Enclosure and non-existence theorems for area stationary currents and currents with mean curvature vector

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Abstract. We discuss certain geometric properties for area stationary currents and currents with integrable mean curvature, so called “enclosure theorems”. As a consequence, we obtain non-existence results for currents with connected support. Finally, we extend these results to currents in submanifolds and state a non-existence result for stationary currents in spheres.

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1. Introduction. Let $\Omega \subset \mathbb{R}^2$ denote a domain and let $X: \Omega \rightarrow \mathbb{R}^3$ be a minimal surface, i.e. a harmonic and conformal mapping of class C^2 . For detailed information on minimal surfaces, we refer to the monographs of Dierkes, Hildebrandt, Sauvigny, and Tromba [4] and [5].

It is well-known that $X(\bar{\Omega})$ is contained in the convex hull of its boundary components $X(\partial\Omega)$. This result holds true for every harmonic mapping due to the maximum principle.

Together with the conformality conditions, Hildebrandt [10] obtained stronger results. In fact, he proved that a minimal surface $X(\bar{\Omega})$ is enclosed by the hyperboloid $H(R) := \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 - z^2 \leq R\}$ for all $R \in \mathbb{R}$ provided $X(\partial\Omega) \subset H(R)$. For $R = 0$, one gets a cone and the theorem still holds true. Furthermore one has that the minimal surface cannot pass through the vertex of the cone. Hildebrandt [10] also stated analogous results for two dimensional H -surfaces.

Again, by using the maximum principle for elliptic equations, Dierkes [3] and Dierkes and Schwab [6] extended these results to compact n -dimensional C^2 -submanifolds $\mathcal{M}$ in $\mathbb{R}^{n+k}$. By regularity, these results immediately lead

to non-existence theorems. In fact, there are no smooth connected minimal submanifolds with boundary components in both disjoint parts of a special cone.

The question arises whether this cone can be enlarged or not. For two dimensional minimal surfaces, this was answered by Osserman and Schiffer [13]. They give the optimal “non-existence” cone. Dierkes [3] proved the corresponding theorem for n -dimensional smooth submanifolds in $\mathbb{R}^{n+1}$. See also the monograph Dierkes et al. [5, Ch. 4] for a complete survey of these results.

We mention that there are more general results of this type in various situations, cf. [1, 9, 19], and [2].

Here we want to address the following question: Do these theorems also generalize to currents? We show in the sequel that this is basically the case. The classical maximum principle will be replaced by a maximum principle of Solomon and White [16]. Maximum principles for singular surfaces in general situations were studied by different authors in the last years. We mention [11, 14], and recently [18] for codimension 1 and [17] where certain varieties of arbitrary codimension are considered.

We use the notation of Simon [15]. Let $n \geq 2, k \geq 1$ be natural numbers and $U \subset \mathbb{R}^{n+k}$ be an open set. We write $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(U)$ for the set of n -dimensional rectifiable currents in U . Our notation slightly differs from most authors as we do not require integer multiplicity θ . As usual, we define the associated Radon measure $\mu_T := \mathcal{H}^n \llcorner \theta$ and $\|\delta T\|$ is the total variation measure. A current with mean curvature $\mathbf{H} = -D_{\mu_T} \|\delta T\| \sigma$ is given by

Definition 1.1. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(U)$ be a current and $\mathbf{H} \in L^1_{\text{loc}}(M \cap U, \mathbb{R}^{n+k}; \mu_T)$. Then T has mean curvature vector $\mathbf{H}$ in U if

$$\int_U \operatorname{div}_M X \, d\mu_T = - \int_U \langle \mathbf{H}, X \rangle \, d\mu_T \quad (1.1)$$

whenever $X \in C^1_c(U \setminus \operatorname{spt} \partial T, \mathbb{R}^{n+k})$.

An important special case is $\mathbf{H}(x) \equiv 0$ leading to the definition of “stationary currents”.

2. Stationary currents.

2.1. Enclosure theorem. The following convex hull property for stationary currents is well-known, cf. [15, Thm. 19.2, Rmk. 34.2].

Theorem 2.1. *Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ with compact support. Then $\operatorname{spt} T \subset \operatorname{conv}(\operatorname{spt} \partial T)$.*

We prove an enclosure result in non-convex sets. Therefore we define for $j = 1, \dots, n-1$, $r_j(x) := \sum_{i=1}^{n+k-j} x_i^2$ and $s_j(x) := \sum_{i=n+k-j+1}^{n+k} x_i^2$, and the quadratic function $q_j(x) = q_j(x_1, \dots, x_{n+k}): \mathbb{R}^{n+k} \rightarrow \mathbb{R}$ given by $q_j(x) := r_j(x) - \frac{n-j}{j} s_j(x)$. Furthermore let $H_j(R) := \{x \in \mathbb{R}^{n+k} : q_j(x) \leq R\}$ be a generalized hyperboloid for some $R \in \mathbb{R}$.

Theorem 2.2 (Enclosure theorem). *Let $\mathcal{H}_j(R)$ be a generalized hyperboloid for $R \in \mathbb{R}$ and $j = 1, \dots, n-1$ congruent to $H_j(R)$. Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ with compact support $\text{spt } T$ and let the boundary values satisfy $\text{spt } \partial T \subset \mathcal{H}_j(R)$. Then we have $\text{spt } T \subset \mathcal{H}_j(R)$.*

Proof. W.l.o.g. assume $\mathcal{H}_j(R) = H_j(R)$. Let $\varepsilon > 0$ be arbitrary and $\gamma \in C^1(\mathbb{R})$ be non-negative and non-decreasing with $\gamma(t) \equiv 0$, $t \leq R + \varepsilon$, and $\gamma(t) > 0$, $\gamma'(t) > 0$ for $t > R + \varepsilon$. Define $\hat{x}: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ by

$$\hat{x}(x) := \left(x_1, \dots, x_{n+k-j}, -\frac{n-j}{j}x_{n+k-j+1}, \dots, -\frac{n-j}{j}x_{n+k} \right)$$

and consider $X(x) := \Psi_{\text{spt } T}(x) \gamma(q_j(x)) \hat{x}(x) \in C_c^1(\mathbb{R}^{n+k} \setminus H_j(R), \mathbb{R}^{n+k})$, where $\Psi_{\text{spt } T}(x) \equiv 1$ in a neighborhood of $\text{spt } T$ is a smooth cut-off function.

