Computing the p -modulus of systems of measures via optimal plans

MALGORZATA CISKA-NIEDZIAŁOMSKA

Abstract. We use the notion of the optimal plan associated with the Fuglede p -modulus of a family of Borel measures to derive formulas for the p -modulus and the extremal function in many special cases. Among others, we deduce Rodin's formula for a family of Hausdorff measures associated with leaves of a foliation defined by a single chart.

Mathematics Subject Classification. 28A25, 28A33.

Keywords. Fuglede p -modulus, Extremal function, Optimal plan, Rodin's formula.

1. Introduction. The notion of p -modulus was introduced by Fuglede [5] in order to study functional completion. Its reciprocal—extremal length, for $p=2$ —had been considered before and is closely related with the study of conformality and quasiconformality of maps on a plane.

Since the work of Fuglede, the p -modulus became a powerful tool in geometric measure theory. For example, Shanmugalingam [7] uses the p -modulus to consider p -harmonic maps on metric measure spaces. In such case, the p -modulus is considered for a family of curves, more precisely, for the family of 1-dimensional Hausdorff measures on continuous curves.

Computation of the p -modulus for a concrete family of measures is not easy and there are not many explicit formulas. The majority of them is related to a family of Hausdorff measures on curves or hypersurfaces and is related to the notion of q -capacity (here p and q are conjugate coefficients).

Recently, Ambrosio, di Marino, and Savare [1] characterised the p -modulus in terms of existence of a certain measure $\mathbf{n}$ on the considered family of measures Σ on X . This measure, called optimal plan, satisfies the following remarkable identity

$$\mathrm{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1}(x) f(x) d\mathbf{m} = \int_\Sigma \int_X f(x) d\mu(x) d\mathbf{n}(\mu), \quad f \geq 0, \quad (1)$$

where $\mathrm{mod}_p(\Sigma)$ is the p -modulus of Σ (with respect to a fixed Borel measure $\mathbf{m}$) and f_Σ is an extremal function, i.e., a function which realizes the p -modulus. It can be shown that, up to a subfamily of modulus zero, f_Σ is unique. Equality (1) allows us to derive formulas for the p -modulus and the extremal function in many special cases.

We begin with a general fact. We consider (1) without the assumption on extremality of f_Σ and under two natural conditions we prove that f_Σ is in fact extremal for the p -modulus of Σ (Proposition 3).

The main part of this note is devoted to formulas for the p -modulus and the extremal function for different families of measures Σ . We heavily rely on (1) and Proposition 3. We rewrite (1) assuming there is a Borel bijection $\Sigma \mapsto Y$ for some Polish space Y and we push forward the optimal plan $\mathbf{n}$ to a probability measure on Y . In a sense, Y counts elements of the family Σ or, in other words, using geometrical language, is a transversal to Σ . We consider the following families:

1. measures which are absolutely continuous with respect to the reference measure $\mathbf{m}$,
2. ‘regular’ family of measures with disjoint supports, in particular,
3. Lebesgue measures associated with leaves of a foliation defined by a single chart (Rodin’s formula)

obtaining new, generalizing, or reproving well known relations.

2. Preliminary facts. Let (X, d) be a Polish space and let $\mathbf{m}$ be a (reference) Borel measure on X . Consider a family of Borel measures Σ . Let $\mathcal{L}(X, \mathbf{m})$ (resp. $\mathcal{L}_+^p(X, \mathbf{m})$) be the set of all (resp. nonnegative) p -integrable functions on X . We say that $f \in \mathcal{L}_+^p(X, \mathbf{m})$ is *admissible* (for Σ or, more precisely, for the p -modulus of Σ) if

$$\int_X f(x) d\mu(x) \geq 1$$

for all $\mu \in \Sigma$. Denote the family of such f by $\mathrm{adm}_p(\Sigma)$. Then the p -modulus of Σ is the following number

$$\mathrm{mod}_p(\Sigma) = \inf_{f \in \mathrm{adm}_p(\Sigma)} \int_X f^p(x) d\mathbf{m}(x)$$

if $\mathrm{adm}_p(\Sigma)$ is nonempty, otherwise we put $\mathrm{mod}_p(\Sigma) = \infty$. An admissible function which realizes the p -modulus is called *extremal*. It can be shown, that up to a subfamily of full p -modulus, there is a unique extremal function [5]. Denote it by f_Σ .

