An estimate for narrow operators on $L^p([0, 1])$

EUGENE SHARGORODSKY AND TEO SHARIA

Abstract. We prove a theorem, which generalises C. Franchetti's estimate for the norm of a projection onto a rich subspace of $L^p([0, 1])$ and the authors' related estimate for compact operators on $L^p([0, 1])$, $1 \leq p < \infty$.

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1. Introduction. For Banach spaces X and Y , let $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ denote the sets of bounded linear and compact linear operators from X to Y , respectively; $\mathcal{B}(X) := \mathcal{B}(X, X)$, $\mathcal{K}(X) := \mathcal{K}(X, X)$; $I \in \mathcal{B}(X)$ denotes the identity operator. An operator $P \in \mathcal{B}(X)$ is called a *projection* if $P^2 = P$. A closed linear subspace $X_0 \subset X$ is called *1-complemented* (in X) if there exists a projection $P \in \mathcal{B}(X)$ such that $P(X) = X_0$ and $\|P\| = 1$.

Let (Ω, Σ, μ) be a nonatomic measure space with $0 < \mu(\Omega) < \infty$. We will use the following notation:

- $\Sigma^+ := \{A \in \Sigma : \mu(A) > 0\}$,
- $\mathbb{I}_A$ is the indicator function of $A \in \Sigma$, i.e. $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{I}_A(\omega) = 0$ if $\omega \notin A$,
- $\mathbf{1} := \mathbb{I}_\Omega$,
- $\mathbf{E}f := \left(\frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu \right) \mathbf{1}$.

We will use the terminology from [6]. A Σ -measurable function g is called a *sign* if it takes values in the set $\{-1, 0, 1\}$, and a *sign on* $A \in \Sigma$ if it is a sign with the support equal to A , i.e. if $g^2 = \mathbb{I}_A$. A sign is of *mean zero* if $\int_\Omega g \, d\mu = 0$.

An operator $T \in \mathcal{B}(L^p(\mu), Y)$, $1 \leq p < \infty$, is called *narrow* if for every $A \in \Sigma^+$ and every $\varepsilon > 0$, there exists a mean zero sign g on A such that $\|Tg\| < \varepsilon$.

Every $T \in \mathcal{K}(L^p(\mu), Y)$ is narrow (see [6, Proposition 2.1]), but there are noncompact narrow operators. Indeed, let $\mathcal{G}$ be a sub- σ -algebra of Σ such that there exists a random variable ξ on $(\Omega, \Sigma, \frac{1}{\mu(\Omega)}\mu)$, which is independent of $\mathcal{G}$ and has a nontrivial Gaussian distribution. Then the corresponding conditional expectation operator $\mathbf{E}^{\mathcal{G}} = \mathbf{E}(\cdot|\mathcal{G}) \in \mathcal{B}(L^p(\mu))$ is narrow (see [6, Corollary 4.25]), but not compact if $\mathcal{G}$ has infinitely many pairwise disjoint elements of positive measure.

Let

$$C_p := \max_{0 \leq \alpha \leq 1} (\alpha^{p-1} + (1-\alpha)^{p-1})^{\frac{1}{p}} \left(\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}} \right)^{1-\frac{1}{p}} \quad (1.1)$$

for $1 < p < \infty$, and $C_1 := 2$.

In the following theorems, $(\Omega, \Sigma, \mu) = ([0, 1], \mathcal{L}, \lambda)$, where λ is the standard Lebesgue measure on $[0, 1]$ and $\mathcal{L}$ is the σ -algebra of Lebesgue measurable subsets of $[0, 1]$.

Our starting point is a result due to C. Franchetti.

Theorem 1.1 ([3, 4]). *Let $P \in \mathcal{B}(L^p([0, 1])) \setminus \{0\}$ be a narrow projection operator, $1 \leq p < \infty$. Then*

$$\|I - P\|_{L^p \rightarrow L^p} \geq \|I - \mathbf{E}\|_{L^p \rightarrow L^p} = C_p. \quad (1.2)$$

The following theorem was proved in [7], where it was used to show that C_p is the optimal constant in the bounded compact approximation property of $L^p([0, 1])$. It implies the inequality in (1.2) in the case when $P \neq 0$ is a finite-rank projection.