Let $\mathcal{T}_x M$ denote the approximate tangent space of T in $x \in M$ (which exists $\mathcal{H}^n$ -a.e.) and $\mathcal{P}_{\mathcal{T}_x M}: \mathbb{R}^{n+k} \rightarrow \mathcal{T}_x M$ the orthogonal projection with matrix representation $(p_{ij})_{i,j=1,\dots,n+k}$ w.r.t. the canonical basis of $\mathbb{R}^{n+k}$. We often abbreviate the projection by $(\cdot)^\top$.

For this vector field X , we have $\text{div}_M X = \langle \nabla_M \gamma, \hat{x} \rangle + \gamma \text{div}_M \hat{x}$ and we calculate the different expressions. Firstly, $\nabla_M \gamma(q_j) = \gamma'(q_j) (Dq_j)^\top = 2\gamma'(q_j) \hat{x}^\top$ which gives $\langle \nabla_M \gamma(q_j), \hat{x} \rangle = 2\gamma'(q_j) \langle \hat{x}^\top, \hat{x} \rangle = 2\gamma'(q_j) |\hat{x}^\top|^2 \geq 0$ and secondly,

$$\begin{aligned} \text{div}_M \hat{x} &= \sum_{i=1}^{n+k} \langle \nabla_M \hat{x}_i, e_i \rangle \\ &= \sum_{i=1}^{n+k-j} \langle \nabla_M x_i, e_i \rangle - \frac{n-j}{j} \sum_{i=n+k-j+1}^{n+k} \langle \nabla_M x_i, e_i \rangle \\ &= \sum_{i=1}^{n+k} p_{ii} - \sum_{i=n+k-j+1}^{n+k} p_{ii} - \frac{n-j}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \\ &= n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \geq 0. \end{aligned}$$

We have used $\text{tr}(\mathcal{P}_{\mathcal{T}_x M}) = n$ in the last equation. Plugging these into (1.1) yields

$$0 = \int_{\mathbb{R}^{n+k}} \gamma(q_j) \left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} + 2\gamma'(q_j) |\hat{x}^\top|^2 d\mu_T.$$

For all $i = n+k-j+1, \dots, n+k$, we have $p_{ii} = 1$ iff $e_i \in \mathcal{T}_x M$. Thus for μ_T -a.e. $x \in M$,

$$\left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} = 0 \text{ iff } p_{ii} = 1 \text{ for all } i = n+k-j+1, \dots, n+k.$$

Otherwise it is strictly positive. Define the set $E := \{x \in M : \mathcal{T}_x M \text{ exists and } e_{n+k-j+1}, \dots, e_{n+k} \in \mathcal{T}_x M\}$. We have on the one hand,

$$\int_{\mathbb{R}^{n+k} \setminus E} \gamma(q_j) \left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} d\mu_T = 0$$

where the expression $\{\dots\}$ is positive and due to the definition of γ , this means $\text{spt } \mu_T \cap (\mathbb{R}^{n+k} \setminus E) \subset \{x \in \mathbb{R}^{n+k} : q_j(x) \leq R + \varepsilon\} = H_j(R + \varepsilon)$. On the other hand, notice that $\{\dots\} = 0$ in E , thus

$$0 = \int_E \gamma(q_j) \left\{ n - \frac{n}{j} \sum_{i=n+k-j+1}^{n+k} p_{ii} \right\} + \gamma'(q_j) |\hat{x}^\top|^2 d\mu_T = \int_E \gamma'(q_j) |\hat{x}^\top|^2 d\mu_T.$$

Now notice that $|\hat{x}^\top|^2 = 0$ iff $\hat{x} \perp \mathcal{T}_x M$, this means in particular $x_{n+k-j+1} = \dots = x_{n+k} = 0$ in E . In other words, if $x_{n+k-j+l} \neq 0$ for (at least) some $l = 1, \dots, j$, then $|\hat{x}^\top|^2 > 0$ in E . Again, because of γ , this establishes

$$\text{spt } \mu_T \cap (E \cap \{x \in \mathbb{R}^{n+k} : x_{n+k-j+l} \neq 0 \text{ for one } l = 1, \dots, j\}) \subset H_j(R + \varepsilon)$$

and combining both results, we arrive at

$$\text{spt } \mu_T \subset H_j(R + \varepsilon) \cup (E \cap \{x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = 0, \dots, x_{n+k} = 0\}). \quad (2.1)$$

Finally, we claim that $E \cap \{x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = 0, \dots, x_{n+k} = 0\} \subset H_j(R + \varepsilon)$. Assume there exists $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n+k}) \in \text{spt } \mu_T$ such that $\tilde{x} \in E$ and $\tilde{x}_{n+k-j+1} = \dots = \tilde{x}_{n+k} = 0$ but $\tilde{x} \notin H_j(R + \varepsilon)$. Choose a function $f \in C_c^0(\mathbb{R}^{n+k})$ which satisfies $f \equiv 0$ in $\{x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = 0, \dots, x_{n+k} = 0\}$ and is strict positive anywhere else in some sufficiently large ball. Then, since $\tilde{x}$ is off the closed hyperboloid together with the fact (2.1), we have that in a small neighborhood points of $\text{spt } T$ fulfill $x_{n+k-j+1} = 0, \dots, x_{n+k} = 0$. By definition of f , this means in particular

$$\lim_{\lambda \searrow 0} \int_{\eta_{\tilde{x}, \lambda}(M)} f(y) \theta(\tilde{x} + \lambda y) d\mathcal{H}^n(y) = 0.$$

Notice that, because of $\tilde{x} \in E$, we have $e_{n+k-j+1}, \dots, e_{n+k} \in \mathcal{T}_{\tilde{x}} M$. In other words, $\mathcal{T}_{\tilde{x}} M \not\subset \{x \in \mathbb{R}^{n+k} : x_{n+k-j+1} = 0, \dots, x_{n+k} = 0\}$.