We say that a certain condition (P) holds for p -a.e. $\mu \in \Sigma$ if there is a subfamily $\Pi \subset \Sigma$ such that (P) holds for every $\mu \in \Pi$ and $\mathrm{mod}_p(\Sigma \setminus \Pi) = 0$. The p -modulus has the following important properties [5]:

- (M1) $\text{mod}_p(\Sigma) = 0$ if and only if there is an admissible function f such that $\int_X f(x) d\mu(x) = \infty$ for every $\mu \in \Sigma$,
 (M2) if $f \in \mathcal{L}^p(X, \mathbf{m})$, then $f \in \mathcal{L}^1(X, \mu)$ for p -a.e. $\mu \in \Sigma$,
 (M3) if $\Pi \subset \Sigma$, then $\text{mod}_p(\Pi) \leq \text{mod}_p(\Sigma)$,
 (M4) if $\Pi \subset \bigcup_i \Sigma_i$, then $\text{mod}_p(\Pi) \leq \sum_i \text{mod}_p(\Sigma_i)$.

The p -modulus is a powerful tool, especially in the context of families of Hausdorff measures on curves, in geometric measure theory [7].

In the article [1], the authors prove the existence of an optimal plan associated with a family Σ . Let us be precise. Firstly, consider a space $\mathcal{M}(X)$ of positive and finite Borel measures on a Polish space X . We can equip this space with the weak- $*$ topology and consider Borel probability measures on $\mathcal{M}(X)$. We say that such a measure $\mathbf{n}$ is a *plan with barycenter in $L^q(X, \mathbf{m})$* , where q is a coefficient conjugate to p , if there is a constant $c(\mathbf{n})$ such that

$$\int_{\mathcal{M}(X)} \int_X f(x) d\mu(x) d\mathbf{n}(\mu) \leq c(\mathbf{n}) \|f\|_p, \quad f \in \mathcal{L}_+(X, \mathbf{m}). \quad (2)$$

The explanation of the use of $L^q(X, \mathbf{m})$ is the following. Define a Borel measure $\underline{\mu}$ on X as follows

$$\underline{\mu}(A) = \int_{\Sigma} \mu(A) d\mathbf{n}(\mu).$$

Then $\underline{\mu}$ is absolutely continuous with respect to $\mathbf{m}$ with a density ρ belonging to $L^q(X, \mathbf{m})$. The authors in [1] prove the existence and remarkable properties of the optimal plan.

Theorem 1 ([1]). *Assume $\Sigma \subset \mathcal{M}(X)$ is a Suslin set such that $\text{mod}_p(\Sigma) > 0$ and $\sup_{\mu \in \Sigma} \mu(X) < \infty$. Then there is a plan $\mathbf{n}$ with barycenter in $L^q(X, \mathbf{m})$ such that $\mathbf{n}$ is concentrated on Σ , $c(\mathbf{n}) = \text{mod}_p(\Sigma)^{-\frac{1}{p}}$ is the best possible constant, and the following formulas hold*

$$\int_X f_{\Sigma}(x) d\mu(x) = 1 \quad \text{for } \mathbf{n}\text{-a.e. } \mu \quad \text{and} \quad \rho = \frac{f_{\Sigma}^{p-1}}{\|f_{\Sigma}\|_p^p},$$

where f_{Σ} is the extremal function for Σ .

Remark 2. Let us recall (see [3]) that a Polish space is Suslin and the Borel image of a Suslin set is again Suslin.

By the definition of ρ , we immediately see that a formula for ρ in the above Theorem implies

$$\int_{\Sigma} \int_X f(x) d\mu(x) d\mathbf{n}(\mu) = \text{mod}_p(\Sigma)^{-1} \int_X f_{\Sigma}^{p-1}(x) f(x) d\mathbf{m}(x) \quad (3)$$

for any nonnegative Borel function f .

