A determinant involving Ramanujan sums and So's conjecture

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Abstract. We compute the determinant of a matrix containing Ramanujan sums associated to the divisors of an integer n , and use this computation to prove a weak version of So's conjecture on circulant graphs with integral spectrum.

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1. Introduction and results. For integers n, m , define the Ramanujan sum

$$c(m, n) = \sum_{(h, n)=1} e^{2\pi i \frac{hm}{n}} = \mu\left(\frac{n}{(n, m)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n, m)}\right)}.$$

Here μ denotes the Möbius function, φ Euler's function, and (n, m) is the greatest common divisor of n and m . Later $\tau(n)$ denotes the number of divisors of n .

Ramanujan sums occur naturally in various problems involving discrete Fourier transforms. Here we only want to stress the relation to arithmetic functions, as described in the book by Schwarz and Spilker [2].

Denote by $\mathcal{D}(n)$ the set of divisors of n . In this note, we compute the determinant of the matrix $C_n = (c(d, t))_{d, t \in \mathcal{D}(n)}$. We prove the following.

Theorem 1. *We have $\det C_n = \pm n^{\frac{\tau(n)}{2}}$.*

Note that as we did not specify an order for the divisors, the determinant is only determined up to a sign. From Theorem 1, we immediately obtain the following.

Corollary 2. *For every n , we have that the matrix C_n is invertible.*

While the calculation of determinants involving arithmetic functions is an area of some activity going back to Smith's computation [3] of $\det((i, j))_{i,j=1}^n$, the main reason that we are interested in this matrix comes from its connection to certain Cayley graphs. Let $S = \{a_1, \dots, a_k\} \subseteq \{0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ be a generating system of $\mathbb{Z}/n\mathbb{Z}$, and $\mathcal{G}$ the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with respect to $\pm S$. So [4] showed that $\mathcal{G}$ has integral spectrum if and only if there exists a set $T \subseteq D(n)$ such that $S = \{i : (i, n) \in T\}$. He conjectured that the spectrum of $\mathcal{G}$ determines n and S . The multi-set of eigenvalues of $\mathcal{G}$ is $\{\sum_{d \in S} c(\ell, \frac{n}{d}) : 1 \leq \ell \leq n\}$. As a weakening of So's conjecture, Sander and Sander [1] conjectured that if S is a set of divisors of n , then the vector $(\sum_{d \in S} c(\ell, \frac{n}{d}))_{\ell=1}^n$ determines S . As $c(\ell, \frac{n}{d})$ depends only on (ℓ, n) , this is equivalent to the statement that the restriction of C_n to $\{0, 1\}^{\tau(n)}$ is injective. We conclude that Corollary 2 implies the following.

Corollary 3. *If S is a set of divisors of the integer n , then the vector $(\sum_{d \in S} c(\ell, \frac{n}{d}))_{\ell=1}^n$ determines S .*

The same result has been obtained independently by Mönius (unpublished).

The function that maps a divisor set S to the multi-set of eigenvalues of the associated Cayley graph can be seen as the composition of the map sending S to the spectral vector and the forgetful map interpreting the vector as a multi-set. So by Corollary 3, So's conjecture is equivalent to the question when the second map is injective. Here we use Corollary 3 to prove the following. As an abbreviation, we put $L(n) = \exp((\log n)^{2/3} \log \log n)$.

Proposition 4. 1. *Suppose that n has no pair of distinct divisors t_1, t_2 such that $\varphi(n/t_1) \leq \varphi(n/t_2) < 2$ with the possible exception of $t_1 = n, t_2 = n/2$. Then So's conjecture holds for n .*
 2. *So's conjecture holds for almost all integers n such that n has no prime divisor $< L(n)$.*

The first statement generalizes the result by Sander and Sander [1] who proved So's conjecture if n is a prime power.

2. Proofs. We prove Theorem 1 by induction over the number of prime divisors of n . For $n = 1$, we have $C_1 = (1)$. Now let n, a be positive integers, p a prime number which does not divide n , we shall compute $\det C_{p^a n}$. Let $d_1, \dots, d_k$ be the divisors of n . We order the divisors of $p^a n$ as

$$d_1, d_2, \dots, d_k; pd_1, pd_2, \dots, pd_k; p^2 d_1, \dots, p^2 d_k; \dots; p^a d_1, \dots, p^a d_k.$$

Then $C_{p^a n}$ can be written as an $(a+1) \times (a+1)$ block matrix

$$\begin{pmatrix} A_{00} & A_{01} & \dots & A_{0a} \\ A_{10} & A_{11} & \dots & A_{1a} \\ \vdots & \vdots & & \vdots \\ A_{a0} & A_{a1} & \dots & A_{aa} \end{pmatrix},$$

where

$$A_{ij} = \left(\mu \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right) \frac{\varphi(p^j d_\nu)}{\varphi \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right)} \right)_{\nu, \mu=1}^k.$$

We now compute A_{ij} . It is clear that $A_{00} = C_n$. If $j \geq i + 2$, then p^2 divides $\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)}$, thus, $A_{ij} = 0$.