This leads to a contradiction to the definition of the approximate tangent space $\theta(\tilde{x}) \int_{\mathcal{T}_{\tilde{x}} M} f(y) d\mathcal{H}^n(y) > 0$. This gives $\text{spt } T = \text{spt } \mu_T \subset H_j(R + \varepsilon)$ and since $\varepsilon > 0$ was arbitrary, the theorem is established. $\square$

2.2. Non-existence theorem.

Definition 2.3. A current $T \in \mathcal{R}_n(U)$ is called connected if $\text{spt } T$ is a connected set.

For $j = 1$ and $R = 0$, we get the cone $H_1(0) = K := \{(x_1, \dots, x_{n+k}) \in \mathbb{R}^{n+k} : x_1^2 + \dots + x_{n+k-1}^2 \leq (n-1)x_{n+k}^2\}$. Let $K^\pm := K \cap \{\pm x_{n+k} > 0\}$ and $T^\pm := T \llcorner \{x \in \mathbb{R}^{n+k} : \pm x_{n+k} > 0\} \in \mathcal{R}_n(\mathbb{R}^{n+k})$. From Theorem 2.2, we deduce that $\text{spt } T^+ \subset \bar{K}^+ = K^+ \cup \{0\}$ and $\text{spt } T^- \subset \bar{K}^- = K^- \cup \{0\}$.

Lemma 2.4. *Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a stationary current in $\mathbb{R}^{n+k}$ and $\text{spt } T \subset K$. Then both currents $T^+ \in \mathcal{R}_n(\mathbb{R}^{n+k})$ and $T^- \in \mathcal{R}_n(\mathbb{R}^{n+k})$ are also stationary in $\mathbb{R}^{n+k}$.*

Proof. We consider T^+ . Let $\varepsilon > 0$ be arbitrary and $\phi_\varepsilon(t) \in C^1(\mathbb{R})$ be non-negative such that $\phi_\varepsilon(t) \equiv 0$, $t \leq 0$, and $\phi_\varepsilon(t) \equiv 1$, $t > \varepsilon$, as well as $0 \leq \phi'_\varepsilon(t) \leq c/\varepsilon$ for all $t \in \mathbb{R}$ for some $c > 0$. Because K^+ is contained in $\{x \in \mathbb{R}^{n+k} : x_{n+k} > 0\}$, it follows $K^+ \cap \{x \in \mathbb{R}^{n+k} : x_{n+k} < \varepsilon\} \subset B_{c\varepsilon}(0)$ for some (other) $c > 0$, which depends only on the angle of the cone, i.e. on the dimension n . From monotonicity for stationary currents, the estimate $\mu_T(B_{c\varepsilon}(0)) \leq c\varepsilon^n$ holds true for small $\varepsilon > 0$. Let X be an arbitrary compactly supported C^1 vector field in $\mathbb{R}^{n+k}$ and $\text{spt } X \cap \text{spt } \partial T^+ = \emptyset$. In view of the stationarity of T , we can test $\phi_\varepsilon(x_{n+k})X$. Thus

$$\begin{aligned} 0 &= \int_{K^+ \cap \{x_{n+k} < \varepsilon\}} \phi'_\varepsilon(x_{n+k}) \langle e_{n+k}^\top, X \rangle d\mu_T + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \text{div}_M X d\mu_T \\ &\leq \frac{c}{\varepsilon} |e_{n+k}^\top| \sup_{\mathbb{R}^{n+k}} |X| \int_{B_{c\varepsilon}(0)} d\mu_T + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \text{div}_M X d\mu_T \\ &\leq \frac{c}{\varepsilon} \mu_T(B_{c\varepsilon}(0)) + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \text{div}_M X d\mu_T \\ &\leq c\varepsilon^{n-1} + \int_{\mathbb{R}^{n+k}} \phi_\varepsilon(x_{n+k}) \text{div}_M X d\mu_T \\ &\xrightarrow{\varepsilon \searrow 0} \int_{\mathbb{R}^{n+k}} \chi_{\{x_{n+k} \geq 0\}} \text{div}_M X d\mu_T = \int_{\mathbb{R}^{n+k}} \text{div}_M X d\mu_{T^+}. \end{aligned}$$

The same inequality holds true for $-X$ and establishes the stationarity of T^+ . □

Applying Theorem 2.1 to T^+ and T^- , respectively, together with the fact that $\text{spt } \partial T^\pm \subset K^\pm$, we get $0 \notin \text{spt } T$. Since all calculations are invariant under translations, we have proved the following theorem.

Theorem 2.5 (Non-existence theorem). *Let $\mathcal{K} = \mathcal{K}^+ \cup \{p_0\} \cup \mathcal{K}^- \subset \mathbb{R}^{n+k}$ be a cone with vertex p_0 which is congruent to K . Then there exists no stationary current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ such that $\text{spt } T$ is compact and connected with boundary values $\text{spt } \partial T \subset \mathcal{K}^+ \cup \mathcal{K}^-$ such that both $\text{spt } \partial T \cap \mathcal{K}^+$ and $\text{spt } \partial T \cap \mathcal{K}^-$ are non-empty.*

Remark. As n increases, the angle of aperture β of K is increasing as well. Precisely we have $\beta = \arctan(\sqrt{n-1})$ and therefore $\beta \rightarrow 90^\circ$ as $n \rightarrow \infty$. The enclosure and non-existence theorems extend the existence theorem for currents with integer multiplicity, cf. [15, Lem. 34.1], as we get more information of the solution, e.g., the shape or the disconnectedness in at least two parts.

