



## Finite groups with few normalizers or involutions

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**Abstract.** The groups having exactly one normalizer are Dedekind groups. All finite groups with exactly two normalizers were classified by Pérez-Ramos in 1988. In this paper we prove that every finite group with at most 26 normalizers of  $\{2, 3, 5\}$ -subgroups is soluble and we also show that every finite group with at most 21 normalizers of cyclic  $\{2, 3, 5\}$ -subgroups is soluble. These confirm Conjecture 3.7 of Zarrin (Bull Aust Math Soc 86:416–423, 2012).

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**1. Introduction and notation.** All groups considered in this paper are finite. We use conventional notions and notations. There have been several results that investigate, in a quantitative way, the influences of properties of subgroups on the structure of the whole group. For example in [2] it was shown that an insoluble group has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to  $A_5$  or  $SL(2, 5)$  and that confirmed a conjecture of Zarrin [17].

We say that a group  $G$  is an  $\mathfrak{N}_n$ -group ( $\mathfrak{N}_n^c$ -group, respectively) if it has exactly  $n$  normalizers of subgroups (normalizers of cyclic subgroups, respectively). The groups belonging to  $\mathfrak{N}_1$  are the Dedekind groups, which are well known. Pérez-Ramos [9] classified  $\mathfrak{N}_2$ -groups. In general, the number of normalizers of subgroups (normalizers of cyclic subgroups) should impact the structure of the group (see [18]). In [18] Zarrin proposed the following conjectures.

**Conjecture 1.** ([18, Conjecture 3.6]) *Every  $\mathfrak{N}_n$ -group with  $n \leq 26$  is soluble.*

**Conjecture 2.** ([18, Conjecture 3.7]) *Every  $\mathfrak{N}_n^c$ -group with  $n \leq 21$  is soluble.*

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Zarrin in [19] confirmed Conjecture 1 for arbitrary groups (not necessarily for finite groups). In fact, he proved that every arbitrary  $\mathfrak{N}_n$ -group with  $n \leq 26$  is soluble. In this paper, we generalize these conjectures in the class of finite groups and we show that it is enough to consider only normalizers of  $\{2, 3, 5\}$ -subgroups of a group. In fact we confirm the solubility of a group with at most 26 normalizers of  $\{2, 3, 5\}$ -subgroups and a group with at most 21 normalizers of cyclic  $\{2, 3, 5\}$ -subgroups (Theorem 3.2).

The classification of simple groups by the number of involutions was investigated for example in [5, 20]. In [20] Zarrin gave a counterexample to Herzog's conjecture on simple groups with the same number of involutions and he formulated his own conjecture on groups with the same number of involutions and the same number of elements of order  $p$ , where  $p$  is an odd prime. In Section 4 we give a few counterexamples to a conjecture of Zarrin [20] and we reformulate his conjecture.

**2. Preliminaries.** Let  $\pi$  be a set of primes, a  $\pi$ -subgroup is any subgroup of a group  $G$  such that the primes dividing its order are all in  $\pi$ . Clearly the trivial subgroup  $\{1\}$  of the group  $G$  is a  $\pi$ -subgroup for any set  $\pi$  of primes and  $N_G(\{1\}) = G$ . A  $\pi'$ -subgroup is any subgroup of a group  $G$  such that none of the prime factors of its order are in  $\pi$ . We say that a group  $G$  is an  $\mathfrak{N}_n(\pi)$ -group ( $\mathfrak{N}_n^c(\pi)$ -group, respectively) if it has exactly  $n$  normalizers of  $\pi$ -subgroups (normalizers of cyclic  $\pi$ -subgroups, respectively). Denote by  $\pi(G)$  the set of all prime divisors of the order of  $G$ .

Let  $p$  be a fixed prime. We write  $\mathbb{P}_p$  to denote the set of primes less than or equal  $p$ . The result [11, Theorem 3.2] implies that  $G \in \mathfrak{N}_1(\mathbb{P}_2)$  if and only if  $G$  is a direct product of a Dedekind 2-group and a group of odd order, hence it is soluble by the Feit-Thompson theorem on solubility of groups of odd order. Note that if  $p$  and  $q$  are primes with  $p < q$ , then  $\mathbb{P}_p \subseteq \mathbb{P}_q$ ,  $\mathfrak{N}_n(\mathbb{P}_q) \subseteq \mathfrak{N}_m(\mathbb{P}_p)$  for some  $m \leq n$ , and  $\mathfrak{N}_s^c(\mathbb{P}_q) \subseteq \mathfrak{N}_r^c(\mathbb{P}_p)$  for some  $r \leq s$ .

Here are some elementary properties of normalizers.