If $j = i + 1$, then

$$\begin{aligned} \mu \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right) \frac{\varphi(p^j d_\nu)}{\varphi \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right)} &= \mu \left(p \frac{d_\nu}{(d_\nu, d_\mu)} \right) \frac{p^{j-1}(p-1)\varphi(d_\nu)}{(p-1)\varphi \left(\frac{d_\nu}{(d_\nu, d_\mu)} \right)} \\ &= -p^{j-1} \mu \left(\frac{d_\nu}{(d_\nu, d_\mu)} \right) \frac{\varphi(d_\nu)}{\varphi \left(\frac{d_\nu}{(d_\nu, d_\mu)} \right)}, \end{aligned}$$

and therefore $A_{ji} = -p^{j-1} A_{00}$.

If $j \leq i$, then

$$\mu \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right) \frac{\varphi(p^j d_\nu)}{\varphi \left(\frac{p^j d_\nu}{(p^j d_\nu, p^i d_\mu)} \right)} = \mu \left(\frac{d_\nu}{(d_\nu, d_\mu)} \right) \frac{p^{j-1}(p-1)\varphi(d_\nu)}{\varphi \left(\frac{d_\nu}{(d_\nu, d_\mu)} \right)},$$

that is, $A_{ji} = (p-1)p^{j-1} A_{00}$. We conclude that up to a permutation of the rows and columns $C_{p^a n}$ equals

$$\begin{pmatrix} C_n & -C_n & 0 & 0 & \dots & 0 \\ C_n & (p-1)C_n & -pC_n & 0 & \dots & 0 \\ C_n & (p-1)C_n & p(p-1)C_n & -p^2C_n & \dots & 0 \\ C_n & (p-1)C_n & p(p-1)C_n & p^2(p-1)C_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & (p-1)C_n & p(p-1)C_n & p^2(p-1)C_n & \dots & -p^{a-1}C_n \\ C_n & (p-1)C_n & p(p-1)C_n & p^2(p-1)C_n & \dots & p^{a-1}(p-1)C_n \end{pmatrix}.$$

We now add the first block column to the second, then the second to the third, and continue in this way. We find that

$$\begin{aligned} \det C_{p^a n} &= \pm \det \begin{pmatrix} C_n & 0 & 0 & 0 & \dots & 0 \\ C_n & pC_n & 0 & 0 & \dots & 0 \\ C_n & pC_n & p^2C_n & 0 & \dots & 0 \\ C_n & pC_n & p^2C_n & p^3C_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & pC_n & p^2C_n & p^3C_n & \dots & 0 \\ C_n & pC_n & p^2C_n & p^3C_n & \dots & p^aC_n \end{pmatrix} \\ &= \pm p^{\frac{a(a+1)}{2}\tau(n)} (\det C_n)^{a+1}. \end{aligned}$$

By our inductive hypothesis, we obtain

$$|\det C_{p^a n}| = p^{\frac{a(a+1)}{2}\tau(n)} n^{\frac{\tau(a)}{2}(a+1)} = (p^a n)^{\frac{(a+1)\tau(n)}{2}} = (p^a n)^{\frac{\tau(p^a n)}{2}},$$

and the proof of Theorem 1 is complete.

As the proofs of Corollary 2 and 3 are obvious, we now turn to the proof of Proposition 4. Suppose there are two different spectral vectors v_1, v_2 associated to the divisor sets S_1, S_2 which are mapped to the same multi-set. Then there is some integer x such that v_1 and v_2 have the same number of entries equal to x , but there exists some index i such that the i -th entry of v_1 is equal to x , while the i -th entry of v_2 is not equal to x . As the level sets of v_1, v_2 are unions of sets of the form $\{i : (i, n) = t\}$ for some divisor t of n , we obtain that there are two non-empty sets T_1, T_2 of divisors of n such that $T_1 \neq T_2$, $\sum_{t \in T_1} \varphi(n/t) = \sum_{t \in T_2} \varphi(n/t)$, and $\sum_{d \in S_1} c(t_1, \frac{n}{d}) = \sum_{d \in S_2} c(t_2, \frac{n}{d})$ for all $t_1 \in T_1, t_2 \in T_2$.