2.1. An important consequence. Let us prove a general consequence of the formula (3).

Now, we give one implication of the formula (3) but without the assumption on extremality of f_Σ . This approach has been partially noticed in the paper [1] but in the context with relation (2) and with different conclusions.

Assume there is a probability measure η on $\mathcal{M}(X)$ such that for any non-negative Borel f ,

$$\int_{\mathcal{M}(X)} \int_X f(x) d\mu(x) d\eta(\mu) = \int_X f_\eta^{p-1}(x) f(x) d\mathbf{m}(x) \quad (4)$$

for some function $f_\eta \in \mathcal{L}_+^p(X, \mathbf{m})$. Then, by the Hölder inequality,

$$\int_{\mathcal{M}(X)} \int_X f(x) d\mu(x) d\eta(\mu) \leq \|f_\eta^{p-1}\|_q \cdot \|f\|_p,$$

hence η is a plan with barycenter in $L^q(X, \mathbf{m})$ with $c(\eta) = \|f_\eta^{p-1}\|_q$. Moreover, if $f \in \mathcal{L}_+^p(X, \mathbf{m})$ is admissible, then

$$\eta(\Sigma) \leq \|f_\eta^{p-1}\|_q \cdot \|f\|_p. \quad (5)$$

Hence, taking the infimum with respect to all admissible functions f , we get

$$\frac{\eta(\Sigma)^p}{\|f_\eta^{p-1}\|_q^p} \leq \text{mod}_p(\Sigma) = \frac{1}{c(\mathbf{n})}.$$

Consider the following two assumptions:

- (E1) the measure η is concentrated on Σ ,
- (E2) there is a constant $c > 0$ such that for every $\mu \in \Sigma$,

$$\int_X f_\eta(x) d\mu(x) = c. \quad (6)$$

Proposition 3. Assume $\Sigma \subset \mathcal{M}(X)$ is a Suslin set such that $\text{mod}_p(\Sigma) > 0$ and $\sup_{\mu \in \Sigma} \mu(X) < \infty$. Assume, moreover, that there is a probability measure η on $\mathcal{M}(X)$ such that (4) holds for some $f_\eta \in \mathcal{L}_+^p(X)$. If (E1), (E2) hold, then

- (i) $c = \text{mod}_p(\Sigma)^{1-q}$,
- (ii) $f_\Sigma = \text{mod}_p(\Sigma)^{q-1} f_\eta$ is extremal for the p -modulus of Σ ,
- (iii) $\eta = \mathbf{n}$, i.e., η is the optimal plan with barycenter in $L^q(X, \mathbf{m})$,
- (iv) (4) is equivalent to (3).

Proof. Substituting $f = f_\eta$ in (4), we have

$$c = \int_X f_\eta^p(x) d\mathbf{m}(x).$$

Thus, by (3) with $f = f_\eta$, we have

$$c = \text{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1}(x) f_\eta(x) d\mathbf{m}(x) \leq \text{mod}_p(\Sigma)^{-\frac{1}{p}} c^{\frac{1}{p}}.$$

Hence

$$c^{\frac{1}{q}} \operatorname{mod}_p(\Sigma)^{\frac{1}{p}} \leq 1.$$

On the other hand, f_Σ is admissible for the p -modulus, so by (5),

$$1 \leq c^{\frac{1}{q}} \operatorname{mod}_p(\Sigma)^{\frac{1}{p}}.$$

Together with the previous inequality, we conclude (i). For the proof of (ii), notice that $c^{-1}f_\eta$ is admissible for the p -modulus of Σ and

$$\operatorname{mod}_p(\Sigma) = c^{1-p} = c^{-p} \int_X f_\eta^p(x) d\mathbf{m}(x) = \int_X (c^{-1}f_\eta)^p(x) d\mathbf{m}(x),$$

which proves the extremality of $c^{-1}f_\eta$. By the uniqueness (up to a subset of zero measure), we get (ii). The remaining conditions follow immediately by (i) and (ii). $\square$

