The Stieltjes condition and multidimensional $\mathcal{K}$ -moment problems

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Abstract. We prove a solvability theorem for the Stieltjes problem on $\mathbb{R}^d$ which is based on the multivariate Stieltjes condition $\sum_{n=1}^{\infty} L(x_j^n)^{-1/(2n)} = +\infty$, $j = 1, \dots, d$. This result is applied to derive a new solvability theorem for the moment problem on unbounded semi-algebraic subsets of $\mathbb{R}^d$.

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1. Introduction. It is well known that the moment problem on unbounded semi-algebraic sets of $\mathbb{R}^d$ leads to new and principal difficulties, concerning the existence and the uniqueness of solutions. A rather general existence theorem is the fibre theorem ([11], see also [2, Chapter 4] or [14, Chapter 13]). As shown in [13], most of the general existence results that are based on positivity conditions can be derived from this theorem. The fibre theorem requires the existence of “sufficiently many” bounded polynomials on the semi-algebraic set.

However, for many unbounded semi-algebraic sets, there are no bounded polynomials except from the constants and positivity on the corresponding preorderings is not enough to ensure the existence of solutions. For such sets, additional assumptions such as growth conditions on the moments are required. The most famous and powerful result in this respect is Nussbaum’s theorem [9]. It says that if L is a linear functional on $\mathbb{R}[x_1, \dots, x_d]$ such that $L(p^2) \geq 0$ for $p \in \mathbb{R}[x_1, \dots, x_d]$ and satisfying the *Carleman condition*

$$\sum_{n=1}^{\infty} L(x_j^{2n})^{-1/(2n)} = +\infty, \quad j = 1, \dots, d, \quad (1)$$

then L is a moment functional with a unique representing measure on $\mathbb{R}^d$.

The aim of this paper is to prove two theorems on the multidimensional moment problem that are based on another growth condition (6), called *Stieltjes condition*. Comparing the powers of $L(x_j^{2^n})$ in both cases, we see that the Stieltjes condition (6) is weaker than the Carleman condition (1). The first main result (Theorem 1) is the counter-part of Nussbaum's theorem for the Stieltjes moment problem. The second main result (Theorem 3) deals with the $\mathcal{K}$ -moment problem for some in general noncompact semi-algebraic set $\mathcal{K}$.

Let $\mathcal{K}$ be a closed subset of $\mathbb{R}^d$. The $\mathcal{K}$ -moment problem asks the following question: *When is a real d -sequence $s = (s_n)_{n \in \mathbb{N}_0^d}$ a $\mathcal{K}$ -moment sequence*, that is, when does there exist a Radon measure μ supported on $\mathcal{K}$ such that

$$s_n = \int_{\mathcal{K}} x^n d\mu(x) \text{ for all } n \in \mathbb{N}_0^d? \quad (2)$$

(Here we tacitly mean that the corresponding integral is finite and equal to s_n and we use the multi-index notation $x^n = x_1^{n_1} \cdots x_d^{n_d}$ for $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$.)

There is a one-to-one correspondence between real d -sequences $s = (s_n)_{n \in \mathbb{N}_0^d}$ and real-valued linear functionals on the polynomial algebra $\mathbb{R}[x_1, \dots, x_d]$ given by $L_s(x^n) = s_n, n \in \mathbb{N}_0^d$. The functional L_s is called the Riesz functional associated with s . Then (2) is equivalent to

$$L_s(p) = \int_{\mathcal{K}} p(x) d\mu(x) \text{ for all } p \in \mathbb{R}[x_1, \dots, x_d]. \quad (3)$$

A linear functional on $\mathbb{R}[x_1, \dots, x_d]$ of this form is called a $\mathcal{K}$ -moment functional. In the special case $\mathcal{K} = \mathbb{R}_+^d$, where $\mathbb{R}_+ := [0, +\infty)$, we call such a functional *Stieltjes moment functional*. If there is no support requirement for the measure, that is, if $\mathcal{K} = \mathbb{R}^d$, the functional is called a *moment functional*. The functional-analytic approach to the moment problem uses the integral representation (3) of functionals rather than the representation (2) of moments.