We now show that under the condition in Proposition 4(1) this cannot happen. Suppose first that $n \in T_1$. We have $c(n, n/d) = \varphi(n/d) \geq c(n/t, n/d)$ for all t , together with the fact that $\sum_{d \in S_1} \varphi(n/d) = \sum_{d \in S_2} \varphi(n/d)$ we obtain that $\sum_{d \in S_1} c(n, n/d) = \sum_{d \in S_2} c(n, n/d)$ implies $(n/d, n/t) = n/d$ for all $d \in S_2$. In other words, we have $t|d$ for all $d \in S_2$. But this contradicts the assumption that S_2 is a generating system of $\mathbb{Z}/n\mathbb{Z}$.

If n is not in T_1 , let $t_0 \in T_1 \Delta T_2$ be minimal. As T_1 and T_2 are different, the symmetric difference is non-empty, that is, t_0 certainly exists. Without loss, we may assume that $t_0 \in T_1$, and we have

$$\begin{aligned} 0 &= \sum_{t \in T_1} \varphi(n/t) - \sum_{t \in T_2} \varphi(n/t) \geq \varphi(n/t_0) - \sum_{\substack{t \in T_1 \cup T_2 \\ t > t_0}} \varphi(n/t) \\ &\geq \varphi(n/t_0) - \frac{\varphi(n/t_0)}{2} - \frac{\varphi(n/t_0)}{4} - \dots - \frac{\varphi(n/t_0)}{2^{\tau(n)}} > 0, \end{aligned}$$

which is impossible. Hence, the first part of the proposition follows.

For the second statement of the proposition, we show that almost all integers n , which have no prime divisor $p < L(n)$, satisfy the assumption of the first statement. Denote by $P^-(n)$ the smallest prime divisor of n , and let $\Phi(x, y)$ be the number of integers $n \leq x$ such that $P^-(n) \leq y$. The following is well known, see e.g. [5, chapter III.6.2] for much more precise results.

Lemma 5. *There exist positive constants c_1, c_2 such that for all real numbers x, y satisfying $2 \leq y \leq \frac{x}{2}$, we have*

$$c_1 \frac{x}{\log y} \leq \Phi(x, y) \leq c_2 \frac{x}{\log y}.$$

We now show that for almost all integers n satisfying $P^-(n) \geq L(n)$, there do not exist divisors $d_1 \neq d_2$ of n such that $\varphi(d_1) \leq \varphi(d_2) < 2\varphi(d_1)$.

If $P^-(n) \geq L(n)$, then for every divisor d of n , we have

$$\frac{\varphi(d)}{d} \geq \left(1 - \frac{1}{L(n)}\right)^{\omega(d)} \geq \left(1 - \frac{1}{L(n)}\right)^{\frac{\log n}{\log 2}} = 1 + o(1),$$

it suffices to estimate the number of integers without small prime divisors, which have divisors d_1, d_2 with $d_1 < d_2 < 3d_1$. To do so, we estimate the number of triplets (n, d_1, d_2) , where $n \in [x, 2x]$, $P^-(n) > L(n)$, and d_1, d_2 are divisors satisfying $d_1 < d_2 < 3d_1$.

Put $y = L(x)$. Then the number of tripels is bounded above by

$$\begin{aligned} & \sum_{k \geq 0} \sum_{\substack{d_1, d_2 \in [2^k y, 6 \cdot 2^k y] \\ d_1 \neq d_2}} |\{n \leq x : P^-(n) \geq y, [d_1, d_2] | n\}| \\ &= \sum_{k \geq 0} \sum_{\substack{d_1, d_2 \in [2^k y, 6 \cdot 2^k y] \\ d_1 \neq d_2 \\ P^-(d_1 d_2) \geq y}} \Phi\left(\frac{x}{[d_1, d_2]}, y\right) \ll \frac{x}{\log y} \sum_{k \geq 0} \sum_{\substack{d_1, d_2 \in [2^k y, 6 \cdot 2^k y] \\ d_1 \neq d_2 \\ P^-(d_1 d_2) \geq y}} \frac{1}{[d_1, d_2]}. \end{aligned}$$

Introducing t as the greatest common divisor of d_1 and d_2 , this quantity becomes

$$\begin{aligned} & \frac{x}{\log y} \sum_{k \geq 0} \sum_{t: P^-(t) \geq y} \sum_{\substack{d'_1, d'_2 \in \left[\frac{2^k y}{t}, 6 \cdot \frac{2^k y}{t}\right] \\ d'_1 \neq d'_2 \\ P^-(d'_1 d'_2) \geq y}} \frac{1}{t d'_1 d'_2} \\ & \ll \frac{x}{\log^3 y} \sum_{k=0}^{\log x} \sum_{t: P^-(t) \geq y} \frac{1}{t} \ll \frac{x}{\log^2 x} \log^4 y = o\left(\frac{x}{\log y}\right), \end{aligned}$$

and the proof is complete.

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