3. Results. Adopt notation from the previous section. Fix $p > 1$ and let Σ be a family of Borel measures such that $\operatorname{mod}_p(\Sigma) > 0$ and $\sup_{\mu \in \Sigma} \mu(X) < \infty$. Assume that the family $\Sigma \subset \mathcal{M}(X)$ can be realized as follows: there is a Borel bijective map $\mu : Y \rightarrow \Sigma$ with a Borel inverse map $\mu^{-1} : \Sigma \rightarrow Y$, where Y is a Polish space. Hence, we may index the family Σ with elements from Y , i.e., $\Sigma = \{\mu_y\}_{y \in Y}$. We may push-forward the optimal plan $\mathbf{n}$ with respect to μ^{-1} to obtain a Borel probability measure $\lambda = \lambda_{\mathbf{n}}$ on Y . Then formula (3) may be rewritten as follows

$$\int_Y \int_X f(x) d\mu_y(x) d\lambda(y) = \operatorname{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1}(x) f(x) d\mathbf{m} \quad (7)$$

for any nonnegative Borel function f .

3.1. Measures absolutely continuous with respect to $\mathbf{m}$. In this section, we provide a formula for the p -modulus in the case when each $\mu_y \in \Sigma$ is absolutely continuous with respect to $\mathbf{m}$.

Proposition 4. Assume there is a Borel function $\alpha = \alpha(x, y) : X \times Y \rightarrow \mathbb{R}_+$ such that $\mu_y = \alpha(\cdot, y)\mathbf{m}$. Put

$$\hat{\alpha}(x) = \int_Y \alpha(x, y) d\lambda(y).$$

Then

$$f_\Sigma(x) = \frac{\hat{\alpha}^{q-1}(x)}{\int_X \hat{\alpha}^q(x) d\mathbf{m}(x)} \quad \text{and} \quad \operatorname{mod}_p(\Sigma) = \left(\int_X \hat{\alpha}^q(x) d\mathbf{m}(x) \right)^{1-p}.$$

Proof. In this case, formula (7) simplifies to

$$\int_Y \int_X \alpha(x, y) f(x) d\mathbf{m} d\lambda(y) = \operatorname{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1}(x) f(x) d\mathbf{m}.$$

Since $f \geq 0$ is arbitrary, by the Fubini–Tonelli theorem, we get

$$\int_Y \alpha(x, y) d\lambda(y) = \text{mod}_p(\Sigma)^{-1} f_\Sigma^{p-1}(x), \quad x \in X.$$

Hence $f_\Sigma(x) = (\text{mod}_p(\Sigma) \hat{\alpha}(x))^{q-1}$. Thus

$$\text{mod}_p(\Sigma) = \text{mod}_p(\Sigma)^q \int_X \hat{\alpha}^q(x) d\mathbf{m},$$

which implies the formula for the p -modulus and for the extremal function. $\square$

If α has separate variables, the formulas are much simpler and, what is probably the most important, we get some additional information about the measure λ .

Corollary 5. *Assume α is of the form $\alpha(x, y) = \beta(x)\gamma(y)$. Then*

$$f_\Sigma(x) = \frac{\beta^{q-1}(x)}{(\int_Y \gamma(y) d\lambda(y))^{q-1} \int_X \beta^q(x) d\mathbf{m}(x)}$$

and

$$\text{mod}_p(\Sigma) = \left(\int_Y \gamma(y) d\lambda(y) \right)^{-p} \left(\int_X \beta^q(x) d\mathbf{m} \right)^{1-p}.$$

Proof. It suffices to notice that

$$\hat{\alpha}(x) = \left(\int_Y \gamma(y) d\lambda(y) \right) \beta(x),$$

and use Proposition 4. $\square$

Corollary 6. *If $\alpha(x, y) = \beta(x)\gamma(y)$, then γ is constant on a subset of Y of full λ -measure.*

Proof. By Theorem 1, we know that $\int_X f_\Sigma(x) d\mu_{y_0}(x) = 1$ for λ -a.e. $y_0 \in Y$. Thus, for λ -a.e. $y_0 \in Y$,

$$1 = \frac{\int_X \beta^{q-1}(x) \beta(x) \gamma(y_0) d\mathbf{m}}{(\int_Y \gamma(y) d\lambda(y))^{q-1} \int_X \beta^q(x) d\mathbf{m}(x)} = \frac{\gamma(y_0)}{\int_Y \gamma(y) d\lambda(y)}.$$

Denoting the denominator on the right hand side of the above formula by c , we have $\gamma(y_0) = c$ for λ -a.e. $y_0 \in Y$. $\square$

3.2. ‘Regular’ family of measures with disjoint supports. Here, we consider a special case of a separate system of measures Σ .