2. A theorem for the Stieltjes moment problem on $\mathbb{R}^d$. In the case $d = 1$, a necessary and sufficient condition for being a Stieltjes moment sequence is that the sequences $(s_n)_{n \in \mathbb{N}_0}$ and $(s_{n+1})_{n \in \mathbb{N}_0}$ are positive semi-definite, or equivalently,

$$L_s(p^2) \geq 0 \text{ and } L_s(xp^2) \geq 0 \text{ for } p \in \mathbb{R}[x]. \quad (4)$$

In dimensions $d \geq 2$, the Stieltjes moment problem is much more subtle and there is no such simple solvability criterion. The counter-part of (4) are the conditions in (5). However, (5) is necessary, but it is not sufficient for L to be a Stieltjes moment functional. The reason for the latter is that there are nonnegative polynomials on $\mathbb{R}_+$ which are not of the form $p_1 + xp_2$, with p_1, p_2 sums of squares.

The following theorem says that the positivity assumption (5) and the Stieltjes condition (6) imply the existence and uniqueness of a solution of the Stieltjes moment problem.

Theorem 1. *Let L be a linear functional on $\mathbb{R}[x_1, \dots, x_d]$ such that*

$$L(p^2) \geq 0 \text{ and } L(x_j p^2) \geq 0 \text{ for } p \in \mathbb{R}[x_1, \dots, x_d], \ j = 1, \dots, d. \quad (5)$$

Suppose that

$$\sum_{n=1}^{\infty} L(x_j^n)^{-1/(2n)} = +\infty \text{ for } j = 1, \dots, d. \quad (6)$$

Then L is a Stieltjes moment functional which has a unique representing measure.

Before we discuss this theorem, we state a well known technical lemma. For the convenience of the reader, we include a proof of the lemma.

Lemma 2. *Let $s = (s_n)_{n \in \mathbb{N}_0}$ be a real sequence with $s_0 = 1$. Suppose that s and the shifted sequence $Es := (s_{n+1})_{n \in \mathbb{N}_0}$ are positive semidefinite, that is,*

$$\sum_{k,l=0}^n s_{k+l} \xi_k \xi_l \geq 0 \text{ and } \sum_{k,l=0}^n s_{k+l+1} \xi_k \xi_l \geq 0 \text{ for all } \xi_0, \xi_1, \dots, \xi_n \in \mathbb{R}, n \in \mathbb{N}_0.$$

Let $m \in \mathbb{N}$. Then we have

$$\sum_{n=1}^{\infty} s_n^{-1/(2n)} = +\infty \quad (7)$$

if and only if

$$\sum_{n=1}^{\infty} s_{nm}^{-1/(2nm)} = +\infty. \quad (8)$$

Proof. Clearly, all summands are nonnegative. Hence, since $(s_{nm}^{-1/(2nm)})_{n \in \mathbb{N}}$ is a subsequence of $(s_n^{-1/(2n)})_{n \in \mathbb{N}}$, it is trivial that (8) implies (7).

We prove the converse. Because the sequences s and Es are positive semidefinite, the Stieltjes theorem (see, e.g., [14, Theorem 3.12]) implies that s is a Stieltjes moment sequence, that is, there is a Radon measure on $\mathbb{R}_+$ such that $s_n = \int_0^\infty x^n d\mu(x)$ for $n \in \mathbb{N}_0$. Clearly, $\mu(\mathbb{R}_+) = s_0 = 1$. Suppose $0 < k < l$. The Hölder inequality, applied with $p = \frac{l}{k}, q = \frac{l}{l-k}$, yields

$$s_k = \int_0^\infty x^k \cdot 1 d\mu \leq \left(\int_0^\infty (x^k)^p d\mu \right)^{1/p} \left(\int_0^\infty 1 d\mu \right)^{1/q} = \left(\int_0^\infty x^l d\mu \right)^{k/l} s_0^{1/q} = s_l^{k/l},$$

so that $s_k^{1/k} \leq s_l^{1/l}$. Hence the sequence $(s_k^{-1/k})_{k \in \mathbb{N}}$ is nonincreasing and therefore

$$\sum_{n=m}^{\infty} s_n^{-1/(2n)} = \sum_{l=0}^{m-1} \sum_{n=1}^{\infty} s_{nm+l}^{-1/(2(nm+l))} \leq m \sum_{n=1}^{\infty} s_{nm}^{-1/(2nm)}.$$

Thus, if the sum in (8) is finite, then the sum in (7) is also finite. That is, (7) implies (8). $\square$

Now suppose L is as in Theorem 1. If $L(1) = 0$, it follows from the Cauchy-Schwarz inequality that $L = 0$, so the assertion is trivial. Otherwise, $L(1) > 0$. Then, setting $L' = L(1)^{-1}L$, L' satisfies the assumptions (5) and (6) and in addition $L'(1) = 1$. (Condition (6) for L' follows from the fact that $\lim_n L(1)^{1/(2n)} = 1$.)

