Subcritical polarisations of symplectic manifolds have degree one

HANSJÖRG GEIGES , KEVIN SPORBECK, AND KAI ZEHMISCH

Abstract. We show that if the complement of a Donaldson hypersurface in a closed, integral symplectic manifold has the homology of a subcritical Stein manifold, then the hypersurface is of degree one. In particular, this demonstrates a conjecture by Biran and Cieliebak on subcritical polarisations of symplectic manifolds. Our proof is based on a simple homological argument using ideas of Kulkarni–Wood.

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1. Donaldson hypersurfaces and symplectic polarisations. Let (M, ω) be a closed, connected, integral symplectic manifold, that is, the de Rham cohomology class $[\omega]_{\text{dR}}$ lies in the image of the homomorphism $H^2(M) \rightarrow H^2_{\text{dR}}(M) = H^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \rightarrow \mathbb{R}$. The cohomology classes in $H^2(M)$ mapping to $[\omega]_{\text{dR}}$ are called *integral lifts*, and by abuse of notation, we shall write $[\omega]$ for any such lift. Following McDuff and Salamon [11, Section 14.5], we call a codimension 2 symplectic submanifold $\Sigma \subset M$ a *Donaldson hypersurface* if it is Poincaré dual to $d[\omega] \in H^2(M)$ for some integral lift $[\omega]$ and some (necessarily positive) integer d . Donaldson [5] has established the existence of such hypersurfaces for any sufficiently large d .

The pair (M, Σ) is called a *polarisation* of (M, ω) , and the number $d \in \mathbb{N}$, the *degree* of the polarisation. Biran and Cieliebak [2] studied these polarisations in the Kähler case, where ω admits a compatible *integrable* almost complex structure J . In that setting, the complement $(M \setminus \Sigma, J)$ admits in a natural way the structure of a Stein manifold.

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As shown recently by Giroux [8], building on work of Cieliebak–Eliashberg, even in the non-Kähler case the complement of a symplectic hypersurface $\Sigma \subset M$ found by Donaldson’s construction admits the structure of a Stein manifold. Here, of course, the complex structure on $M \setminus \Sigma$ does not, in general, extend over Σ . Complements of Donaldson hypersurfaces are also studied in [4].

2. Subcritical polarisations. The focus of Biran and Cieliebak [2] lay on *subcritical* polarisations of Kähler manifolds, which means that $(M \setminus \Sigma, J)$ admits a plurisubharmonic Morse function φ all of whose critical points have, for $\dim M = 2n$, index less than n . (They also assumed that φ coincides with the plurisubharmonic function defining the natural Stein structure outside a compact set containing all critical points of φ .)

More generally, McDuff and Salamon [11, p. 504] propose the study of polarisations (M, Σ) where the complement $M \setminus \Sigma$ is homotopy equivalent to a subcritical Stein manifold (of finite type). We relax this condition a little further and call (M, Σ) *homologically subcritical* if $M \setminus \Sigma$ has the *homology* of a subcritical Stein manifold, that is, of a CW-complex containing finitely many cells up to dimension at most $n - 1$. This means that there is some $\ell \leq n - 1$ such that $H_k(M \setminus \Sigma)$ vanishes for $k \geq \ell + 1$ and $H_\ell(M \setminus \Sigma)$ is torsion-free.

Motivated by the many examples they could construct, Biran and Cieliebak [2, p. 751] conjectured that subcritical polarisations necessarily have degree 1. They suggested an approach to this conjecture using either symplectic or contact homology. A rough sketch of a proof along these lines, in the language of symplectic field theory, was given by Eliashberg–Givental–Hofer [6, p. 661]. A missing assumption $c_1(M \setminus \Sigma) = 0$ of that argument and a few more details—still short of a complete proof—were added by J. He [9, Proposition 4.2], who appeals to Gromov–Witten theory and polyfolds.

Here is our main result, which entails the conjecture of Biran–Cieliebak.

Theorem 1. *Let (M, ω) be a closed, integral symplectic manifold, and $\Sigma \subset M$ a compact symplectic submanifold of codimension 2, Poincaré dual to the integral cohomology class $d[\omega]$ for some (positive) integer d . If (M, Σ) is homologically subcritical, then $d[\omega]/\text{torsion}$ is indivisible in $H^2(M)/\text{torsion}$. In particular, $d = 1$.*

Notation. The free part of $H^2(M)$ is not a well-defined subgroup of $H^2(M)$, but the torsion part is, and so is the quotient group $H^2(M)/\text{torsion}$, which is free abelian. We write $d[\omega]/\text{torsion}$ for the class represented by $d[\omega]$ in this quotient. This class is determined by $[\omega]_{\text{dR}}$ and does not depend on the choice of integral lift $[\omega]$.

Our proof is devoid of any sophisticated machinery. The assumption on (M, Σ) to be homologically subcritical guarantees the surjectivity of a certain homomorphism in homology described by Kulkarni and Wood [10]; this implies the claimed indivisibility.

3. The Kulkarni–Wood homomorphism. We consider a pair (M, Σ) consisting of a closed, connected, oriented manifold M of dimension $2n$, and a compact,

oriented hypersurface $\Sigma \subset M$ of codimension 2. No symplectic assumptions are required in this section.