**Lemma 2.1.** *The following statements about a group  $G$  are true.*

- (1) *If  $H$  and  $K$  are subgroups of  $G$ , then  $N_K(H) = N_G(H) \cap K$ .*
- (2) *If  $H$  is a subgroup of  $G$  and  $K$  is a normal subgroup of  $G$ , then  $N_G(H)K/K \subseteq N_{G/K}(HK/K)$ . In particular, if  $K \subseteq H$ , then equality holds.*

Assume that  $\pi$  is a set of primes. Let  $\bar{H}$  be a  $\pi$ -subgroup of  $G/\Phi(G)$ , then there exists a subgroup  $H$  such that  $\Phi(G) \subseteq H$  and  $H/\Phi(G) = \bar{H}$ . By [6, Satz III.3.6 and Satz III.3.8]  $\Phi(G)$  is a nilpotent group such that  $\pi(\Phi(G)) \subseteq \pi(G/\Phi(G))$ . Therefore a Hall  $\pi'$ -subgroup of  $\Phi(G)$ , say  $F$ , is normal in  $G$ . Hence by [10, 9.1.2] there exists a  $\pi$ -subgroup  $K$  of  $G$  such that  $H = KF$  and  $\bar{H} = H/\Phi(G) = K\Phi(G)/\Phi(G)$ . Clearly, if  $\bar{L}$  is a cyclic  $\pi$ -subgroup of  $G/\Phi(G)$ , then there exists a cyclic  $\pi$ -subgroup  $L$  of  $G$  such that  $\bar{L} = L\Phi(G)/\Phi(G)$ . Therefore the above lemma has the following easy consequences.

**Lemma 2.2.** *Let  $G$  be a group and let  $\pi$  be a set of primes. Suppose that  $G \in \mathfrak{N}_n^c(\pi) \cap \mathfrak{N}_t(\pi)$  and  $K < G$ . Then*

- (1)  $n \leq t$ ;
- (2)  $K \in \mathfrak{N}_m^c(\pi) \cap \mathfrak{N}_l(\pi)$  for some  $m \leq n$  and  $l \leq t$ ;
- (3)  $G/\Phi(G) \in \mathfrak{N}_r^c(\pi) \cap \mathfrak{N}_s(\pi)$  for some  $r \leq n$  and  $s \leq t$ .

For the proof of the main theorem we will need the following lemma.

**Lemma 2.3** [2, Lemma 2.2]. *Let  $G$  be a non-soluble group whose maximal subgroups are soluble. Then  $G/\Phi(G)$  is a minimal simple group.*

**3. Groups with at most 26 normalizers of subgroups.**

**Lemma 3.1.** *Let  $G$  be a minimal simple group.*

- (1) If  $G \cong A_5 \cong PSL(2, 5) \cong PSL(2, 4)$ , then  $G \in \mathfrak{N}_{22}^c(\mathbb{P}_5) \cap \mathfrak{N}_{27}(\mathbb{P}_5)$ ;
- (2) If  $G \not\cong A_5$ , then  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  where  $n \geq 38$ .

*Proof.* By [15] (see also [6, Bemerkung II.7.5])  $G$  is isomorphic to one of the following groups:

- (I)  $PSL(2, 2^p)$ , where  $p$  is a prime;
- (II)  $PSL(2, 3^p)$ , where  $p$  is an odd prime;
- (III)  $PSL(2, p)$ , where  $p > 3$  is a prime and  $p^2 - 1 \not\equiv 0 \pmod{5}$ ;
- (IV) a Suzuki group  $Sz(2^p)$ , where  $p$  is an odd prime;
- (V)  $PSL(3, 3)$ .

Clearly  $N_G(\{1\}) = G$ . By a calculation with the GAP [14] or [3] we obtain the statement (1) and we notice that

- if  $G \cong PSL(2, 7)$ , then  $G \in \mathfrak{N}_{50}^c(\mathbb{P}_5)$ ;
- if  $G \cong PSL(2, 8)$ , then  $G \in \mathfrak{N}_{38}^c(\mathbb{P}_5)$ ;
- if  $G \cong PSL(2, 32)$ , then  $G \in \mathfrak{N}_{530}^c(\mathbb{P}_5)$ .

Assume that  $G \cong PSL(2, 2^p)$ ,  $p > 5$ . Let  $P$  be a Sylow 2-subgroup of  $G$  and  $x$  be an involution of  $P$ . Then by [6, Satz II.8.2] or [4, XII.260]  $P$  is an elementary abelian 2-group of order  $2^p$ ,  $P = C_G(\langle x \rangle) = N_G(\langle x \rangle)$  and  $G$  has  $2^p + 1$  Sylow 2-subgroups. It follows that  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  for some  $n > 2^p + 1 \geq 2^7 + 1 = 129$ .