Theorem 1.2 ([7]). *Let $1 \leq p < \infty$, $\gamma \in \mathbb{C}$, and let $T \in \mathcal{K}(L^p([0, 1]))$. Then*

$$\|I - T\|_{L^p \rightarrow L^p} + \inf_{\|u\|_{L^p}=1} \|(\gamma I - T)u\|_{L^p} \geq \|I - \gamma \mathbf{E}\|_{L^p \rightarrow L^p}. \quad (1.3)$$

In particular,

$$\|I - T\|_{L^p \rightarrow L^p} + \inf_{\|u\|_{L^p}=1} \|(I - T)u\|_{L^p} \geq C_p. \quad (1.4)$$

The following theorem is the main result of the paper. It generalises both Theorems 1.1 and 1.2.

Theorem 1.3. *Estimates (1.3) and (1.4) hold for all narrow operators $T \in \mathcal{B}(L^p([0, 1]))$, $1 \leq p < \infty$.*

In the case $p = 1$, inequality (1.4) (for all narrow operators) follows from a Daugavet-type result due to A.M. Plichko and M.M. Popov ([5, §9, Theorem 8], see also [6, Corollary 6.4]):

$$\|I - T\| = 1 + \|T\| \quad \text{for every narrow operator } T \in \mathcal{B}(L^1([0, 1])).$$

Indeed,

$$\begin{aligned} & \|I - T\|_{L^1 \rightarrow L^1} + \inf_{\|u\|_{L^1}=1} \|(I - T)u\|_{L^1} \\ & \geq \|I - T\|_{L^1 \rightarrow L^1} + 1 - \sup_{\|u\|_{L^1}=1} \|Tu\|_{L^1} \\ & = 1 + \|T\|_{L^1 \rightarrow L^1} + 1 - \|T\|_{L^1 \rightarrow L^1} = 2 = C_1. \end{aligned}$$

2. Proof of Theorem 1.3. It follows from the definition of a narrow operator that if $T \in \mathcal{B}(L^p(\mu))$ is narrow and $S \in \mathcal{B}(L^p(\mu))$, then $ST \in \mathcal{B}(L^p(\mu))$ is narrow (see [6, Proposition 1.8]). On the other hand, there are $S, T \in \mathcal{B}(L^p(\mu))$ such that T is narrow but TS is not (see [6, Proposition 5.1]). The following lemma shows that the latter cannot happen if S is a multiplication operator.

Lemma 2.1. *Let $X = L^p(\mu)$, $g \in L^\infty(\mu)$, and $T \in \mathcal{B}(X, Y)$ be a narrow operator. Then the operator $TgI \in \mathcal{B}(X, Y)$ is also narrow.*

Proof. There is nothing to prove if $T = 0$. Suppose $T \neq 0$. Take any $A \in \Sigma^+$ and any $\varepsilon > 0$. There exists a simple function $g_0 = \sum_{k=1}^M a_k \mathbb{I}_{B_k} \neq 0$ such that

$$\|g - g_0\|_{L^\infty} < \frac{\varepsilon}{2\|T\|(\mu(\Omega))^{1/p}}.$$

Here $M \in \mathbb{N}$, $B_k \in \Sigma$, $k = 1, \dots, M$, are pairwise disjoint, $\mu(B_k) > 0$, $\cup_{k=1}^M B_k = \Omega$, $a_k \in \mathbb{C}$.

Let $A_k := A \cap B_k$. If $\mu(A_k) > 0$, let h_k be a mean zero sign on A_k such that

$$\|Th_k\| < \frac{\varepsilon}{2\sum_{k=1}^M |a_k|}.$$

Let

$$h := \sum_{\{k: \mu(A_k) > 0\}} h_k.$$

It is clear that h is a mean zero sign on A and

$$\begin{aligned} \|TgIh\| &\leq \|Tg_0h\| + \|T(g - g_0)h\| \\ &\leq \left\| T \sum_{\{k: \mu(A_k) > 0\}} a_k h_k \right\| + \|T\| \|g - g_0\|_{L^\infty} \|h\|_{L^p} \\ &< \sum_{\{k: \mu(A_k) > 0\}} |a_k| \|Th_k\| + \|T\| \frac{\varepsilon}{2\|T\|(\mu(\Omega))^{1/p}} (\mu(\Omega))^{1/p} \\ &\leq \frac{\varepsilon}{2\sum_{k=1}^M |a_k|} \sum_{\{k: \mu(A_k) > 0\}} |a_k| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

The above lemma and its proof remain valid if X is a Köthe F-space on (Ω, Σ, μ) (see [6, Section 1.3]). Similarly, the following lemma and its proof remain valid if X is a rearrangement-invariant Banach space on $([0, 1], \mathcal{L}, \lambda)$ with absolutely continuous norm. This lemma is a minor modification of [6, Theorem 2.21] and [5, § 8, Proposition 5].