The proof of the following is the same as in [3, Cor.3].

Corollary 2.6 (Necessary conditions). *Let $B_1, B_2 \subset \mathbb{R}^{n+k}$ be two closed sets and suppose there exists a connected, stationary current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ in $\mathbb{R}^{n+k}$ with $\text{spt } \partial T \subset B_1 \cup B_2$ and that both $\text{spt } \partial T \cap B_1 \neq \emptyset$ and $\text{spt } \partial T \cap B_2 \neq \emptyset$. Then we have:*

- (i) *If $B_i := \{x \in \mathbb{R}^{n+k} : |x - y_i| \leq \delta_i\}, i = 1, 2$, are closed balls with centers y_i and radii δ_i and $R := |y_1 - y_2|$, then $R \leq \left(\frac{n}{n-1}\right)^{1/2} (\delta_1 + \delta_2)$.*
- (ii) *If B_1 and B_2 are arbitrary compact sets of diameter d_1 and d_2 , separated by a slab of width $r > 0$, then $r \leq \frac{1}{2} \left(\frac{2n(n+k)}{(n-1)(n+k-1)}\right)^{1/2} (d_1 + d_2)$.*

2.3. Optimal results in codimension one. The question arises if it is possible to “enlarge” the cone and still prove non-existence. We restrict ourselves to codimension $k = 1$. We use Dierkes’ [3] construction of n -dimensional catenoids enclosing a cone with a larger angle of aperture. Therefore we show that the enclosing procedure also holds true for currents.

Consider a curve $(x, y(x))$ in $\mathbb{R}^2$ and its rotational symmetric n -dimensional graph $\{(x, y(x), \omega) \in \mathbb{R} \times \mathbb{R}^n : x \in [x_0, x_1], \omega \in \mathcal{S}^{n-1}\}$. Its area is proportional to the one dimensional integral $\int_{x_0}^{x_1} y^{n-1}(x) \sqrt{1 + y'(x)^2} dx$. Stationary solutions of this variational integral correspond to n -dimensional minimal submanifolds in $\mathbb{R}^{n+1}$. Results from calculus of variation give for a solution $y(x)$, the family of inverse functions $x(y) = a \int_{n-1\sqrt{a}}^y \frac{1}{\sqrt{\xi^{2(n-1)} - a^2}} d\xi + c$ for all $a > 0, c \in \mathbb{R}$, and for all $y \geq n-1\sqrt{a}$. Then every member of the one-parameter family $x = g(y, a), a > 0$, is tangent to the half line $y = \tau_0 x, x > 0$ (independent of a), where $\tau_0 := (z_0^{2(n-1)} - 1)^{1/2}$ and z_0 is the unique solution of

$$\frac{z}{\sqrt{z^{2(n-1)} - 1}} = \int_1^z \frac{d\xi}{\sqrt{\xi^{2(n-1)} - 1}}.$$

Furthermore every point of the half line is contained in exactly one member of this family.

Let $f(\cdot, a)$ be the family of inverse functions and extend f by even reflection, $f(x, a) = f(-x, a)$ for $x \leq 0$, then we have a smooth function $f(\cdot, a) : \mathbb{R} \rightarrow \mathbb{R}$. Define $\rho := (x_1^2 + \dots + x_n^2)^{1/2}$, then $\mathcal{M}_a := \{x \in \mathbb{R}^{n+1} : \rho = f(x_{n+1}, a)\}$, $a > 0$, is a family of smooth n -dimensional minimal submanifolds in $\mathbb{R}^{n+1}$. Furthermore $C_a := \{x \in \mathbb{R}^{n+1} : \rho \leq f(x_{n+1}, a)\}$, $a > 0$, “encloses” by construction the cone $K^* := K \cup \{0\} := \{x \in \mathbb{R}^{n+1} : \rho < \pm \tau_0 x_{n+1}\} \cup \{0\}$.

Proposition 2.7 (Enclosure result.) *Let $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ be a stationary current in $\mathbb{R}^{n+1}$ with compact support and $\text{spt } \partial T \subset K = \{x \in \mathbb{R}^{n+1} : \rho < \pm \tau_0 x_{n+1}\}$. Then we have $\text{spt } T \subset \bar{K}$.*

Proof. We will show $\text{spt } T \subset C_a$ for all $a > 0$. Because of $\bigcap_{a>0} C_a = \bar{K}$, the proposition is then established. Let us assume $\text{spt } T \not\subset C_a$ for some $a > 0$ and consider $\eta_{0,\lambda\#} T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ with minimal $\lambda > 1$ s.t. $\text{spt}(\eta_{0,\lambda\#} T) \subset C_a$. Then there exists $p \in \text{spt}(\eta_{0,\lambda\#} T)$ with $p \in \partial C_a$. The contracted current lies completely on one side of the smooth submanifold ∂C_a and touches it at least

in p . We want to apply the maximum principle of Solomon and White [16]. Therefore let us abbreviate for the fixed λ : $\eta(x) := \eta_{0,\lambda}(x)$ and $\tilde{T} := \eta_{0,\lambda\#}T = \eta_{\#}T$.

Claims:

- (i) If $\text{spt } \partial T \subset K$, then we have $\text{spt } \partial \tilde{T} \subset K$.
- (ii) $\mathcal{T}_x M$ exists in x iff $\mathcal{T}_{\eta(x)}\eta(M)$ exists in $\eta(x)$ and we have equality.
- (iii) Let T be stationary in $\mathbb{R}^{n+1}$, then so is $\tilde{T} = \tau(\eta(M), \theta_{\tilde{T}}, \xi_{\tilde{T}})$.

