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Correction

Correction to: Holomorphic curves in Shimura varieties

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Correction to: Arch. Math. 111 (2018), 379–388 https://doi.org/10.1007/s00013-018-1227-4.

This erratum addresses an error in the application of a theorem of Hwang and To in the cited paper.

Erwan Rousseau pointed out that in the proof of [2, Theorem 3.2] the result of [3] about volumes of analytic sets in Hermitian symmetric domains cannot be applied as the set considered in the proof is not in general analytic. To fix this, it is enough to substitute [2, Theorem 3.6] with Proposition 0.7.

We start by recalling some definitions about o-minimal structures and cell decomposition.

Fix an o-minimal expansion $\tilde{\mathbb{R}}$ of the field of real numbers. By definable, we mean definable in $\tilde{\mathbb{R}}$.

Definition 0.1. Let $i_0, \ldots, i_n \in \{0,1\}$. A $(i_1, \ldots, i_n)$-*cell* is a subset of $\mathbb{R}^n$ defined inductively as follows:

- A (0)-cell is a point,
- a (1)-cell is a segment,
- an $(i_1, \ldots, i_{n-1}, 0)$-cell is the graph of a definable function on an $(i_1, \ldots, i_{n-1})$-cell,
- an $(i_1, \ldots, i_{n-1}, 1)$-cell is defined as the set

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n | f(x_1, \ldots, x_{n-1}) < x_n < g(x_0, \ldots, x_{n-1}) \text{ and } (x_1, \ldots, x_{n-1}) \in C\} \quad (0.1)$$

where C is an $(i_1, \ldots, i_{n-1})$-cell and f and g are definable functions on C.

Definition 0.2. A cell is said to be *analytic* if the definable functions used in its inductive definition can be chosen analytic.

Definition 0.3. A *decomposition* of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ defined inductively as follows:

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- A decomposition of $\mathbb{R}$ is a partition of the type
$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty), \{a_1\}, \ldots, \{a_k\}\}, \tag{0.2}$$
- a decomposition of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many cells $\{A_i\}_{i\in I}$ such that the set of projections $\{\pi(A_i)\}_{i\in I}$ is a decomposition of $\mathbb{R}^{n-1}$; here π is the projection onto the first $n-1$-coordinates.

Theorem 0.4 (Cell decomposition). *Given definable sets $A_1, \ldots, A_l$ in $\mathbb{R}^m$, there is a decomposition of $\mathbb{R}^n$ which partitions each of the A_i.*

Definition 0.5. The o-minimal structure $\tilde{\mathbb{R}}$ is said to admit *analytic cell decomposition* if it satisfies the cell decomposition theorem with the additional requirement that the cells can all be chosen analytic.

A result of van den Dries and Miller in [1, Section 8] implies.

Theorem 0.6 (Analytic cell decomposition). *The o-minimal structures $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$ admit analytic cell decomposition.*

The proof of the proposition below follows the guideline of [4, Theorem 2.7].

Proposition 0.7. *Assume $\tilde{\mathbb{R}}$ admits analytic cell decomposition. Let U be a connected $\tilde{\mathbb{R}}$-definable subset of $\mathcal{X}$ of dimension 2 such that $\dim_{\mathbb{R}}(\bar{U} \cap \partial\mathcal{X}) = 1$. Fix a point $x_0 \in \mathcal{X}$. Then there exist real numbers c_1, c_2 such that for any $R > 0$ sufficiently large,*
$$Vol(B(x_0, R) \cap U) \geq c_1 \exp(c_2 R), \tag{0.3}$$
where $B(x_0, R)$ is the geodesic ball in $\mathcal{X}$ of centre x_0 and radius R.

Proof. In the course of the proof, we will use the following notation:
- $\Delta_{\alpha,\beta} = \{r \exp i\theta | 0 \leq r < 1 \text{ and } \alpha < \theta < \beta\}$ where $0 < \alpha < \beta$ are real numbers,
- $C_{\alpha,\beta} = \{\exp i\theta | \alpha < \theta < \beta\}$,
- $\bar{\Delta}_{\alpha,\beta} = \Delta_{\alpha,\beta} \cup C_{\alpha,\beta}$.

Since $\tilde{\mathbb{R}}$ admits analytic cell decomposition, we may fix an analytic cell U' of dimension 2 such that $\bar{U} \cap \partial\mathcal{X}$ is a connected real analytic curve. This implies that we can find positive real numbers α, β and a real analytic map $\psi : \Delta_{\alpha,\beta} \to U'$ with the following properties:
- $\psi(\Delta_{\alpha,\beta})$ is contained in U',
- ψ extends to a real analytic function in a neighbourhood of $\bar{\Delta}_{\alpha,\beta}$ such that $\psi(C_{\alpha,\beta}) \subset \bar{U}' \cap \partial\mathcal{X}$ is a non-constant real analytic curve.

Let ω be the Kähler form on $\mathcal{X}$ associated with its Bergmann metric. By [4, Lemma 2.8], we have
$$\psi^*\omega = s\omega_\Delta + \eta, \tag{0.4}$$
where ω_Δ is the Poincaré metric on the unit disc Δ and η is a smooth $(1,1)-$form in a neighbourhood of $C_{\alpha,\beta}$. Now given $R, c_3 > 0$, consider the set
$$I^R_{\alpha,\beta} = \left\{z \in \Delta_{\alpha,\beta} | c_3 e^{-R-1} \leq d_{\Delta,e}(z, \partial\Delta) \leq c_3 e^{-R}\right\}. \tag{0.5}$$

This set is an annulus sector inside the unit circle. The main point in considering this set is that as R tends to infinity, the hyperbolic distance of $I^R_{\alpha,\beta}$ from the origin tends to infinity and its volume is exponential in R. We now use [4, Lemma 2.4 and Lemma 2.8] to see that there exists a constant $c_3 > 0$ such that the image $\psi(I^R_{\alpha,\beta})$ in $\mathcal{X}$ is contained in the geodesic ball $B_{\mathcal{X},h}(x_0, R)$. We are now ready to calculate the volumes. Let $R > 0$ be sufficiently large, then

$$\mathrm{Vol}\,(U \cap B(x_0, R)) \geq \int\limits_{I^R_{\alpha,\beta}} \omega_\Delta \geq c_1 \exp(c_2 R) \tag{0.6}$$

for some positive constant $c_1, c_2 > 0$. □

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