Lemma 2.2. *Let $X = L^p([0, 1])$ and $T \in \mathcal{B}(X, Y)$ be a narrow operator. Then there exists a 1-complemented subspace X_0 of X isometrically isomorphic to X such that $\mathbf{1} \in X_0$ and the restriction $T|_{X_0}$ of T to X_0 is a compact operator.*

Proof. Take a mean zero sign $\bar{g}_1$ on $[0, 1]$ and set $T_1 := T\bar{g}_1I$. The operator T_1 is narrow according to Lemma 2.1. The proof of [6, Theorem 2.21] (with T_1 in place of T and with $\varepsilon \geq 2\|T_1\bar{g}_1\|$, $\varepsilon_1 > \|T_1\bar{g}_1\|$) shows that there exists

a 1-complemented (see the proof of [5, §8, Proposition 5]) subspace X_1 of X isometrically isomorphic to X such that $\bar{g}_1 \in X_1$ and the restriction $T_1|_{X_1}$ of T_1 to X_1 is a compact operator. Let $X_0 := \bar{g}_1 X_1$. Since $\bar{g}_1^2 = \mathbf{1}$, the operator of multiplication $\bar{g}_1 I$ is an isometric isomorphism of X_0 onto X_1 and of X onto itself. Let $P_1 \in \mathcal{B}(X)$ be a projection onto X_1 such that $\|P_1\| = 1$. Then $P_0 := \bar{g}_1 P_1 \bar{g}_1 I \in \mathcal{B}(X)$ is a projection onto X_0 such that $\|P_0\| = 1$. Hence X_0 is 1-complemented (this follows also from [1, Theorem 4] since X_0 is isometrically isomorphic to $X = L^p([0, 1])$). Further, $\mathbf{1} = \bar{g}_1^2 \in \bar{g}_1 X_1 = X_0$ and $T|_{X_0} = T_1|_{X_1} \bar{g}_1 I|_{X_0}$ is compact.

Proof of Theorem 1.3. Take an arbitrary $\varepsilon > 0$. Let

$$\delta := \inf_{\|u\|_{L^p}=1} \|(\gamma I - T)u\|_{L^p}. \quad (2.1)$$

There exists $u_0 \in L^p([0, 1])$ such that $\|u_0\|_{L^p} = 1$ and $\|(\gamma I - T)u_0\|_{L^p} < \delta + \varepsilon$. Then there exists an approximation $h := \sum_{k=1}^M a_k \mathbb{I}_{A_k}$ of u_0 such that A_k , $k = 1, \dots, M$, $M \in \mathbb{N}$, are pairwise disjoint Borel sets of positive measure, $\cup_{k=1}^M A_k = [0, 1]$, $a_k \in \mathbb{C}$, and

$$\|\gamma h - Th\|_{L^p([0,1])} \leq \delta + 2\varepsilon, \quad \|h\|_{L^p([0,1])} = 1. \quad (2.2)$$

Partition $[0, 1]$ into subintervals I_k of length $\lambda(A_k)$, $k = 1, \dots, M$. Since $(A_k, \mathcal{L}, \lambda)$ is isomorphic (modulo sets of measure 0) to $(I_k, \mathcal{L}, \lambda)$ (see, e.g., [2, Theorem 9.2.2 and Corollary 6.6.7]), one can easily derive from Lemma 2.2 the existence, for each k , of a 1-complemented subspace X_k of $L^p(A_k)$ isometrically isomorphic to $L^p(I_k)$ such that $\mathbb{I}_{A_k} \in X_k$ and $T|_{X_k}$ is a compact operator. Let