Claim (i) and (ii) are proved by direct calculations. For the third statement, note that we have for the Jacobian, $J_M \eta = \lambda^{-n}$, cf. [15, Ch. 12]. Let $Y \in C_c^1(\mathbb{R}^{n+1} \setminus \text{spt}(\partial(\eta_{0,\lambda\#}T)), \mathbb{R}^{n+1})$ be arbitrary. Then $X(x) := Y \circ \eta(x)$ is of class $C_c^1(\mathbb{R}^{n+1} \setminus \text{spt}(\partial T), \mathbb{R}^{n+1})$. We calculate the variation of the contracted current

$$\begin{aligned} \delta \tilde{T}(Y) &= \int_{\mathbb{R}^{n+1}} \text{div}_{\eta(M)} Y(y) \, d\mu_{\tilde{T}}(y) \\ &= \int_{\eta(M)} \text{div}_{\eta(M)} Y(y) \, \theta_{\tilde{T}}(y) \, d\mathcal{H}^n(y) \\ &= \int_{\eta(M)} \text{div}_{\eta(M)} (X \circ \eta^{-1}(y)) \, \theta_T \circ \eta^{-1}(y) \, d\mathcal{H}^n(y). \end{aligned}$$

We have for the first factor

$$\begin{aligned} \text{div}_{\eta(M)} (X \circ \eta^{-1}(y)) &= \sum_{i=1}^{n+1} \langle \mathcal{P}_{\mathcal{T}_y \eta(M)} [D(X_i \circ \eta^{-1}(y))], e_i \rangle \\ &= \sum_{i=1}^{n+1} \langle \mathcal{P}_{\mathcal{T}_y \eta(M)} [DX_i \circ \eta^{-1}(y) \mathbb{I}_{n+1} \lambda], e_i \rangle \\ &= \lambda \sum_{i=1}^{n+1} \langle \mathcal{P}_{\mathcal{T}_{\eta(x)} \eta(M)} [DX_i \circ \eta^{-1}(y)], e_i \rangle \\ &= \lambda \sum_{i=1}^{n+1} \langle \mathcal{P}_{\mathcal{T}_{\eta(x)} \eta(M)} [DX_i], e_i \rangle \circ \eta^{-1}(y) \\ &= \lambda (\text{div}_M X) \circ \eta^{-1}(y) \end{aligned}$$

where we have used the second claim. Finally, we get by applying the area formula

$$\begin{aligned} \delta \tilde{T}(Y) &= \lambda \int_{\eta(M)} (\text{div}_M X \, \theta_T) \circ \eta^{-1} \, d\mathcal{H}^n = \lambda \int_M \text{div}_M X \, \theta_T \, J_M \eta \, d\mathcal{H}^n \\ &= \lambda \lambda^{-n} \int_M \text{div}_M X \, \theta_T \, d\mathcal{H}^n = \lambda^{-n+1} \int_M \text{div}_M X \, d\mu_T = 0. \end{aligned}$$

Thus $\tilde{T}$ is stationary in $\mathbb{R}^{n+1}$ because Y was arbitrary. Now we are able to apply the maximum principle [16] since the support of T is compact. Therefore

we get coincidence of $\text{spt } \tilde{T}$ and ∂C_a in an open subset. This is a contradiction to claim (i). $\square$

With *exactly* the same argument as in Section 2.2, we can prove the non-existence theorem. Furthermore it is optimal because the cone is enclosed by a family of minimal submanifolds.

Theorem 2.8 (Optimal non-existence theorem). *Let $\mathcal{K} = \mathcal{K}^+ \cup \{p_0\} \cup \mathcal{K}^- \subset \mathbb{R}^{n+1}$ be a cone with vertex p_0 congruent to $K^\pm \cup \{0\} = \{x \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 - \tau_0^2 x_{n+1}^2 < 0, \pm x_{n+1} > 0\} \cup \{0\}$. Then there is no stationary current $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ with compact support such that $\text{spt } T$ is connected and the boundary fulfills $\text{spt } \partial T \subset \mathcal{K}^+ \cup \mathcal{K}^-$ such that $\text{spt } \partial T \cap \mathcal{K}^+$ as well as $\text{spt } \partial T \cap \mathcal{K}^-$ is non-empty.*

3. Currents with mean curvature vector.

3.1. Enclosure theorem. The enclosure Theorem 2.2 naturally extends to currents with mean curvature vector $\mathbf{H} \in L^1_{\text{loc}}(\mu_T)$ under appropriate conditions. To this end, we define the quadratic function $q_j(x)$ with an additional parameter $b \in [0, 1]$. Thus $q_j(x) := r_j(x) - (n-j)/j b s_j(x)$ and again $H_j(R) := \{x \in \mathbb{R}^{n+k} : q_j(x) \leq R\}$.

Theorem 3.1 (Enclosure theorem). *Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a current in $\mathbb{R}^{n+k}$ with mean curvature vector $\mathbf{H}(x)$ and compact support $\text{spt } T$. It satisfies $\text{spt } \partial T \subset H_j(R)$ for $R \in \mathbb{R}$, $j = 1, \dots, n-1$, and $b \in [0, 1]$. The mean curvature vector $\mathbf{H}$ fulfills*

$$b + |\mathbf{H}(x)| \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T - \text{a.e. } x \in \mathbb{R}^{n+k} \setminus H_j(R). \quad (3.1)$$

Then $\text{spt } T \subset H_j(R)$.