Thus we can assume without loss of generality in Theorem 1 that $L(1) = 1$. Then, for $j = 1, \dots, d$, the sequence $(s_n)_{n \in \mathbb{N}_0} = (L(x_j^n))_{n \in \mathbb{N}_0}$ satisfies the assumptions of Lemma 2. Therefore, by this lemma, condition (6) is equivalent to

$$\sum_{n=1}^{\infty} L(x_j^{mn})^{-1/(2nm)} = +\infty \text{ for one (then for all) } m \in \mathbb{N}. \quad (9)$$

The case $m = 4$ is used in the proof of Theorem 1 in Sect. 4 below.

The proof of Theorem 1 given in Sect. 4 is essentially based on the operator-theoretic approach to the moment problem, see [14, Section 12.5] or [17]. While Carleman's condition in Nussbaum's theorem leads to quasi-analytic vectors [9], Theorem 1 uses Stieltjes vectors, see (17) for the definition of a Stieltjes vector. The crucial operator-theoretic result is the following: If a nonnegative symmetric operator has a dense set of Stieltjes vectors, then it is essentially self-adjoint. This theorem was first proved by Nussbaum [10] and later independently by Masson and McClary [8], see also [12, Theorem 7.15].

The Carleman condition goes back to T. Carleman's classical work [4]. It is now a standard tool in the theory of the classical moment problem and used in many papers such as [1, 3, 5, 7, 9, 16]. The multidimensional classical moment problem on $\mathbb{R}^d$ and its interplay with real algebraic geometry are treated in [14] and [2]. Applications of the moment problem are developed in [6].

3. A theorem on the $\mathcal{K}$ -moment problem for semi-algebraic sets. Theorem 1 is essentially used to derive the following result concerning the $\mathcal{K}$ -moment problem for semi-algebraic sets.

Theorem 3. *Let $\mathbf{f} = \{f_1, \dots, f_m\}$ be a finite set of polynomials $f_j \in \mathbb{R}[x_1, \dots, x_d]$, $d \in \mathbb{N}$, which generate the polynomial algebra $\mathbb{R}[x_1, \dots, x_d]$. Suppose that L is a linear functional on $\mathbb{R}[x_1, \dots, x_d]$ satisfying*

$$L(p^2) \geq 0 \text{ and } L(f_j p^2) \geq 0 \text{ for } p \in \mathbb{R}[x_1, \dots, x_d], j = 1, \dots, m, \quad (10)$$

and

$$\sum_{n=1}^{\infty} L(f_j^n)^{-1/(2n)} = +\infty \text{ for } j = 1, \dots, m. \quad (11)$$

Then L is a moment functional. It has a unique representing measure. This measure is supported on the semi-algebraic set

$$\mathcal{K}(\mathbf{f}) := \{x \in \mathbb{R}^d : f_1(x) \geq 0, \dots, f_m(x) \geq 0\}. \quad (12)$$

In Theorem 3, we assumed that the polynomials $f_1, \dots, f_m$ generate the whole polynomial algebra $\mathbb{R}[x_1, \dots, x_d]$. Without this assumption, the proof of Theorem 3 in Sect. 4 remains valid and shows that $L(q) = \int_{\mathcal{K}(\mathbf{f})} q(\lambda) d\mu(\lambda)$ for all polynomials q in the subalgebra generated by $f_1, \dots, f_m$.

The power and usefulness of Theorem 3 is nicely illustrated by the following example.

Example 4. Suppose that $d = 2$ and $m = 2$. We fix a number $k \in \mathbb{N}$ and define $f_1(x) = x_2 - x_1^k$, $f_2(x) = x_1$. Obviously, the polynomials f_1, f_2 generate the polynomial algebra $\mathbb{R}[x_1, x_2]$. The corresponding semi-algebraic set $\mathcal{K}(\mathbf{f})$ is the part of the plane above the curve $x_2 = x_1^k$ lying in the positive quadrant. In the case $k = 1$, we get the cone in the positive quadrant bounded by the x_2 -axis and the line $x_2 = x_1$.