$$X_0 := \{f \in L^p([0, 1]) : f|_{A_k} \in X_k, k = 1, \dots, M\}.$$

It is easy to see that X_0 is 1-complemented and isometrically isomorphic to $L^p([0, 1])$, and that $T_0 := T|_{X_0}$ is a compact operator. Let $J : L^p([0, 1]) \rightarrow X_0$ be an isometric isomorphism and $P_0 \in \mathcal{B}(L^p([0, 1]))$ be a projection onto X_0 such that $\|P_0\| = 1$. Then $T_1 := J^{-1}P_0 T_0 J \in \mathcal{K}(L^p([0, 1]))$ and it follows from Theorem 1.2 that

$$\begin{aligned} & \|I - T_0\|_{X_0 \rightarrow L^p} + \inf_{f \in X_0, \|f\|_{L^p}=1} \|(\gamma I - T_0)f\|_{L^p} \\ & \geq \|P_0(I - T_0)\|_{X_0 \rightarrow X_0} + \inf_{f \in X_0, \|f\|_{L^p}=1} \|P_0(\gamma I - T_0)f\|_{L^p} \\ & = \|J^{-1}P_0(I - T_0)J\|_{L^p \rightarrow L^p} + \inf_{\varphi \in L^p, \|\varphi\|_{L^p}=1} \|J^{-1}P_0(\gamma I - T_0)J\varphi\|_{L^p} \\ & = \|I - T_1\|_{L^p \rightarrow L^p} + \inf_{\varphi \in L^p, \|\varphi\|_{L^p}=1} \|(\gamma I - T_1)\varphi\|_{L^p} \geq \|I - \gamma \mathbf{E}\|_{L^p \rightarrow L^p}. \end{aligned}$$

Since $h \in X_0$, it follows from (2.2) that

$$\delta + 2\varepsilon \geq \inf_{f \in X_0, \|f\|_{L^p}=1} \|(\gamma I - T_0)f\|_{L^p}.$$

Hence

$$\begin{aligned} \|I - T\|_{L^p \rightarrow L^p} + \delta + 2\varepsilon & \geq \|I - T_0\|_{X_0 \rightarrow L^p} + \inf_{f \in X_0, \|f\|_{L^p}=1} \|(\gamma I - T_0)f\|_{L^p} \\ & \geq \|I - \gamma \mathbf{E}\|_{L^p \rightarrow L^p} \end{aligned}$$

and

$$\|I - T\|_{L^p \rightarrow L^p} + \inf_{\|u\|_{L^p}=1} \|(\gamma I - T)u\|_{L^p} + 2\varepsilon \geq \|I - \gamma \mathbf{E}\|_{L^p \rightarrow L^p}$$

for all $\varepsilon > 0$ (see (2.1)). $\square$

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References

- [1] Ando, T.: Contractive projections in L_p -spaces. *Pac. J. Math.* **17**, 391–405 (1966)
- [2] Bogachev, V.I.: *Measure Theory*, Vol. I and II. Springer, Berlin (2007)
- [3] Franchetti, C.: The norm of the minimal projection onto hyperplanes in $L^p[0, 1]$ and the radial constant. *Boll. Un. Mat. Ital. B (7) 4(4)*, 803–821 (1990)
- [4] Franchetti, C.: Lower bounds for the norms of projections with small kernels. *Bull. Aust. Math. Soc.* **45**(3), 507–511 (1992)
- [5] Plichko, A.M., Popov, M.M.: Symmetric function spaces on atomless probability spaces. *Dissertationes Math.* **306**, 85 pp. (1990)
- [6] Popov, M., Randrianantoanina, B.: *Narrow Operators on Function Spaces and Vector Lattices*. de Gruyter Studies in Mathematics, vol. 45. de Gruyter, Berlin (2013)
- [7] Shargorodsky, E., Sharia, T.: Sharp estimates for conditionally centred moments and for compact operators on L^p spaces. [arXiv:2008.06925](https://arxiv.org/abs/2008.06925) (2020)

EUGENE SHARGORODSKY
 Department of Mathematics
 King's College London
 Strand, London WC2R 2LS
 United Kingdom
 e-mail: eugene.shargorodsky@kcl.ac.uk

and

Fakultät Mathematik
Technische Universität Dresden
01062 Dresden
Germany

TEO SHARIA
Department of Mathematics Royal Holloway
University of London
Egham Surrey TW20 0EX
United Kingdom
e-mail: t.sharia@rhul.ac.uk

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