Proof. The proof is similar to the case $\mathbf{H} \equiv 0$. Define

$$\hat{x}(x) := \left(x_1, \dots, x_{n+k-j}, -\frac{n-j}{j} b x_{n+k-j+1}, \dots, -\frac{n-j}{j} b x_{n+k} \right)$$

and consider $X(x) := \Psi_{\text{spt } T}(x) \gamma(q_j(x)) \hat{x}(x)$. Note that the additional term can be estimated directly with the Cauchy–Schwarz inequality $\langle \mathbf{H}(x), \hat{x}(x) \rangle \geq -|\mathbf{H}(x)| |\hat{x}(x)| = -|\mathbf{H}(x)| \left[r_j(x) + \left(\frac{n-j}{j} \right)^2 b^2 s_j(x) \right]^{1/2}$. Plugging all terms into (1.1) gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{n+k}} \langle \nabla_M \gamma(q_j), \hat{x} \rangle + \text{div}_M \hat{x} \gamma(q_j) + \gamma(q_j) \langle \hat{x}, \mathbf{H} \rangle d\mu_T \\ &\geq \int_{\mathbb{R}^{n+k}} \gamma(q_j) \left\{ n - \left(1 + \frac{n-j}{j} b \right) \sum_{i=n+k-j+1}^{n+k} p_{ii} - |\mathbf{H}| \left[r_j + \left(\frac{n-j}{j} \right)^2 b^2 s_j \right]^{1/2} \right\} \\ &\quad + 2\gamma'(q_j) |\hat{x}^\top|^2 d\mu_T. \end{aligned}$$

We can estimate the expression $\{...\}$ pointwise

$$\begin{aligned} n - \left(1 + \frac{n-j}{j} b\right) \sum_{i=n+k-j+1}^{n+k} p_{ii} - |\mathbf{H}| \left[r_j + \left(\frac{n-j}{j}\right)^2 b^2 s_j \right]^{1/2} \\ \geq n - \left(1 + \frac{n-j}{j} b\right) j - |\mathbf{H}| \left[r_j + \left(\frac{n-j}{j}\right)^2 b^2 s_j \right]^{1/2} \\ = (n-j) \left((1-b) - |\mathbf{H}| \left[\frac{r_j}{(n-j)^2} + \left(\frac{b}{j}\right)^2 s_j \right]^{1/2} \right) \end{aligned} \quad (3.2)$$

and now notice that the resulting term is non-negative in $\mathbb{R}^{n+k} \setminus H_j(R)$ for μ_T -a.e. x . Next we integrate over the sets E and $\mathbb{R}^{n+k} \setminus E$ where E is defined exactly as above. In the latter case, we evidently have strict inequality in (3.2) as not all p_{ii} are equal to one. Thus $\text{spt } \mu_T \cap (\mathbb{R}^{n+k} \setminus E) \subset H_j(R)$. The proof is now finished as before. $\square$

We mention two sufficient characterizations for the mean curvature vector $\mathbf{H}$ such that the condition (3.1) is satisfied. Both are easier to check.

Lemma 3.2. *The condition (3.1) is fulfilled provided*

- (i) $q := |x| |\mathbf{H}(x)| < 1$ for μ_T -a.e. $x \in \mathbb{R}^{n+k} \setminus H_j(R)$ and $b \leq 1 - q$ or
- (ii) $q := |x| |\mathbf{H}(x)| < n - j$ for μ_T -a.e. $x \in \mathbb{R}^{n+k} \setminus H_j(R)$ and $b \leq \min \left\{ \frac{1}{n-1}, 1 - \frac{q}{n-j} \right\}$.

Proof. A calculation gives

$$b + |\mathbf{H}| \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq b + |\mathbf{H}| |x| = b + q \leq 1$$

thus i) is established. Estimations of b yield the second claim

$$\begin{aligned} b + |\mathbf{H}| \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} &= b + |\mathbf{H}| \frac{1}{n-j} \left[r_j(x) + \left(\frac{n-j}{j}\right)^2 b^2 s_j(x) \right]^{1/2} \\ &\leq 1 - \frac{1}{n-j} q + |\mathbf{H}| \frac{1}{n-j} \left[r_j(x) + \left(\frac{n-j}{j}\right)^2 \left(\frac{1}{n-1}\right)^2 s_j(x) \right]^{1/2} \\ &\leq 1 - \frac{1}{n-j} q + |\mathbf{H}| \frac{1}{n-j} |x| = 1 - \frac{1}{n-j} q + \frac{1}{n-j} q = 1. \end{aligned}$$

$\square$

3.2. Non-existence theorem. Now let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a current with $0 \in \text{spt } T \subset K := H_1(0)$. Notice, for any current $T = \tau(M, \theta, \xi)$, we have an associated rectifiable varifold $V = v(M, \theta)$ just by dropping the orientation. In fact, we have as in [15].

Proposition 3.3. *Let V be a rectifiable varifold in some neighborhood of the origin. Suppose the density exists in 0 and for $V_j := \eta_{0, \lambda_j} \# V, \lambda_j \searrow 0$, we have $\mu_{V_j} \rightarrow \mu_C$ in $\mathbb{R}^{n+k}$ such that μ_C is associated to a rectifiable varifold C which is stationary in $\mathbb{R}^{n+k}$. Then C is a cone.*

Theorem 3.4. *Let $V = v(M, \theta)$ and $C = v(N, \vartheta)$ be two varifolds as above. If $\text{spt } V \subset K := \{x \in \mathbb{R}^{n+k} : x_1^2 + \dots + x_{n+k-1}^2 \leq (n-1)b x_{n+k}^2\}$, then we have $\text{spt } C \subset K$ as well.*

Lemma 3.5. *Let $(\mu_j)_{j \in \mathbb{N}}, \mu$ be Radon measures with $\mu_j \rightarrow \mu, j \rightarrow \infty$, in $\mathbb{R}^{n+k}$ (weak convergence of Radon measures). Assume for some closed set A , we have $\text{spt } \mu_j \subset A$ for all $j \in \mathbb{N}$. Then $\text{spt } \mu \subset A$.*