Proposition 7. *Assume there is a Borel map $\alpha : L \times Y \rightarrow X$ and Borel functions $\beta, \gamma : L \times Y \rightarrow \mathbb{R}_+$, where (L, ν) is a Polish space with a Borel measure ν , such that*

$$\alpha(\cdot, y)_\#(\beta(\cdot, y)\nu) = \mu_y, \quad \alpha_\#(\gamma(\nu \otimes \lambda)) = \mathbf{m}.$$

Then, for λ -a.e. y_0 ,

$$f_{\Sigma}(\alpha(z, y)) = \left(\int_L \frac{\beta^q(z, y_0)}{\gamma^{q-1}(z, y_0)} d\nu(z) \right)^{-1} \frac{\beta^{q-1}(z, y)}{\gamma^{q-1}(z, y)} \quad (8)$$

and

$$\text{mod}_p(\Sigma) = \left(\int_L \frac{\beta^q(z, y_0)}{\gamma^{q-1}(z, y_0)} d\nu(z) \right)^{1-p}. \quad (9)$$

Proof. By the assumptions, formula (7) simplifies to (for any nonnegative Borel map f)

$$\begin{aligned} & \int_{L \times Y} f(\alpha(z, y)) \beta(z, y) d\nu(z) d\lambda(y) \\ &= \text{mod}_p(\Sigma)^{-1} \int_{L \times Y} f_{\Sigma}^{p-1}(\alpha(z, y)) f(\alpha(z, y)) \gamma(z, y) d\nu(z) d\lambda(y). \end{aligned}$$

Hence

$$\beta(z, y) = \text{mod}_p(\Sigma)^{-1} f_{\Sigma}^{p-1}(\alpha(z, y)) \gamma(z, y).$$

Therefore

$$f_{\Sigma}(\alpha(z, y)) = \text{mod}_p(\Sigma)^{q-1} \left(\frac{\beta(z, y)}{\gamma(z, y)} \right)^{q-1}.$$

By Theorem 1, for λ -a.e. y , we have

$$\text{mod}_p(\Sigma) = \left(\int_L \frac{\beta^q(z, y)}{\gamma^{q-1}(z, y)} d\nu(z) \right)^{1-p}.$$

This also implies the formula for f_{Σ} . $\square$

Remark 8. Notice that the function of the variable y on the right hand side of (9) is constant on a set of full λ -measure in Y .

Let us prove, in a sense, the converse statement by using Proposition 3.

Proposition 9. Assume there is a Borel bijective map $\alpha : L \times Y \rightarrow X$ and Borel positive functions $\beta, \gamma : L \times Y \rightarrow \mathbb{R}$, where (L, ν) and (Y, η) are Polish spaces with a Borel measure ν and a Borel probability measure η , such that

$$\alpha(\cdot, y)_{\#}(\beta(\cdot, y)\nu) = \mu_y, \quad \alpha_{\#}(\gamma(\nu \otimes \eta)) = \mathbf{m}.$$

If, additionally, $\frac{\beta}{\gamma} \circ \alpha^{-1} \in \mathcal{L}^q(X, \mathbf{m})$ and the function

$$y \mapsto \int_L \frac{\beta^q(z, y)}{\gamma^{q-1}(z, y)} d\nu(z)$$

is constant, then the extremal function f_{Σ} for the p -modulus of Σ and the p -modulus $\text{mod}_p(\Sigma)$ are given by (8) and (9) (for any y_0).

