Strictly elliptic operators with generalized Wentzell boundary conditions on continuous functions on manifolds with boundary

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Abstract. We prove that strictly elliptic operators with generalized Wentzell boundary conditions generate analytic semigroups of angle $\frac{\pi}{2}$ on the space of continuous functions on a compact manifold with boundary.

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1. Introduction. We start from a strictly elliptic differential operator A_m with domain $D(A_m)$ on the space $C(\overline{M})$ of continuous functions on a smooth, compact, orientable Riemannian manifold $(\overline{M}, g)$ with smooth boundary ∂M . Moreover, let C be a strictly elliptic differential operator on the boundary, take $\frac{\partial^a}{\partial \nu^g} : D(\frac{\partial^a}{\partial \nu^g}) \subset C(\overline{M}) \rightarrow C(\partial M)$ to be the outer conormal derivative, and functions $\eta, \gamma \in C(\partial M)$ with η strictly positive and a constant $q > 0$. In this setting, we define the operator $A^B \subset A_m$ with *generalized Wentzell boundary conditions* by requiring

$$\begin{aligned} f \in D(A^B) : & \iff f \in D(A_m) \cap D\left(\frac{\partial^a}{\partial \nu^g}\right), \quad A_m f|_{\partial M} \\ & = q \cdot C f|_{\partial M} - \eta \cdot \frac{\partial^a}{\partial \nu^g} f + \gamma \cdot f|_{\partial M}. \end{aligned} \quad (1.1)$$

On a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, Favini, Goldstein, Goldstein, Obrecht, and Romanelli [8] showed that for $A_m = \Delta_\Omega$ and $C = \Delta_{\partial\Omega}$ the operator A^B generates an analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{\Omega})$. In a preprint Goldstein, Goldstein, and Pierre [9] generalized this statement to arbitrary elliptic differential operators of the form $A_m f := \sum_{l,k=1}^n \partial_l(a^{kl}\partial_k f)$ and $C\varphi := \sum_{l,k=1}^n \partial_l(\alpha^{kl}\partial_k \varphi)$.

Our main theorem (Theorem 4.6) generalizes these results to arbitrary strictly elliptic operators A_m and C on smooth, compact, orientable Riemannian manifolds with smooth boundary.

Consider a half-ball $B_1^+(0) := \{x \in \mathbb{R}^n : x_n \geq 0, |x| \leq 1\} \subset \mathbb{R}^n$. With the restriction g of the metric of $\mathbb{R}^n$ to $(B_1^+(0), g)$, we obtain a smooth, compact, orientable Riemannian manifold $B_1^+(0)$ with smooth boundary. It is not the closure of a domain in $\mathbb{R}^n$ since the boundary is only $\partial B_1^+(0) = \{x \in \mathbb{R}^n : x_n = 0, |x| \leq 1\}$.

The situation $q = 0$ on bounded, smooth domains in $\mathbb{R}^n$ was studied by Engel and Fragnelli [5] and on smooth, compact, orientable Riemannian manifolds in [3].

For $q = 0$, the boundary condition is a partial differential equation of first order whereas for $q > 0$ it is a partial differential equation of second order. Using the theory developed in [5] and [2], this yields two different abstract Dirichlet-to-Neumann operators: In the case $q = 0$, it is a pseudo differential operator of first order, in the case $q > 0$, it is an elliptic differential operator of second order perturbed by a pseudo differential operator of first order.

The paper is organized as follows. In the second section, we introduce the abstract setting from [5] and [2] for our problem. In the third section, we study the special case that A_m is the Laplace-Beltrami operator and B is the normal derivative. In the last section, we generalize to arbitrary strictly elliptic operators and their conormal derivatives.

Throughout the whole paper, we use the Einstein notation for sums and write $x_i y^i$ shortly for $\sum_{i=1}^n x_i y^i$. Moreover, we denote by $\hookrightarrow$ a continuous and by $\xhookrightarrow{c}$ a compact embedding.

2. The abstract setting. As in [5, Sect. 2], the basis of our investigation is the following.

Abstract setting 2.1. Consider

- (i) two Banach spaces X and ∂X , called *state* and *boundary space*, respectively;
- (ii) a densely defined *maximal operator* $A_m : D(A_m) \subset X \rightarrow X$;
- (iii) a *boundary (or trace) operator* $L \in \mathcal{L}(X, \partial X)$;
- (iv) a *feedback operator* $B : D(B) \subseteq X \rightarrow \partial X$.

