being arbitrary. $\square$

Making use of local coordinate charts and applying Theorem 2, we immediately obtain the following.

Theorem 4. *Let $f: M \rightarrow \mathbb{R}$ be a function defined on an analytic (resp. a Nash) manifold M of dimension $n \geq 3$. Assume that the restriction $f|_N$ is an analytic (resp. a Nash) function for every analytic (resp. Nash) submanifold $N \subset M$ homeomorphic to $\mathbb{S}^2$. Then f is an analytic (resp. a Nash) function.*

The analytic case in Theorem 4 is contained in [3, Theorem 1].

Replacing in Theorem 1 (resp. Theorem 2) $\mathcal{F}_2(X)$ by $\mathcal{F}_1(X)$ (resp. $\mathcal{E}_2(\mathbb{B}^n)$ by $\mathcal{E}_1(\mathbb{B}^n)$), one would get a false statement.

Counterexample 5. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y, z) = \begin{cases} \frac{x^8 + y(x^2 - y^3)^2 + z^4}{x^{10} + (x^2 - y^3)^2 + z^2} & \text{for } (x, y, z) \neq (0, 0, 0), \\ 0 & \text{for } (x, y, z) = (0, 0, 0). \end{cases}$$

Then the restriction of f is an analytic (resp. a Nash) function on each non-singular analytic (resp. Nash) curve in $\mathbb{R}^3$, but f is not even continuous at $(0, 0, 0)$.

To establish the first part of the assertion, it suffices to prove that for any nonsingular analytic curve $C \subset \mathbb{R}^3$ passing through $(0, 0, 0)$, the restriction $f|_C$ is analytic at $(0, 0, 0)$. Since C is nonsingular, it has near $(0, 0, 0)$ a local analytic parametrization

$$x = x(t), \quad y = y(t), \quad z = z(t) \text{ for } t \text{ near } 0 \in \mathbb{R},$$

where $x(0) = y(0) = z(0) = 0$, and at least one of the analytic functions $x(t)$, $y(t)$, $z(t)$ has a zero of order 1 at $t = 0$. It is not hard to check that the function $f(x(t), y(t), z(t))$ is analytic for t near $0 \in \mathbb{R}$. Thus $f|_C$ is analytic at $(0, 0, 0)$, as required.

Clearly, the function f is not continuous at $(0, 0, 0)$ since on the curve $x^2 - y^3 = 0$, $z = 0$, it is equal to $\frac{1}{x^2}$ away from $(0, 0, 0)$.

However, for C^∞ functions, we have the following version of Theorem 1.

Theorem 6. *Let $f: X \rightarrow \mathbb{R}$ be a C^∞ function defined on an irreducible nonsingular algebraic set $X \subset \mathbb{R}^m$ of dimension $n \geq 2$. Assume that the restriction $f|_C$ is an analytic (resp. a Nash) function for every algebraic curve $C \in \mathcal{F}_1(X)$. Then f is an analytic (resp. a Nash) function.*

Proof. We argue as in the proof of Theorem 1, substituting Theorem 7 below for Theorem 2. $\square$

Theorem 7. *Let $f: \mathbb{B}^n \rightarrow \mathbb{R}$ be a C^∞ function defined on the unit open ball $\mathbb{B}^n \subset \mathbb{R}^n$. Assume that $n \geq 2$ and the restriction $f|_{\Sigma^1}$ is an analytic (resp. a Nash) function for every Euclidean 1-sphere $\Sigma^1 \in \mathcal{E}_1(\mathbb{B}^n)$. Then f is an analytic (resp. a Nash) function.*

Proof. To begin with, we consider the analytic case, assuming that the restriction $f|_{\Sigma^1}$ is an analytic function for every Euclidean 1-sphere $\Sigma^1 \in \mathcal{E}_1(\mathbb{B}^n)$.

First we prove analyticity of f on the punctured unit ball $\mathbb{B}^n \setminus \{0\}$. This is equivalent to proving analyticity of the function

$$g := f \circ \mu|_{\mathbb{R}^n \setminus \overline{\mathbb{B}^n}}: \mathbb{R}^n \setminus \overline{\mathbb{B}^n} \rightarrow \mathbb{R}.$$

The problem is local, so fix a point $b \in \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$. Our goal is to show that g is analytic at b . Evidently, g is of class C^∞ and its restriction to any affine line contained in $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ is analytic. Let

$\mathcal{L} :=$ the set of all affine lines contained in $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ and passing through b .