Assume that  $G \cong PSL(2, p)$ ,  $p$  is a prime,  $p^2 - 1 \not\equiv 0 \pmod{5}$  and one of the following holds:

- (a)  $p \equiv -3 \pmod{8}$  and  $p > 5$ ;
- (b)  $p \equiv 1 \pmod{8}$ .

By [4, XII.260]  $G$  has a subgroup  $M$  such that  $M$  is isomorphic to the dihedral group of order  $p - 1$ ,  $M$  contains a Sylow 2-subgroup of  $G$ ,  $|Z(M)| = 2$ , and  $N_G(Z(M)) = M$ . By [4, XII.260] there is a unique conjugacy class of self-normalizing  $\frac{p(p+1)}{2}$  subgroups isomorphic to  $M$ . Hence  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  where  $n > \frac{p(p+1)}{2} \geq \frac{13(13+1)}{2} = 91$ .

Assume that  $G \cong PSL(2, q)$  and one of the following holds:

- (i)  $q = 3^p$ , where  $p$  is an odd prime,
- (ii)  $q$  is a prime,  $q \equiv 3 \pmod{8}$ ,  $q^2 - 1 \not\equiv 0 \pmod{5}$  and  $q > 3$ ,
- (iii)  $q$  is a prime,  $q \equiv -1 \pmod{8}$ ,  $q^2 - 1 \not\equiv 0 \pmod{5}$  and  $q > 7$ .

By [4, XII.260]  $G$  has a subgroup  $M$  such that  $M$  is isomorphic to the dihedral group of order  $q + 1$ ,  $M$  contains a Sylow 2-subgroup of  $G$ ,  $|Z(M)| = 2$ , and

$N_G(Z(M)) = M$ . By [4, XII.260] there is a unique conjugacy class of self-normalizing  $\frac{q(q-1)}{2}$  subgroups isomorphic to  $M$ . Hence  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  where  $n > \frac{q(q-1)}{2} \geq \frac{23(23-1)}{2} = 253$ .

Assume that  $G \cong Sz(2^p)$ , where  $p$  is an odd prime. Let  $P$  be a Sylow 2-subgroup of  $G$  and let  $x \in Z(P)$  be an involution. By [12,13] the centralizer of any involution of  $G$  is a 2-group. Hence  $P = C_G(\langle x \rangle) = N_G(\langle x \rangle)$ . By [16, Theorem 4.1]  $G$  has  $2^{2p} + 1$  Sylow 2-subgroups. It follows that  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  for some  $n > 2^{2p} + 1 \geq 2^6 + 1 = 65$ .

Assume that  $G \cong PSL(3, 3)$ . By [7, Lemma XII.5.1] if  $x$  is an involution in the centre of Sylow 2-subgroup of  $G$ , then  $N_G(\langle x \rangle) = C_G(\langle x \rangle) \cong GL(2, 3)$ . Hence a calculation with GAP [14] or [3] there is a unique conjugacy class of self-normalizing 117 subgroups isomorphic to  $N_G(\langle x \rangle)$ . Hence  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$ , where  $n > 117$ . □

**Remark 3.1.** Since the orders of groups  $PSL(2, 7)$ ,  $PSL(2, 8)$ , and  $PSL(2, 32)$  are not divisible by 5, we obtain in fact that if  $G$  is a minimal simple group and  $G \not\cong A_5$ , then  $G \in \mathfrak{N}_n^c(\mathbb{P}_3)$  for  $n \geq 38$ .

**Theorem 3.2.** *Let  $G$  be a group. Assume that one of the following conditions holds:*

- (1)  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  with  $n \leq 21$ ;
- (2)  $G \in \mathfrak{N}_n^c(\mathbb{P}_5)$  with  $n \leq 26$ .

*Then  $G$  is soluble.*

*Proof.* Arguing by contradiction, assume that  $G$  is a non-soluble group of minimal order such that  $G$  satisfies the condition (i), where  $i \in \{1, 2\}$ . Then by Lemma 2.2 (2) every maximal subgroup of  $G$  is soluble. Hence by Lemmas 2.2 (3) and 2.3  $G/\Phi(G)$  is a minimal simple group satisfying the condition (i), contrary to Lemma 3.1. □

By Theorem 3.2 we obtain

**Corollary 3.3.**

- (1) *Every  $\mathfrak{N}_n$ -group with  $n \leq 26$  is soluble.*
- (2) *Every  $\mathfrak{N}_n^c$ -group with  $n \leq 21$  is soluble.*