In this case, assumptions (5) and (6) take the following form:

$$L(p^2) \geq 0, \quad L((x_2 - x_1^k)p^2) \geq 0, \quad L(x_1 p^2) \geq 0 \quad \text{for all } p \in \mathbb{R}[x_1, x_2], \quad (13)$$

$$\sum_{n=1}^{\infty} L((x_2 - x_1^k)^n)^{-1/(2n)} = +\infty, \quad \sum_{n=1}^{\infty} L(x_1^n)^{-1/(2n)} = +\infty. \quad (14)$$

Then Theorem 3 states that each linear functional L on $\mathbb{R}[x_1, x_2]$ satisfying (13) and (14) is a moment functional with a unique representing measure and this measure is supported on $\mathcal{K}(\mathbf{f})$.

Let us try to treat the moment problem for this semi-algebraic set $\mathcal{K}(\mathbf{f})$ by other methods. It is easily verified that the constant polynomials are the only polynomials that are bounded on the set $\mathcal{K}(\mathbf{f})$. Hence the fibre theorem [13] does not apply to the set $\mathcal{K}(\mathbf{f})$.

Another possibility is to assume in addition to (13) the multivariate Carleman condition, that is,

$$\sum_{n=1}^{\infty} L(x_1^{2n})^{-1/(2n)} = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} L(x_2^{2n})^{-1/(2n)} = +\infty. \quad (15)$$

Then L is a moment functional by Nussbaum's theorem with a unique representing measure and by Lasserre's localization theorem ([7], see also [14, Theorem 14.25]), this measure is supported on $\mathcal{K}(\mathbf{f})$.

As already noted above, the Stieltjes condition (6) is weaker than the Carleman condition (1). Comparing the two approaches mentioned in the preceding paragraphs shows another advantage of Theorem 3 over Nussbaum's theorem. Assumption (14) of Theorem 3 requires growth conditions of the functional L at the boundary curves $x_2 = x_1^k$ and $x_1 = 0$ of the set $\mathcal{K}(\mathbf{f})$, while the Carleman assumptions (15) are growth conditions of L at the coordinate axis. That is, Theorem 3 contains weaker growth assumptions concerning the powers of $L(x_j^n)$ and it is better adapted to the geometric form of the semi-algebraic set $\mathcal{K}(\mathbf{f})$.

4. Proofs of Theorems 1 and 3. Throughout this proof, we abbreviate the polynomial algebras $\mathbb{R}[y_1, \dots, y_m]$ by $\mathbb{R}_m[\underline{y}]$, $\mathbb{C}[y_1, \dots, y_m]$ by $\mathbb{C}_m[\underline{y}]$, and $\mathbb{R}[x_1, \dots, x_d]$ by $\mathbb{R}_d[\underline{x}]$.

As noted in Sect. 2, we can assume without loss of generality that $L(1) = 1$. We extend the $\mathbb{R}$ -linear functional L on $\mathbb{R}_d[\underline{x}]$ to a $\mathbb{C}$ -linear functional $L_{\mathbb{C}}$ on $\mathbb{C}_d[\underline{x}]$ by setting

$$L_{\mathbb{C}}(p_1 + ip_2) = L(p_1) + iL(p_2), \quad p_1, p_2 \in \mathbb{R}_d[\underline{x}].$$

Note that $\mathbb{C}_d[\underline{x}]$ is a complex $*$ -algebra with involution $(p_1 + ip_2)^+ := p_1 - ip_2$ for $p_1, p_2 \in \mathbb{R}_d[\underline{x}]$. Since $(p_1 + ip_2)^+(p_1 + ip_2) = p_1^2 + p_2^2$, the first condition of assumption (5) implies that $L_{\mathbb{C}}(p^+p) \geq 0$ for $p \in \mathbb{C}_d[\underline{x}]$, that is, $L_{\mathbb{C}}$ is a positive linear functional on the complex unital $*$ -algebra $\mathbb{C}_d[\underline{x}]$. Hence the GNS construction applies to this functional $L_{\mathbb{C}}$.