Proof. We show $x \notin A$ implies the existence of a $\rho > 0$ such that $\mu(B_\rho(x)) = 0$. From the hypothesis, we have $\text{dist}(x, A) > \varepsilon > 0$ and therefore define $\rho := \frac{1}{3} \text{dist}(x, A)$. Notice, because of $\text{spt } \mu_j \subset A$ for all $j \in \mathbb{N}$, it evidently follows $B_{2\rho}(x) \cap \text{spt } \mu_j = \emptyset$ for all $j \in \mathbb{N}$. Now choose a continuous function $\phi : \mathbb{R}^{n+k} \rightarrow [0, 1]$ satisfying $\phi|_{B_\rho(x)} \equiv 1$ and $\phi|_{\mathbb{R}^{n+k} \setminus B_{2\rho}(x)} \equiv 0$. Since

$$0 \leq \mu(B_\rho(x)) \leq \liminf_{j \rightarrow \infty} \mu_j(B_\rho(x)) \leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}^{n+k}} \phi \, d\mu_j = \liminf_{j \rightarrow \infty} \int_{\text{spt } \mu_j} \phi \, d\mu_j = 0,$$

this completes the proof. $\square$

Proof of Theorem 3.4. Again, define $V_j := \eta_{0, \lambda_j} \# V$, then we know for every j that $\text{spt } V_j = \text{spt}(\eta_{0, \lambda_j} \# V) \subset \eta_{0, \lambda_j}(\text{spt}(V)) \subset \eta_{0, \lambda_j}(K) = K$. From Proposition 3.3, we obtain $\mu_{V_j} \rightarrow \mu_C, j \rightarrow \infty$, and Lemma 3.5 gives $\text{spt } C = \text{spt } \mu_C \subset K$ as required. $\square$

We consider the two restricted varifolds $C^\pm := C \llcorner \{x : \pm x_{n+k} > 0\} = v(N \cap \{x : \pm x_{n+k} > 0\}, \vartheta|_{N \cap \{x : \pm x_{n+k} > 0\}})$ satisfying $\text{spt } C^+ \subset \bar{K}^+ = K^+ \cup \{0\} \subset \{x \in \mathbb{R}^{n+k} : x_{n+k} \geq 0\}$ and $\text{spt } C^-$ is obviously contained in $\{x \in \mathbb{R}^{n+k} : x_{n+k} \leq 0\}$. Both varifolds are still cones and literally as in the proof of Lemma 2.4, it follows that both C^+ and C^- are stationary in $\mathbb{R}^{n+k}$. Hence we can apply the following lemma to conclude a contradiction. For a proof, see Simon [15, Thm. 36.5 & Rmk. 36.6].

Lemma 3.6. *Let $C = v(N, \vartheta)$ be a rectifiable varifold stationary in $\mathbb{R}^{n+k}$ and satisfying $\eta_{0, \lambda} \# C = C$ for all $\lambda > 0$. Furthermore assume $\text{spt } C \subset \bar{\mathfrak{H}}$ where $\mathfrak{H}$ is some open half space in $\mathbb{R}^{n+k}$ with $0 \in \partial \mathfrak{H}$. Then $\text{spt } C \subset \partial \mathfrak{H}$.*

This gives $\text{spt } C^\pm \subset \{x_{n+k} = 0\}$ in contradiction to $\text{spt } C^\pm \subset \bar{K} = K$ because by construction evidently $C \not\equiv 0$. Notice in particular this means that the support of a stationary tangent cone cannot pass through the vertex of the cone K and therefore the origin cannot be contained in the support of the current T . In order to fulfill the assumptions of Proposition 3.3, we assume $\mathbf{H} \in L_{\text{loc}}^p(U, \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$ and $\theta \geq 1$ μ_T -a.e. in U for some small neighborhood U of the origin with $U \cap \text{spt } \partial T = \emptyset$. This gives the existence of an area stationary tangent cone in every $x \in \text{spt } T \cap U$, see [15, Ch. 8]. Finally, this establishes

Theorem 3.7 (Non-existence theorem). *Let $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ be a current with compact support $\text{spt } T$ and mean curvature vector $\mathbf{H}$. Assume for some $b \in [0, 1]$,*

$$\text{spt } \partial T \subset K^\pm := \left\{ x \in \mathbb{R}^{n+k} : \sum_{i=1}^{n+k-1} x_i^2 \leq (n-1)b x_{n+k}^2, \pm x_{n+k} > 0 \right\}$$

such that both $\text{spt } \partial T \cap K^+$ and $\text{spt } \partial T \cap K^-$ are non-empty. Furthermore define $r_1(x) := x_1^2 + \cdots + x_{n+k-1}^2$ and suppose the mean curvature vector $\mathbf{H}$ satisfies on the one hand

$$b + |\mathbf{H}| \left[\frac{r_1(x)}{(n-1)^2} + b^2 x_{n+k}^2 \right]^{1/2} \leq 1 \text{ for } \mu_T - \text{a.e. } x \in \mathbb{R}^{n+k} \setminus K \quad (3.3)$$

and on the other hand, in some neighborhood U of the origin, $\mathbf{H} \in L_{\text{loc}}^p(U, \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$ and $\theta(x) \geq 1$ for μ_T -a.e. $x \in U$. Then $\text{spt } T$ cannot be connected.

Remark. Instead of condition (3.3), one of the requirements of Lemma 3.2 with $j = 1$ and $R = 0$ can also be fulfilled. The condition $\theta \geq 1$ μ_T -a.e. is automatically given for currents with integer multiplicity.