**Remark 3.2.** Let  $G$  be a group.

- (1) By a calculation with the GAP [14] or [3] we also note that if  $G \cong A_5$ , then  $G \in \mathfrak{N}_6^c(\mathbb{P}_2) \cap \mathfrak{N}_{16}^c(\mathbb{P}_2)$  and  $G \in \mathfrak{N}_{11}(\mathbb{P}_2) \cap \mathfrak{N}_{21}(\mathbb{P}_3)$ .
- (2) It can be proved that  $G$  is soluble if one of the following conditions holds:
  - (i)  $G \in \mathfrak{N}_n^c(\mathbb{P}_2)$  with  $n \leq 5$ ;
  - (ii)  $G \in \mathfrak{N}_n^c(\mathbb{P}_3)$  with  $n \leq 15$ ;
  - (iii)  $G \in \mathfrak{N}_n(\mathbb{P}_2)$  with  $n \leq 10$ ;
  - (iv)  $G \in \mathfrak{N}_n(\mathbb{P}_3)$  with  $n \leq 20$ .
- (3) It can be also proved that  $G$  is a non-soluble  $\mathfrak{N}_{27}$ -group if and only if one of the following holds:
  - (i)  $G \cong A_5$ ;
  - (ii)  $G \cong SL(2, 5)$ ;

- (iii)  $G \cong K \times L$ , where  $K \cong A_5$  and  $L$  is a Dedekind  $\{3, 5\}'$ -group;
- (iv)  $G \cong K \times L$ , where  $K \cong SL(2, 5)$  and  $L$  is an abelian  $\{3, 5\}'$ -group and a Sylow 2-subgroup of  $G$  is a Dedekind group.

**4. Groups with the same number of elements of prime order.** Let  $I_k(G)$  be the number of elements of order  $k$  in  $G$ . In [20] Zarrin gave a counterexample to Herzog’s conjecture on simple groups with the same number of involutions [5] and he formulated his own conjecture on groups with the same number of involutions and the same number of elements of order  $p$ , where  $p$  is an odd prime. There seems to be some typographical errors in Zarrin’s conjecture and it should be formulated in the following way.

**Conjecture 3.** *Let  $S$  and  $G$  be non-abelian simple groups such that  $I_2(G) = I_2(S)$  and  $I_p(G) = I_p(S)$  for some odd prime divisor  $p$ . Then  $|G| = |S|$ .*

If we remove the condition that  $G$  is non-abelian simple, then we can point out a few counterexamples to the conjecture as it is formulated in [20]. In examples (3)–(4) we use the notation as in [3].

- (1) Assume that  $S$  is an arbitrary non-abelian simple group and  $G = S \times H$ , where  $H$  is an arbitrary non-trivial  $\pi(S)'$ -group. Then  $I_p(G) = I_p(S)$  for every prime  $p \in \pi(S)$  and  $|G| = |S| \cdot |H|$ .
- (2) Assume that  $S = A_5$  and  $G = \mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b} \times \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^d} \times \mathbb{Z}_{5^m} \times \mathbb{Z}_{5^n}$ , where  $a, b, c, d, m, n$  are positive integers. Then  $I_2(S) = I_2(G) = 15$ ,  $I_5(S) = I_5(G) = 24$  and  $|S| = 60 \neq 2^{a+b+c+d} \cdot 5^{m+n} = |G|$ .
- (3) Assume that  $S = Sz(8)$  and  $G = Aut(S) = Sz(8) : 3$ . Then by [3]  $I_p(S) = I_p(G)$  for every  $p \in \pi(S) = \pi(G) \setminus \{3\}$  and  $|G| = 3 \cdot |S|$ .
- (4) Assume that  $S = PSL(2, 25)$  and  $G = PSL(2, 25) \cdot 2$ . Then by [3]  $I_p(S) = I_p(G)$  for every  $p \in \pi(G) = \pi(S)$  and  $|G| = 2 \cdot |S|$ .

**Remarks added in the revised version of the paper.** We also would like to point out that in [8] one can find a few conjectures which are related to Conjecture 3. In [8] Moretó also showed that some families of simple groups are essentially determined just by the number of elements of order  $p$ , where  $p$  is the largest prime divisor of the order of the simple group. In [5] Herzog proved that simple groups with  $k$  involutions, where  $k \equiv 1 \pmod{4}$ , are completely determined. During the preparation of the revised version of the paper, I have also noticed the paper of Anabanti [1] with a counterexample to Conjecture 3. Therefore, in view of Herzog’s conjecture [5], Conjecture 3 and the counterexamples to these conjectures [1, 20] it might seem reasonable to consider the following problem.

**Problem 4.1.** Find the smallest positive integer  $k$  for which the following statement is correct.

*Let  $S$  and  $G$  be non-abelian simple groups. Assume that there exist prime divisors  $p_1, \dots, p_k$  of the orders of the groups  $S$  and  $G$  such that  $2 = p_1 < \dots < p_k$  and  $I_{p_i}(G) = I_{p_i}(S)$  for all  $i \in \{1, \dots, k\}$ . Then  $|G| = |S|$ .*

By [1, 20] it follows that  $k > 2$ .

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