Using these spaces and operators, we define the operator $A^B : D(A^B) \subset X \rightarrow X$ with abstract *generalized Wentzell boundary conditions* as

$$A^B f := A_m f, \quad D(A^B) := \{f \in D(A_m) \cap D(B) : L A_m f = B f\}. \quad (2.1)$$

For an interpretation of Wentzell boundary conditions as “dynamic boundary conditions”, we refer to [5, Sect. 2].

In the sequel, we need the following operators.

Notation 2.2. The kernel of L is a closed subspace and we consider the restriction $A_0 \subset A_m$ given by

$$A_0 : D(A_0) \subset X \rightarrow X, \quad D(A_0) := \{f \in D(A_m) : L f = 0\}.$$

The *abstract Dirichlet operator associated with A_m* is, if it exists,

$$L_0^{A_m} := (L|_{\ker(A_m)})^{-1} : \partial X \rightarrow \ker(A_m) \subseteq X,$$

i.e. $L_0^{A_m} \varphi = f$ is the unique solution of the abstract Dirichlet problem

$$\begin{cases} A_m f = 0, \\ Lf = \varphi. \end{cases} \quad (2.2)$$

If it is clear which operator A_m is meant, we simply write L_0 .

Finally, we introduce the *abstract Dirichlet-to-Neumann operator associated with (A_m, B)* , defined by

$$N^{A_m, B} \varphi := BL_0^{A_m} \varphi, \quad D(N^{A_m, B}) := \{\varphi \in \partial X : L_0^{A_m} \varphi \in D(B)\}.$$

If it is clear which operators A_m and B are meant, we write $N = N^{A_m, B}$ and call it the (abstract) Dirichlet-to-Neumann operator.

3. Laplace–Beltrami operator with generalized Wentzell boundary conditions.

Take now as maximal operator $A_m : D(A_m) \subset C(\overline{M}) \rightarrow C(\overline{M})$ the Laplace–Beltrami operator Δ_M^g with domain $D(A_m) := \left\{ f \in \bigcap_{p>1} W_{loc}^{2,p}(M) \cap C(\overline{M}) : A_m f \in C(\overline{M}) \right\}$. Moreover, consider another strictly elliptic differential operator $C : D(C) \subset C(\partial M) \rightarrow C(\partial M)$ in divergence form on the boundary space. To this end, take real valued functions

$$\alpha_j^k = \alpha_k^j \in C^\infty(\partial M), \quad \beta_j \in C(\partial M), \quad \gamma \in C(\partial M), \quad 1 \leq j, k \leq n,$$

such that α_j^k are strictly elliptic, i.e.

$$\alpha_j^k(q) g^{jl}(q) X_k(q) X_l(q) > 0$$

for all co-vectorfields X_k, X_l on ∂M with $(X_1(q), \dots, X_n(q)) \neq (0, \dots, 0)$. Let $\alpha = (\alpha_j^k)_{j,k=1,\dots,n}$ denote the 1-1-tensorfield and $\beta = (\beta_j)_{j=1,\dots,n}$. Moreover, we denote by $|\alpha|$ the determinate of α and define $C : D(C) \subset C(\partial M) \rightarrow C(\partial M)$ by

$$\begin{aligned} C\varphi &:= \sqrt{|\alpha|} \operatorname{div}_g \left(\frac{1}{\sqrt{|\alpha|}} \alpha \nabla_{\partial M}^g \varphi \right) + \langle \beta, \nabla_{\partial M}^g \varphi \rangle + \gamma \cdot \varphi, \\ D(C) &:= \left\{ \varphi \in \bigcap_{p>1} W^{2,p}(\partial M) : C\varphi \in C(\partial M) \right\}. \end{aligned} \quad (3.1)$$

In order to define the feedback operator, we first consider $B_0 : D(B_0) \subset C(\overline{M}) \rightarrow C(\partial M)$ given by

$$B_0 f := -g(a \nabla_M^g f, \nu_g), \quad D(B_0) := \left\{ f \in \bigcap_{p>1} W_{loc}^{2,p}(M) \cap C(\overline{M}) : B_0 f \in C(\partial M) \right\}.$$

This leads to the feedback operator $B : D(B) \subset C(\overline{M}) \rightarrow C(\partial M)$ given by

$$\begin{aligned} Bf &:= q \cdot CLf - \eta \cdot g(\nabla_M^g f, \nu_g), \\ D(B) &:= \{f \in D(A_m) \cap D(B_0) : Lf \in D(C)\}, \end{aligned}$$

where $L: C(\overline{M}) \rightarrow C(\partial M)$, $f \mapsto f|_{\partial M}$ denotes the trace operator and $q > 0$ and $\eta \in C(\overline{M})$ is positive. Using these operators A_m and B , we define the operator A^B with Wentzell boundary conditions on $C(\overline{M})$ as in (2.1).