4. Enclosure and non-existence theorems for currents in submanifolds. Here we discuss an important modification of stationarity in Euclidean spaces. More generally we let $\mathcal{N}$ be an $(n + l)$ -dimensional C^2 -submanifold of $\mathbb{R}^{n+k}$ for $0 \leq l \leq k$. We denote by $B_y: \mathcal{T}_y \mathcal{N} \times \mathcal{T}_y \mathcal{N} \rightarrow (\mathcal{T}_y \mathcal{N})^\perp$ the second fundamental form of $\mathcal{N}$ at y . Then T is called stationary in $\mathcal{N}$ if the first variational formula holds true with $\mathbf{H}_M := \sum_{i=1}^n B_x(\tau_i, \tau_i) = \text{tr}(B_x)|_{\mathcal{T}_x M}$ and $\{\tau_1, \dots, \tau_n\}$ is any orthonormal basis for the approximate tangent space $\mathcal{T}_x M$ of T in x , cf. Simon [15, Def. 16.4].

Theorem 4.1 (General enclosure and non-existence theorem). *Let $\mathcal{N}$ be an $(n + l)$ -dimensional embedded C^2 -submanifold of $\mathbb{R}^{n+k}$ for $0 \leq l \leq k$. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k})$, $M \subset \mathcal{N}$ be a stationary current in $\mathcal{N}$ with compact support. Furthermore for some $R \in \mathbb{R}$, assume*

$$b + |\mathbf{H}_M(x)| \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T - \text{a.e. } x \in \mathcal{N} \setminus H_j(R). \quad (4.1)$$

If $\text{spt } \partial T \subset H_j(R)$, then $\text{spt } T \subset H_j(R)$.

Let (4.1) be true for $j = 1$ and $R = 0$ and assume for $\varepsilon > 0$, on the one hand, $\theta \geq 1$ μ_T -a.e. in $B_\varepsilon(0)$ and on the other hand, $\mathbf{H}_M \in L_{\text{loc}}^p(B_\varepsilon(0), \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$. Furthermore let $\text{spt } \partial T \subset K^\pm$ such that both $\text{spt } \partial T \cap K^+ \neq \emptyset$ and $\text{spt } \partial T \cap K^- \neq \emptyset$. Then $\text{spt } T$ cannot be a connected set.

We want to estimate the abstract curvature $\mathbf{H}_M$ against some quantity depending only on the submanifold $\mathcal{N}$. We restrict ourselves to the case $l = k - 1$. As usual, we can define the principal curvatures $\kappa_1, \dots, \kappa_{n+k-1}$ of $\mathcal{N}$ which we order with respect to their absolute value: $|\kappa_1| \geq \cdots \geq |\kappa_{n+k-1}|$. We define $\Lambda_n := |\kappa_1| + \cdots + |\kappa_n|$ what we call the n -mean curvature of $\mathcal{N}$. This number slightly differs from the usual n -mean curvature used in barrier principles of higher codimension, cf. [6, 7]. We are now able to proceed similarly as in [12] and arrive at the important estimate: For μ_T -a.e. $x \in M$, we have $|\mathbf{H}_M(x)| = |\text{tr}(B_x)|_{\mathcal{T}_x M}| \leq \Lambda_n(x)$. This is just a conclusion of the following general result where we skip the proof.

Lemma 4.2. *Let $Q: \mathfrak{V} \rightarrow \mathbb{R}$ be a quadratic form of an N -dimensional Euclidean vector space $\mathfrak{V}$ with ordered eigenvalues $|\kappa_1| \geq \dots \geq |\kappa_N|$. Then we have for every n -dimensional subspace $\mathfrak{W} \subset \mathfrak{V}$, the estimate $|\operatorname{tr}(Q)|_{\mathfrak{W}}| \leq |\kappa_1| + \dots + |\kappa_n|$.*

So we get results under conditions just depending on the ambient space.

Corollary 4.3 (Enclosure and non-existence theorem in hypersurfaces). *Let $\mathcal{N}$ be an $(n+k-1)$ -dimensional embedded C^2 -submanifold of $\mathbb{R}^{n+k}$. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k})$, $M \subset \mathcal{N}$ be a stationary current in $\mathcal{N}$ with compact support. Furthermore for some $R \in \mathbb{R}$, assume*

$$b + \Lambda_n(x) \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \text{ for } \mu_T - \text{a.e. } x \in \mathcal{N} \setminus H_j(R). \quad (4.2)$$

If $\operatorname{spt} \partial T \subset H_j(R)$, then $\operatorname{spt} T \subset H_j(R)$.

Let (4.2) be true for $j = 1$ and $R = 0$ and assume for $\varepsilon > 0$, on the one hand, $\theta \geq 1$ μ_T -a.e. in $B_\varepsilon(0)$ and on the other hand, $\Lambda_n \in L^p_{\text{loc}}(B_\varepsilon(0), \mathbb{R}^{n+k}; \mu_T)$ for some $p > n$. Furthermore let $\operatorname{spt} \partial T \subset K^\pm$ such that both $\operatorname{spt} \partial T \cap K^+ \neq \emptyset$ and $\operatorname{spt} \partial T \cap K^- \neq \emptyset$. Then $\operatorname{spt} T$ cannot be a connected set.

Example 4.4 (Non-existence of stationary currents in spheres). Define the shifted sphere $\mathcal{S} = \mathcal{S}^{n+k-1}(x_0, R) := \{x \in \mathbb{R}^{n+k} : |x - x_0| = R\}$ with $0 \in \mathcal{S}$. Let $T = \tau(M, \theta, \xi) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with integer multiplicity and $M \subset \mathcal{S}$ be a stationary current in $\mathcal{S}$. Assume $\operatorname{spt} T$ is compact, $q := \sup_{x \in \operatorname{spt} T} |x| \frac{n}{R} < 1$, and suppose $\operatorname{spt} \partial T \subset K := \{x \in \mathbb{R}^{n+k} : x_1^2 + \dots + x_{n+k-1}^2 \leq (n-1)(1-q)x_{n+k}^2\}$ such that both $\operatorname{spt} \partial T \cap (K \cap \{x_{n+k} > 0\}) \neq \emptyset$ and $\operatorname{spt} \partial T \cap (K \cap \{x_{n+k} < 0\}) \neq \emptyset$. Then T cannot be connected.

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