The union C of all lines in $\mathcal{L}$ is a cone in $\mathbb{R}^n$ and the set $U := C \setminus \{b\}$ is open in $\mathbb{R}^n$. Consider the series $\Sigma_k P_k$ of homogeneous polynomials in n variables $x = (x_1, \dots, x_n)$, where

$$P_k(x) = \frac{1}{k!} \sum_{|\alpha|=k} \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(b) x_1^{\alpha_1} \dots x_n^{\alpha_n},$$

$$\alpha = (\alpha_1, \dots, \alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n.$$

By construction, for each vector line $\Lambda \subset \mathbb{R}^n$ with $b + \Lambda \in \mathcal{L}$, the series $\Sigma_k P_k(x)$ converges to $g(b + x)$ for all $x \in \Lambda$ in a neighborhood of $0 \in \Lambda$. Hence, in view of Lemma 8 below, the series $\Sigma_k P_k(x)$ converges to an analytic function $\gamma: B \rightarrow \mathbb{R}$ defined on an open ball $B \subset \mathbb{R}^n$ centered at the origin. Clearly, the function

$$\gamma_b: B_b \rightarrow \mathbb{R}, \gamma_b(b + x) := \gamma(x) \text{ for } x \in B,$$

is analytic on the ball $B_b := b + B$ centered at b . Since $g = \gamma_b$ on $B_b \cap L$ for all $L \in \mathcal{L}$, it follows that

$$g = \gamma_b \text{ on } B_b \cap C.$$

We claim that $g = \gamma_b$ in a neighborhood of b in $\mathbb{R}^n$. Indeed, fix an affine line $L_0 \in \mathcal{L}$ and let K be any affine line contained in $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$ that is parallel to $L_0 \in \mathcal{L}$ and satisfies $K \cap B_b \cap U \neq \emptyset$. The functions g and γ_b coincide on $K \cap B_b \cap U$, hence the analytic functions $g|_{K \cap B_b}$ and $\gamma_b|_{K \cap B_b}$ are equal. The union of the sets $K \cap B_b$, for all K as above, is a neighborhood of b in $\mathbb{R}^n$. It follows that g and γ_b are equal in a neighborhood of b in $\mathbb{R}^n$, as claimed. So f is an analytic function on $\mathbb{B}^n$, except possibly at the origin.

To prove that f is analytic at the origin, we use the inversion

$$\mu_a: \mathbb{R}^n \setminus \{a\} \rightarrow \mathbb{R}^n \setminus \{a\}, \mu_a(x) = \mu(x - a) + a,$$

centered at a point $a \in \mathbb{B}^n \setminus \{0\}$. Let $U_a := \mu_a(\mathbb{B}^n \setminus \{a\})$. The function

$$h := f \circ (\mu_a|_{U_a})^{-1} : U_a \rightarrow \mathbb{R}$$

is of class $\mathcal{C}^\infty$, analytic except possibly at $\mu_a(0)$, and its restriction is analytic on every affine line passing through $\mu_a(0)$ and contained in U_a . Arguing as above, we show that the Taylor series of h at $\mu_a(0)$ converges to h in a neighborhood of $\mu_a(0)$ in $\mathbb{R}^n$. Thus f is analytic at the origin, and therefore everywhere on $\mathbb{B}^n$.

In the Nash case, we proceed as in the proof of Theorem 2. $\square$

We have used the following result, see [4, Lemma 3] for the proof.

Lemma 8. *Let $\sum_k^\infty P_k$ be a series of real homogenous polynomials in n variables, $\deg P_k = k$. Assume that there exists a nonempty open subset $\Omega \subset \mathbb{S}^{n-1}$ such that for every point $a \in \Omega$, one can find a constant $\rho_a > 0$ such that the series $\sum_k^\infty P_k(x)$ converges at $x = \rho_a a$. Then there exist constants $c > 0, r > 0$ such that*

$$|P_k(z)| \leq \frac{c}{2^k} \text{ for } z \in \mathbb{C}^n, \|z\| \leq r, k \geq 0.$$

In particular, the function $z \mapsto \sum_k^\infty P_k(z)$ is holomorphic in the ball $\|z\| < r, z \in \mathbb{C}^n$.

Working on local coordinate charts, we derive from Theorem 7 the following.

Theorem 9. *Let $f: M \rightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function defined on an analytic (resp. a Nash) manifold M of dimension $n \geq 2$. Assume that the restriction $f|_C$ is an analytic (resp. a Nash) function for every analytic (resp. Nash) submanifold $C \subset M$ homeomorphic to $\mathbb{S}^1$. Then f is an analytic (resp. a Nash) function.*

One can compare Theorems 6 and 9 with the following example.

Example 10. The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy \exp\left(-\frac{1}{x^2+y^2}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

is of class $\mathcal{C}^\infty$ and analytic with respect to each variable separately. However, f is not analytic as a function of two variables.

Acknowledgements. The second named author was partially supported by the National Science Center (Poland) under Grant Number 2018/31/B/ST1/01059.

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Received: 7 July 2021

Accepted: 8 September 2021.