Note that the feedback operator B can be splitted into

$$B = q \cdot CL + \eta \cdot B_0.$$

The following proof is inspired by [7] and similar to [2, Ex. 5.3].

Lemma 3.1. *The operator B is relatively A_0 -bounded of bound 0.*

Proof. Since $D(A_0) \subset \ker(L)$, the operators B and $\eta \cdot B_0$ coincide on $D(A_0)$. Hence it remains to prove the statement for the operator B_0 . By [13, Chap. 5., Thm. 1.3] and the closed graph theorem, we obtain

$$[D(A_0)] \hookrightarrow W^{2,p}(M).$$

Rellich's embedding (see [1, Thm. §3 2.10, Part III.]) implies

$$W^{2,p}(M) \xhookrightarrow{c} C^{1,\alpha}(M) \xhookrightarrow{c} C^1(\overline{M})$$

for $p > \frac{n-1}{1-\alpha}$, where n denotes the dimension of $\overline{M}$. So we obtain

$$[D(A_0)] \xhookrightarrow{c} C^1(\overline{M}) \hookrightarrow C(\overline{M}).$$

Therefore, by Ehrling's lemma (cf. [12, Thm. 6.99]), for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_{C^1(\overline{M})} \leq \varepsilon \|f\|_{A_0} + C_\varepsilon \|f\|_X$$

for every $f \in D(A_0)$. Since $B_0 \in \mathcal{L}(C^1(\overline{M}), \partial X)$, this implies the claim. $\square$

Lemma 3.2. *The operator N^{Δ_m, B_0} is relatively C -bounded of bound 0.*

Proof. Let $W := -(\Delta_{\partial M}^g)^{\frac{1}{2}}$ and remark that by the proof of [3, Thm. 3.8], there exists a relatively W -bounded perturbation P of bound 0 such that

$$N^{\Delta_m, B_0} = W + P.$$

Therefore [11, Thm. 3.8] implies that N^{Δ_m, B_0} is relatively $\Delta_{\partial M}^g$ -bounded of bound 0. Using the (uniform) ellipticity of C , there exists a constant $\Lambda > 0$ such that

$$\|\Delta_{\partial M}^g \varphi\|_{C(\partial M)} \leq \Lambda \cdot \|C\varphi\|_{C(\partial M)}$$

for $\varphi \in D(C) = D(\Delta_{\partial M}^g)$. Hence N^{Δ_m, B_0} is relatively C -bounded of bound 0. $\square$

Now the abstract results of [2] lead to the desired result.

Theorem 3.3. *The operator A^B with Wentzell boundary conditions associated to the Laplace-Beltrami operator $\Delta_m = \Delta_M^g$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{M})$.*

Proof. We verify the assumptions of [2, Thm. 4.3]. Remark that by [3, Lem. 3.6] and Lemma 3.1 above, the Dirichlet operator $L_0 \in \mathcal{L}(C(\partial M), C(\overline{M}))$ exists and B is relatively A_0 -bounded of bound 0. By multiplicative perturbation, we assume without loss of generality that $q = 1$. Now [4, Thm. 1.1] implies that A_0 is sectorial of angle $\frac{\pi}{2}$ on $C(\overline{M})$ and has compact resolvent. Moreover, by [4, Cor. 3.6], the operator C generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(\partial M)$. Finally, the claim follows by [2, Thm. 4.3]. $\square$

4. Elliptic operators with generalized Wentzell boundary conditions. Consider a strictly elliptic differential operator $A_m: D(A_m) \subset C(\overline{M}) \rightarrow C(\overline{M})$ in divergence form on the boundary space. To this end, let

$$a_j^k = a_k^j \in C^\infty(\overline{M}), \quad b_j \in C_c(M), \quad c \in C(\overline{M}), \quad 1 \leq j, k \leq n,$$

be real-valued functions, such that a_j^k are strictly elliptic, i.e.