We briefly recall the corresponding GNS representation (see, e.g., [15, Section 3.5] for a detailed exposition). To simplify the notation, let us suppress the index $\mathbb{C}$ and write L instead of $L_{\mathbb{C}}$. The GNS representation π_L of L acts on a complex inner product space $(\mathcal{D}_L, \langle \cdot, \cdot \rangle)$. Let $\mathcal{H}_L$ denote the Hilbert space completion of $\mathcal{D}_L$. The representation π_L has an algebraically cyclic vector φ_L , that is, $\mathcal{D}_L = \pi_L(\mathbb{C}_d[\underline{x}])\varphi_L$, and it satisfies

$$L(p) = \langle \pi_L(p)\varphi_L, \varphi_L \rangle, \quad p \in \mathbb{C}_d[\underline{x}]. \quad (16)$$

For $p, q \in \mathbb{C}_d[\underline{x}]$ and $\eta, \zeta \in \mathcal{D}_L$, we have $\pi_L(p)\pi_L(q)\eta = \pi_L(pq)\eta$, $\pi_L(1)\eta = \eta$, and

$$\langle \pi_L(p)\eta, \zeta \rangle = \langle \eta, \pi_L(p^+)\zeta \rangle.$$

In the subsequent proof, we will freely use these properties.

Our proof is essentially based on the concept of a Stieltjes vector. Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. A vector φ of $\mathcal{H}$ is called a *Stieltjes vector* for T ([10], see also [12, Definition 7.1]) if $\varphi \in \mathcal{D}(T^n)$ for all $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \|T^n \varphi\|^{-1/2n} = +\infty. \quad (17)$$

The crucial step of the proof of Theorem 1 is the next lemma. It shows that assumption (6) implies that the domain $\mathcal{D}_L$ contains enough Stieltjes vectors for the representation operators $\pi_L(x_j)$.

Lemma 5. *Suppose S is a symmetric operator on $\mathcal{D}_L$ which leaves $\mathcal{D}_L$ invariant and commutes with all operators of $\pi_L(\mathbb{R}_d[\underline{x}])$. Then $S\varphi_L$ is a Stieltjes vector for $\pi_L(x_j)$, $j = 1, \dots, d$. In particular, φ_L is a Stieltjes vector for each operator $\pi_L(x_j)$.*

Proof. Let $j = 1, \dots, d$ and $n \in \mathbb{N}$. Using the assumptions on the symmetric operator S and basic properties of the GNS representation π_L , we derive

$$\begin{aligned} \|\pi_L(x_j)^n S\varphi_L\|^4 &= \langle \pi_L(x_j)^n S\varphi_L, \pi_L(x_j)^n S\varphi_L \rangle^2 \\ &= \langle \pi_L(x_j)^{2n} S\varphi_L, S\varphi_L \rangle^2 = \langle \pi_L(x_j)^{2n} \varphi_L, S^2 \varphi_L \rangle^2 \\ &\leq \|\pi_L(x_j)^{2n} \varphi_L\|^2 \|S^2 \varphi_L\|^2 \\ &= \langle \pi_L(x_j)^{2n} \varphi_L, \pi_L(x_j)^{2n} \varphi_L \rangle \|S^2 \varphi_L\|^2 \\ &= \langle \pi_L(x_j^{4n}) \varphi_L, \varphi_L \rangle \|S^2 \varphi_L\|^2 \leq L(x_j^{4n}) (1 + \|S^2 \varphi_L\|)^2 \end{aligned}$$

and hence

$$\|\pi_L(x_j)^n \varphi_L\|^{1/(2n)} \leq L(x_j^{4n})^{1/(8n)} (1 + \|S^2 \varphi_L\|)^{1/(8n)}. \quad (18)$$

Since $\lim_{n \rightarrow \infty} (1 + \|S^2 \varphi_L\|)^{1/(8n)} = 1$, there is a constant $c > 0$ such that

$$(1 + \|S^2 \varphi_L\|)^{1/(8n)} \leq c \text{ for all } n \in \mathbb{N}. \quad (19)$$

Combining (18) and (19) yields

$$\sum_{n=1}^{\infty} \|\pi_L(x_j)^n \varphi\|^{-1/(2n)} \geq c^{-1} \sum_{n=1}^{\infty} L(x_j^{4n})^{-1/(8n)}.$$