Proof. By assumptions and the Fubini–Tonelli theorem, we have

$$\begin{aligned} \int_Y \int_X f(x) d\mu_y(x) d\eta(y) &= \int_L \int_Y f(\alpha(z, y)) \beta(z, y) d\eta(y) d\nu(z) \\ &= \int_X \frac{\beta}{\gamma} \circ \alpha^{-1}(x) f(x) d\mathbf{m}(x) \end{aligned}$$

for any Borel function $f \geq 0$. Thus (4) holds with $f_\eta = \left(\frac{\beta}{\gamma} \circ \alpha^{-1}\right)^{q-1}$. Moreover, for any $y \in Y$,

$$\int_X f_\eta(x) d\mu_y(x) = \int_L \frac{\beta^q(z, y)}{\gamma^{q-1}(z, y)} d\nu(z).$$

Hence, we may apply Proposition 3. $\square$

Example 10 (Rodin’s formula). The original Rodin formula [6] is the formula for the extremal function for the p -modulus of a family of curves on the Euclidean plane, which are images of horizontal lines in the rectangle $[0, 1] \times [0, b]$. It was generalized to arbitrary dimension [2] (still for a family of curves) and recently, by the author and K. Niedziałowski, to any family of k -dimensional surfaces defined by a single chart [4]. Let us recall this result and show how it can be obtained from Proposition 9.

Theorem 11 ([4]). *Let L and Y be two domains in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively. Let $\alpha : L \times Y \rightarrow \Omega$ be a C^1 -smooth diffeomorphism onto $\Omega \subset \mathbb{R}^n$. Denote by Σ the family of m -dimensional surfaces σ_y , $y \in Y$, being the images of L with respect to f , $\sigma_y = \alpha(L, y)$. Then the extremal function for the p -modulus of Σ is the following*

$$f_\Sigma(x) = \frac{1}{l(z)} \left(\frac{|J_\alpha^z|}{|J_\alpha|} \right)^{q-1} \circ \alpha^{-1}(x), \quad x = \alpha(z, y), \quad (10)$$

where

$$l(y) = \int_L \left(\frac{|J_\alpha^z|}{|J_\alpha|} \right)^q J_\alpha dz. \quad (11)$$

Moreover, the p -modulus of Σ equals

$$\text{mod}_p(\Sigma) = \int_Y l(y)^{1-p} dy. \quad (12)$$

Recall that J_α^z is the Jacobian of a map $z \mapsto \alpha(z, y)$ with fixed z , whereas J_α is the ‘full’ Jacobian of α .

Let $\nu = dz$ be the Lebesgue measure on L and $\mathbf{m} = dx$ be the Lebesgue measure on Ω and let μ_y be the Lebesgue measure on σ_y . Let us define β and γ by

$$\beta(z, y) = |J^z \alpha(z, y)|, \quad \gamma(z, y) = \frac{|J_\alpha(z, y)|}{\rho(y)},$$

where $\rho(y) = cl(y)^{1-p}$ and c is a constant such that the measure

$$\eta = \rho dy$$

is a probability measure on Y , i.e., $c^{-1} = \int_Y l(y)^{1-p} dy$. Since, by the Fubini–Tonelli theorem, for any nonnegative Borel f ,

$$\begin{aligned} \int_Y \int_L f(\alpha(z, y)) \gamma(z, y) d\nu(z) d\eta(y) &= \int_Y \int_L f(\alpha(z, y)) |J_\alpha| dz dy \\ &= \int_X f(x) d\mathbf{m}(x), \end{aligned}$$

we have $\alpha_\#(\gamma \nu \otimes \eta) = \mathbf{m}$. Clearly, $\alpha(\cdot, y)_\#(\beta(\cdot, y)\nu) = \mu_y$. Moreover,

$$\int_L \frac{\beta^q(z, y)}{\gamma^{q-1}(z, y)} d\nu(z) = \int_L c^{q-1} \frac{|J_\alpha^z(z, y)|^q}{|J_\alpha(z, y)|^{q-1}} l(y)^{-1} dz = c^{q-1}$$

for any $y \in Y$. Thus the assumptions of Proposition 9 are satisfied. Hence (12) and (11) hold.

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MAŁGORZATA CİSKA-NIEDZIALOMSKA

Department of Mathematics and Computer Science

University of Łódź

ul. Banacha 22

90-238 Łódź

Poland

e-mail: `malgorzata.niedzialomska@wmii.uni.lodz.pl`

Received: 22 January 2020