$$a_j^k(q)g^{jl}(q)X_k(q)X_l(q) > 0$$

for all co-vectorfields X_k, X_l on $\overline{M}$ with $(X_1(q), \dots, X_n(q)) \neq (0, \dots, 0)$. Let $a = (a_j^k)_{j,k=1,\dots,n}$ be the 1-1-tensorfield and $b = (b_j)_{j=1,\dots,n}$. Then we define $A_m: D(A_m) \subset C(\overline{M}) \rightarrow C(\overline{M})$ by

$$\begin{aligned} A_m f &:= \sqrt{|a|} \operatorname{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + c \cdot f, \\ D(A_m) &:= \left\{ \varphi \in \bigcap_{p>1} W_{loc}^{2,p}(M) \cap C(\overline{M}) : A_m f \in C(\overline{M}) \right\}. \end{aligned} \quad (4.1)$$

Note that, since $\overline{M}$ is compact, every strictly elliptic operator is uniformly elliptic (and of course vice versa).

We consider a $(2, 0)$ -tensorfield on $\overline{M}$ given by

$$\tilde{g}^{kl} = a_i^k g^{il}.$$

Its inverse $\tilde{g}$ is a $(0, 2)$ -tensorfield on $\overline{M}$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $\overline{M}$. We denote $\overline{M}$ with the old metric by $\overline{M}^g$ and with the new metric by $\overline{M}^{\tilde{g}}$ and remark that $\overline{M}^{\tilde{g}}$ is a smooth, compact, orientable Riemannian manifold with smooth boundary ∂M . Since the differentiable structures of $\overline{M}^g$ and $\overline{M}^{\tilde{g}}$ coincide, the identity

$$\operatorname{Id}: \overline{M}^g \longrightarrow \overline{M}^{\tilde{g}}$$

is a C^∞ -diffeomorphism. Hence the spaces

$$\begin{aligned} X &:= C(\overline{M}) := C(\overline{M}^{\tilde{g}}) = C(\overline{M}^g) \\ \text{and} \quad \partial X &:= C(\partial M) := C(\partial \overline{M}^{\tilde{g}}) = C(\partial \overline{M}^g) \end{aligned}$$

coincide. Moreover, [10, Prop. 2.2] implies that the following spaces coincide

$$\begin{aligned}
 L^p(M) &:= L^p(M^{\tilde{g}}) = L^p(M^g), \\
 W^{k,p}(M) &:= W^{k,p}(M^{\tilde{g}}) = W^{k,p}(M^g), \\
 L^p_{loc}(M) &:= L^p_{loc}(M^{\tilde{g}}) = L^p_{loc}(M^g), \\
 W^{k,p}_{loc}(M) &:= W^{k,p}_{loc}(M^{\tilde{g}}) = W^{k,p}_{loc}(M^g), \\
 L^p(\partial M) &:= L^p(\partial M^{\tilde{g}}) = L^p(\partial M^g), \\
 W^{k,p}(\partial M) &:= W^{k,p}(\partial M^{\tilde{g}}) = W^{k,p}(\partial M^g), \\
 L^p_{loc}(\partial M) &:= L^p_{loc}(\partial M^{\tilde{g}}) = L^p_{loc}(\partial M^g), \\
 W^{k,p}_{loc}(\partial M) &:= W^{k,p}_{loc}(\partial M^{\tilde{g}}) = W^{k,p}_{loc}(\partial M^g)
 \end{aligned} \tag{4.2}$$

for all $p > 1$ and $k \in \mathbb{N}$. Denote by $\hat{A}_m$ the maximal operator defined in (4.1) with $b_j = c = 0$ and by $\hat{C}$ the operator given in (3.1) for $\beta_j = \gamma = 0$. Moreover, denote the corresponding feedback operator by $\hat{B}$.

Next, we look at the operators A_m , B_0 , and C with respect to the new metric $\tilde{g}$.

Lemma 4.1. *The operator $\hat{A}_m$ and the Laplace-Beltrami operator $\Delta_M^{\tilde{g}}$ coincide on $C(\overline{M})$.*

Proof. Using local coordinates, we obtain

$$\begin{aligned}
 \hat{A}_m f &= \frac{1}{\sqrt{|g|}} \sqrt{|a|} \partial_j \left(\sqrt{|g|} \frac{1}{\sqrt{|a|}} a_l^j g^{kl} \partial_k f \right) \\
 &= \frac{1}{\sqrt{|\tilde{g}|}} \partial_j \left(\sqrt{|\tilde{g}|} \tilde{g}^{kl} \partial_k f \right) = \Delta_m^{\tilde{g}} f
 \end{aligned}$$

for $f \in D(\hat{A}_m) = D(\Delta_m^{\tilde{g}})$ since $|g| = |a| \cdot |\tilde{g}|$. □

Now we compare the maximal operators A_m and $\hat{A}_m$.