Since we have assumed that $L(1) = 1$, the discussion after Lemma 2 in Sect. 2 applies and shows that assumption (6) implies (9). Setting $m = 4$ in (9) it follows that the right hand side of the preceding inequality is infinite and so is the left hand side. By (17), this shows that $S\varphi_L$ is a Stieltjes vector for $\pi_L(x_j)$. Setting $S = I$, we conclude that φ_L is a Stieltjes vector for $\pi_L(x_j)$. $\square$

Lemma 6. *The operators $T_j := \overline{\pi_L(x_j)}$, $j = 1, \dots, d$, are pairwise strongly commuting self-adjoint operators.*

Proof. First we verify that $\pi_L(x_j) \geq 0$. For this, let $p \in \mathbb{C}_m[y]$. Then we can write $p = p_1 + ip_2$ with $p_1, p_2 \in \mathbb{R}_d[x]$ and $p^+p = p_1^2 + p_2^2$. Therefore,

$$\langle \pi_L(x_j) \pi(p) \varphi_L, \pi(p) \varphi_L \rangle = \langle \pi_L(x_j (p_1^2 + p_2^2)) \varphi_L, \varphi_L \rangle = L(x_j (p_1^2 + p_2^2)) \geq 0$$

by assumption (5). Thus $\pi_L(x_j) \geq 0$. Therefore, since each vector of $\mathcal{D}_L$ is a Stieltjes vector for $\pi_L(x_j)$ by Lemma 5, it follows from Nussbaum's theorem in [10] (see also [12, Theorem 7.15]) that $\pi_L(x_j)$ is essentially self-adjoint. This means that its closure T_j is self-adjoint. Since $\pi_L(x_j) \geq 0$, T_j is positive. Hence $\mathcal{E}_j := (T_j + I)\mathcal{D}_L = \pi_L(x_j + 1)\mathcal{D}_L$ is dense in $\mathcal{H}_L$ for $j = 1, \dots, x$ by [12, Proposition 3.15].

We prove that the self-adjoint operators T_j and T_k strongly commute.

Clearly, the symmetric operator $\pi_L(x_k + 1)$ commutes with all operators of $\pi_L(\mathbb{R}_d[x])$. Therefore, Lemma 5 applies with $S := \pi_L(x_k + 1)$ and shows that all vectors of $\mathcal{E}_k = \pi_L(x_k + 1)\mathcal{D}_L$ are Stieltjes vectors for the symmetric operator $\pi_L(x_j)$. Since $\mathcal{E}_k$ is dense as noted above, $T_j[\mathcal{E}_k = \pi_L(x_j)]\mathcal{E}_k$ is essentially self-adjoint again by [12, Theorem 7.15]. Hence, by [12, Proposition 3.15], $\mathcal{E}_{jk} := (T_j + I)\mathcal{E}_k = (T_j + I)(T_k + I)\mathcal{D}_L$ is dense in $\mathcal{H}_L$. Let $\psi \in \mathcal{E}_{jk}$. By the definition of $\mathcal{E}_{jk}$, there is a vector $\varphi \in \mathcal{D}_L$ such that $\psi = (T_j + I)(T_k + I)\varphi = (T_k + I)(T_j + I)\varphi$. Then $(T_k + I)^{-1}(T_j + I)^{-1}\psi = \varphi = (T_j + I)^{-1}(T_k + I)^{-1}\psi$. This shows that the operators $(T_j + I)^{-1}$ and $(T_k + I)^{-1}$ commute on the dense linear subspace $\mathcal{E}_{jk}$. Since T_j and T_k are positive self-adjoint operators, these operators are bounded, so they commute everywhere on $\mathcal{H}_L$. That is, the resolvent operators $(T_j + I)^{-1}$ of T_j and $(T_k + I)^{-1}$ of T_k commute. Hence the self-adjoint operators T_j and T_k commute strongly by [12, Theorem 5.27]. $\square$

Thus, $\{T_1, \dots, T_d\}$ is a d -tuple of strongly commuting positive self-adjoint operators acting on the Hilbert space $\mathcal{H}_L$. Therefore, by the multidimensional spectral theorem [12, Theorem 5.23], there exists a unique spectral measure E on the Borel σ -algebra of $\mathbb{R}^d$ such that $T_j = \int \lambda_j dE(\lambda)$, $j = 1, \dots, d$.