Write $i: \Sigma \rightarrow M$ for the inclusion map. The Poincaré duality isomorphisms on M and Σ from cohomology to homology, given by capping with the fundamental class, are denoted by PD_M and PD_Σ , respectively.

In their study of the topology of complex hypersurfaces, Kulkarni and Wood [10] used the following composition, which we call the *Kulkarni–Wood homomorphism*:

$$\begin{aligned} \Phi_{\text{KW}}: H^k(M) &\xrightarrow{i^*} H^k(\Sigma) \xrightarrow{\text{PD}_\Sigma} H_{2n-2-k}(\Sigma) \xrightarrow{i_*} \\ &\xrightarrow{i_*} H_{2n-2-k}(M) \xrightarrow{\text{PD}_M^{-1}} H^{k+2}(M). \end{aligned}$$

Lemma 2. *The Kulkarni–Wood homomorphism equals the cup product with the cohomology class $\sigma := \text{PD}_M^{-1}(i_*[\Sigma]) \in H^2(M)$.*

Proof. For $\alpha \in H^k(M)$, we compute

$$\begin{aligned} \Phi_{\text{KW}}(\alpha) &= \text{PD}_M^{-1} i_* \text{PD}_\Sigma i^* \alpha = \text{PD}_M^{-1} i_* (i^* \alpha \cap [\Sigma]) \\ &= \text{PD}_M^{-1} (\alpha \cap i_* [\Sigma]) = \text{PD}_M^{-1} (\alpha \cap \text{PD}_M(\sigma)) \\ &= \text{PD}_M^{-1} (\alpha \cap (\sigma \cap [M])) = \text{PD}_M^{-1} ((\alpha \cup \sigma) \cap [M]) \\ &= \alpha \cup \sigma; \end{aligned}$$

all the formulae for cup and cap products used in this chain of identities can be found in [3, Theorem VI.5.2]. $\square$

Lemma 3. *If the complement $M \setminus \Sigma$ has the homology type of a CW-complex of dimension ℓ for some $\ell \leq n-1$, then $\Phi_{\text{KW}}: H^k(M) \rightarrow H^{k+2}(M)$ is surjective in the range $\ell-1 \leq k \leq 2n-\ell-2$.*

Proof. Write $\nu\Sigma$ for an open tubular neighbourhood of Σ in M . By homotopy, excision, duality, and the universal coefficient theorem, we have

$$\begin{aligned} H_k(M, \Sigma) &\cong H_k(M, \nu\Sigma) \cong H_k(M \setminus \nu\Sigma, \partial(\nu\Sigma)) \\ &\cong H^{2n-k}(M \setminus \nu\Sigma) \\ &\cong FH_{2n-k}(M \setminus \Sigma) \oplus TH_{2n-k-1}(M \setminus \Sigma), \end{aligned}$$

where F and T denote the free and the torsion part, respectively. This vanishes for $2n-k-1 \geq \ell$, that is, for $k \leq 2n-\ell-1$. It follows that the homomorphism $i_*: H_{2n-2-k}(\Sigma) \rightarrow H_{2n-2-k}(M)$ is surjective for $2n-2-k \leq 2n-\ell-1$, or $k \geq \ell-1$.

Similarly (or directly by the Poincaré–Lefschetz duality), we have

$$H^k(M, \Sigma) \cong H_{2n-k}(M \setminus \Sigma),$$

which vanishes for $2n-k \geq \ell+1$, that is, for $k \leq 2n-\ell-1$. Hence, the homomorphism $i^*: H^k(M) \rightarrow H^k(\Sigma)$ is surjective for $k+1 \leq 2n-\ell-1$, that is, for $k \leq 2n-\ell-2$. $\square$

4. Proof of Theorem 1. Under the assumptions of Theorem 1, the homomorphism $\Phi_{\text{KW}}: H^k(M) \rightarrow H^{k+2}(M)$ is surjective at least in the range $n - 2 \leq k \leq n - 1$; simply set $\ell = n - 1$ in Lemma 3. Thus, we can pick an even number $k = 2m$ in this range. The free part of $H^{2m+2}(M)$ is non-trivial since this cohomology group contains the element $[\omega]^{m+1}$ of infinite order.

On the other hand, Φ_{KW} is given by the cup product with $d[\omega]$, as shown in Lemma 2, so we have the commutative diagram

$$\begin{array}{ccc} H^2(M) & \xrightarrow{\Phi_{\text{KW}}} & H^{2m+2}(M) \\ \downarrow & & \downarrow \\ H^2(M)/\text{torsion} & \xrightarrow{\cup(d[\omega]/\text{torsion})} & H^{2m+2}(M)/\text{torsion}. \end{array}$$

If $d[\omega]/\text{torsion}$ were divisible, the map on free abelian groups at the bottom of this diagram would not be surjective, and neither would be Φ_{KW} .

Remark 4. The real Euler class of the circle bundle $\partial(\nu\Sigma)$ equals $d[\omega]_{\text{dR}}$, and the natural Boothby–Wang contact structure on this bundle has an exact convex filling by the complement $M \setminus \nu\Sigma$, see [7, Lemma 3], [8, Proposition 5], and [4, Lemma 2.2]. With [1, Theorem 1.2], the condition ‘homologically subcritical’ of Theorem 1 may be replaced by assuming the existence of *some* subcritical Stein filling of this Boothby–Wang contact structure.

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