Lemma 4.2. *The operators A_m and $\hat{A}_m$ differ only by a relatively bounded perturbation of bound 0.*

Proof. Using (4.2), we define

$$P_1 f := b_l g^{kl} \partial_k f$$

for $f \in D(A_m) \cap D(\hat{A}_m)$. Since $b_l \in C_c(M)$, there exist compact sets $K_l := \text{supp}(b_l)$. Let $K := \bigcup_{l=1}^n K_l$ and note that it is a compact set and every b_l and hence $P_1 f$ vanishes outside of K . We define

$$(\hat{A}_m)|_K f := \Delta_m^{\tilde{g}} f$$

$$\begin{aligned}
 D((\hat{A}_m)|_K) &:= \{f \in C(K) : \text{there exists a function } \tilde{f} \in D(\hat{A}_m) \\
 &\quad \text{such that } \tilde{f}|_K = f\}.
 \end{aligned}$$

Morrey's embedding ([1, Thm. §3 2.10, Part III.]) implies

$$[D((\hat{A}_m)|_K)] \xhookrightarrow{c} C^1(K) \hookrightarrow C(K). \tag{4.3}$$

Moreover, we obtain

$$\begin{aligned}\|P_1 f\|_{C(\overline{M})} &\leq \sup_{q \in \overline{M}} |b_l(q) g^{kl}(q) (\partial_k f)(q)| \\ &= \sup_{q \in K} |b_l(q) g^{kl}(q) (\partial_k f)(q)| \\ &\leq C \sum_{k=1}^n \|(\partial_k f)|_K\|_{C(K)}\end{aligned}$$

and therefore $P_1 \in \mathcal{L}(C^1(K), C(\overline{M}))$. Hence $D(\hat{A}_m) = D(\tilde{A}_m)$. By (4.3), we conclude from Ehrling's lemma (see [12, Thm. 6.99]) that

$$\begin{aligned}\|P_1 f\|_{C(\overline{M})} &\leq C \|f|_K\|_{C^1(K)} \leq \varepsilon \|(\hat{A}_m)|_K f|_K\|_{C(K)} \\ &\quad + \varepsilon \|f|_K\|_{C(K)} + C(\varepsilon) \|f|_K\|_{C(K)} \\ &\leq \varepsilon \|\hat{A}_m f\|_{C(\overline{M})} + \tilde{C}(\varepsilon) \|f\|_{C(\overline{M})}\end{aligned}$$

for $f \in D(\hat{A}_m)$ and all $\varepsilon > 0$. Hence P_1 is relatively A_m -bounded of bound 0. Finally remark that

$$P_2 f := c \cdot f, \quad D(P_2) := C(\overline{M})$$

is bounded and that

$$\tilde{A}_m f = \hat{A}_m f + P_1 f + P_2 f$$

for $f \in D(\hat{A}_m)$. □

Lemma 4.3. *The operators B_0 and the negative conormal derivative $-\frac{\partial}{\partial \nu^{\tilde{g}}}$ coincide.*

Proof. Since the Sobolev spaces coincide, we compute in local coordinates

$$\begin{aligned}B_0 f &= -g_{ij} g^{jl} a_l^k \partial_k f g^{im} \nu_m \\ &= -g_{ij} \tilde{g}^{jl} \partial_k f g^{im} \nu_m \\ &= -\tilde{g}_{ij} \tilde{g}^{jl} \partial_k f \tilde{g}^{im} \nu_m \\ &= -\frac{\partial}{\partial \nu^{\tilde{g}}} f\end{aligned}$$

for $f \in D(B) = D(\frac{\partial^{\tilde{g}}}{\partial \nu})$. □

Define $\tilde{C}: D(\tilde{C}) \subset C(\partial M) \rightarrow C(\partial M)$ by

$$\tilde{C}\varphi := \sqrt{|\tilde{\alpha}|} \operatorname{div}_{\tilde{g}} \left(\frac{1}{\sqrt{|\tilde{\alpha}|}} \tilde{\alpha} \nabla_{\partial M}^{\tilde{g}} \varphi \right), \quad D(\tilde{C}) := \{\varphi \in W^{2,p}(\partial M) : C\varphi \in C(\partial M)\},$$

where $\tilde{\alpha}(q) := a(q)^{-1} \cdot \alpha(q)$.