Now we proceed as in the standard operator approach to the moment problem (see, e.g., [14, p. 305]). We define a Radon measure μ on $\mathbb{R}^d$ by

$\mu(\cdot) := \langle E(\cdot)1, 1 \rangle$. The vectors of the domain $\mathcal{D}_L$, in particular the polynomial 1, are contained in the domains of all operator polynomials $p(T_1, \dots, T_d)$ for $p \in \mathbb{C}_d[\underline{x}]$ and we have $\pi_L(p) \subseteq p(T_1, \dots, T_d)$. Therefore, using the functional calculus and equation (16), we derive

$$\begin{aligned} \int p(\lambda) d\mu(\lambda) &= \int p(\lambda) d\langle E(\lambda)1, 1 \rangle = \langle p(T_1, \dots, T_d)1, 1 \rangle \\ &= \langle \pi_L(p)\varphi_L, \varphi_L \rangle = L(p). \end{aligned}$$

That is, L is a moment functional. Because the self-adjoint operators T_j are positive, the spectral measure E , hence the measure μ , is supported on $\mathbb{R}_+^d$, that is, L is a Stieltjes moment functional. Since all operators $\pi_L(x_j)$, $j = 1, \dots, d$, are essentially self-adjoint, μ is determinate by [14, Theorem 14.2], that is, μ is the unique representing measure for L . This completes the proof of Theorem 1.

Now we turn to the proof of Theorem 3. We define a unital algebra homomorphism $\theta : \mathbb{R}_m[\underline{y}] \mapsto \mathbb{R}_d[\underline{x}]$ by $\theta(y_j) = f_j$, $j = 1, \dots, m$, and a linear functional $\tilde{L}$ on $\mathbb{R}_m[\underline{y}]$ by $\tilde{L}(p) := L(\theta(p))$, that is, $\tilde{L}(p(y_1, \dots, y_m)) = L(p(f_1, \dots, f_m))$ for $p \in \mathbb{R}_m[\underline{y}]$. The assumptions (10) and (11) on L imply that the functional $\tilde{L}$ satisfies the assumptions (5) and (6). Therefore, by Theorem 1, $\tilde{L}$ is a Stieltjes moment functional on $\mathbb{R}_m[\underline{y}]$, that is, there exists a Radon measure ν on $\mathbb{R}_+^m$ such that

$$\tilde{L}(p) = \int_{\mathbb{R}_+^m} p(t) d\nu(t) \text{ for } p \in \mathbb{R}_m[\underline{y}]. \quad (20)$$

Let $\tau : \mathcal{K}(\mathbf{f}) \mapsto \mathbb{R}_+^m$ denote the mapping defined by $\tau(\lambda) = (f_1(\lambda), \dots, f_m(\lambda))$. That τ maps $\mathcal{K}(\mathbf{f})$ into $\mathbb{R}_+^m$ follows from the definition (12) of the semi-algebraic set $\mathcal{K}(\mathbf{f})$. Let μ denote the pull-back of the measure ν with respect to the mapping τ . Using (20), we derive for $p \in \mathbb{R}_m[\underline{y}]$,

$$\begin{aligned} L(p(f_1, \dots, f_m)) &\equiv L(\theta(p)) = \tilde{L}(p) = \int_{\mathbb{R}_+^m} p(y_1, \dots, y_m) d\nu(y) \\ &= \int_{\mathcal{K}(\mathbf{f})} p(\tau(\lambda)) d\mu(\lambda) = \int_{\mathcal{K}(\mathbf{f})} p(f_1(\lambda), \dots, f_m(\lambda)) d\mu(\lambda). \end{aligned}$$

By assumption, the polynomials $f_1, \dots, f_m$ generate the algebra $\mathbb{R}_d[\underline{x}]$. Therefore, θ is surjective and the preceding equality implies that $L(q) = \int_{\mathcal{K}(\mathbf{f})} q(\lambda) d\mu(\lambda)$ for all (!) polynomials $q \in \mathbb{R}_d[\underline{x}]$. By construction, the measure μ is supported on the semi-algebraic set $\mathcal{K}(\mathbf{f})$. It has finite moments because $L(q^2) = \int q(\lambda)^2 d\mu < +\infty$ for $q \in \mathbb{R}[\underline{x}]$. Thus L is a moment functional. The uniqueness of the representing measure μ of L follows at once from the uniqueness of the representing measure ν of $\tilde{L}$ by Theorem 1. Now the proof of Theorem 3 is complete.

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