Lemma 4.4. *The operators $\hat{C}$ and $\tilde{C}$ coincide on $C(\partial M)$.*

Proof. An easy calculation shows

$$\frac{|\tilde{g}|}{|\tilde{\alpha}|} = \frac{|g|}{|\alpha|},$$

$$\tilde{\alpha}_l^k \tilde{g}^{lj} = \alpha_l^k g^{lj}.$$

Hence we obtain in local coordinates

$$\begin{aligned} \tilde{C}\varphi &= \sqrt{\frac{|\tilde{\alpha}|}{|\tilde{g}|}} \partial_k \left(\sqrt{\frac{|\tilde{g}|}{|\tilde{\alpha}|}} \tilde{\alpha}_l^k \tilde{g}^{li} \partial_i \varphi \right) \\ &= \sqrt{\frac{|\alpha|}{|g|}} \partial_k \left(\sqrt{\frac{|g|}{|\alpha|}} \alpha_l^k g^{li} \partial_i \varphi \right) \\ &= \sqrt{|\alpha|} \operatorname{div}_g \left(\frac{1}{|\alpha|} \alpha \nabla^j \varphi \right) = \hat{C}\varphi \end{aligned}$$

for $\varphi \in D(\hat{C}) = D(\tilde{C})$. □

Next we compare the operators C and $\hat{C}$.

Lemma 4.5. *The operators C and $\hat{C}$ differ only by a relatively bounded perturbation of bound 0.*

Proof. Denote by

$$P\varphi := \langle \beta, \nabla_{\partial M}^g \rangle + \gamma \cdot \varphi \text{ for } f \in D(P) := C^1(\partial M)$$

and note that $P \in \mathcal{L}(C^1(\partial M), C(\partial M))$. The Sobolev embeddings and the closed graph theorem imply

$$[D(C)] \xhookrightarrow{c} C^1(\partial M) \hookrightarrow C(\partial M).$$

Finally, the claim follows by Ehrling's lemma (cf. [12, Thm. 6.99]). □

Now we are prepared to prove our main theorem.

Theorem 4.6. *The operator A^B with Wentzell boundary conditions generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{M})$.*

Proof. Since $\tilde{C}$ is a strictly elliptic differential operator in divergence form on $C(\partial M)$, we obtain by Theorem 3.3 that the Laplace-Beltrami operator with Wentzell boundary conditions given by

$$(\Delta_M^{\tilde{g}} f)|_{\partial M} = q \cdot \tilde{C}f|_{\partial M} - \eta \frac{\partial \tilde{g}}{\partial \nu} f$$

generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{M})$. Now Lemma 4.1, Lemma 4.3, and Lemma 4.4 imply that the operator $\hat{A}^{\hat{B}}$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{M})$. Note that A_m and $\hat{A}_m$ differ only by a relatively A_m -bounded perturbation of bound 0 by Lemma 4.2. By Lemma 4.5, one obtains that the perturbation on the boundary is relatively $\hat{C}$ -bounded. Now the claim follows from [2, Thm. 4.2]. □

Remark 4.7. Theorem 4.6 generalizes the main theorem in [9] for the case $p = \infty$.

Corollary 4.8. *The initial-value boundary problem*

$$\begin{cases} \frac{d}{dt}u(t, q) = A_m u(t, q), & t \geq 0, \quad q \in \overline{M}, \\ \frac{d}{dt}\varphi(t, q) = B u(t, q), & t \geq 0, \quad q \in \partial M, \\ u(t, x) = \varphi(t, x), & t \geq 0, \quad x \in \partial M, \\ u(0, q) = u_0(q), & q \in \overline{M}, \end{cases}$$

on $C(\overline{M})$ is well-posed. Moreover, the solution $\begin{pmatrix} u(t) \\ \varphi(t) \end{pmatrix} \in C^\infty(M) \times C^\infty(\partial M)$ for $t > 0$ depends analytically on the initial value $\begin{pmatrix} u_0 \\ u_0|_{\partial M} \end{pmatrix}$ and is governed by a compact and analytic semigroup, which can be extended to the right half plane.

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