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Correction

Correction to: Quantum j-invariant in positive characteristic I: definition and convergence

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Correction to: Arch. Math. 107 (2016), 23–35
https://doi.org/10.1007/s00013-016-0911-5.

The absolute value calculation given by Theorem 3 is incorrect: In this erratum, we correct it.[1] Let $f \in k_\infty - k$, $\varepsilon = q^{l-\sum_{i=1}^{N+1}\deg(a_i)}$. Write $\beta_{N+1} := \bar{\mathfrak{q}}_{N+1}/\bar{\mathfrak{q}}_N$, where $\bar{\mathfrak{q}}_i$ is the ith best approximation of f, normalized so that it is monic [1, p. 29].

Theorem 3. $|j_\varepsilon(f)| = \begin{cases} q^{q^2} & \textit{if } l = 1 \textit{ and } |\beta_{N+1}| > q, \\ q^{q^2+q-1} & \textit{otherwise.} \end{cases}$

Note 1. Corollary 1 of [1] is not affected by this correction. Moreover, Theorem 3 is not used in the papers [2–4], so this correction has no impact on them either.

We follow here the more efficient method developed in [3]. By [1, Lemma 1]

$$\frac{\Lambda_\varepsilon(f)}{\bar{\mathfrak{q}}_N} = \mathrm{span}_{\mathbb{F}_q}\{1, \alpha_1, \alpha_2 \dots\} := \mathrm{span}_{\mathbb{F}_q}\{1, \dots, T^{l-1}, \beta_{N+1}, \dots\}. \quad (1)$$

Lemma 1 [3, Lemma 5]. *Let*

$$\Omega_i(q^n - 1) := \sum (c_0 + \cdots + c_{i-1}\alpha_{i-1} + \alpha_i)^{1-q^n},$$

[1] This correction has been incorporated into the arXiv version of [1], arXiv:1512.03780.

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where the sum is over $c_0, \ldots, c_{i-1} \in \mathbb{F}_q$*. When* $i = 1$*,*

$$\Omega_1(q^n - 1) = \frac{\alpha_1^{q^n} - \alpha_1}{\prod_{c \in \mathbb{F}_q}(c + \alpha_1^{q^n})} \quad \textit{and} \quad |\Omega_1(q^n - 1)| = |\alpha_1|^{q^n(1-q)}.$$

More generally, for all $i \geq 1$*,* $|\Omega_i(q^n - 1)| \leq |\alpha_i|^{q^n(1-q)}$*.*

Proof. Write $\alpha = \alpha_1$. Then

$$\Omega_1(q^n - 1) = \sum_c \frac{c + \alpha}{c + \alpha^{q^n}} = \frac{\sum_c (c + \alpha) \prod_{d \neq c}(d + \alpha^{q^n})}{\prod_c (c + \alpha^{q^n})}. \tag{2}$$

Denote by $s_i(c)$ the ith elementary symmetric function on $\mathbb{F}_q - \{c\}$. Thus, $s_0(c) = 1$, $s_1(c) = \sum_{d \neq c} d$, etc. Then, the numerator of (2) may be written as

$$\sum_{j=0}^{q-1} \left(\sum_c (c + \alpha) s_{q-1-j}(c) \right) \alpha^{jq^n}.$$

Note that there is no constant term, and the coefficient of α is $\prod_{d \neq 0} d = -1$. Now

$$\sum_c c s_{q-2}(c) = \sum_{c \neq 0} c s_{q-2}(c) = \sum_{c \neq 0} c \prod_{d \neq 0, c} d = \sum_{c \neq 0} c \cdot (-c^{-1}) = (q-1)(-1) = 1,$$

which is the coefficient of α^{q^n}. Moreover, $\sum_c s_{q-2}(c) = s_{q-2}(0) - \sum_{c \neq 0} c^{-1} = 0$, so the α^{q^n+1} term vanishes. For $i < q - 2$, we claim that

$$\sum_c c s_i(c) = 0 = \sum_c s_i(c).$$

When $i = 0$, $s_0(c) = 1$ for all c, the terms $\alpha^{q^n(q-1)}$, $\alpha^{q^n(q-1)+1}$ have coefficients $\sum_c c = \sum_c 1 = 0$ and so vanish. When $i = 1$, we have $q > 3$, so $s_1(c) = -c$ and

$$\sum_c s_1(c) = -\sum_c c = 0 = -\sum_c c^2 = \sum_c c s_1(c)$$

since the sums occurring above are power sums over $\mathbb{F}_q$ of exponent $1, 2 < q-1$. For general $i < q - 2$, we have $q > i + 2$ and

$$\sum_c s_i(c) = \sum_c P(c)$$

where $P(X)$ is a polynomial over $\mathbb{F}_q$ of degree $i < q - 2$. Hence, $\sum_c P(c) = \sum_c cP(c) = 0$, since, again, these are sums of powers of c of exponent less than $q - 1$. Thus, the numerator is $\alpha^{q^n} - \alpha$ and the absolute value claim follows immediately. When $i > 1$, for each $\vec{c}$, let $\vec{c}_+ = (c_1, \ldots, c_{i-1})$ and write $\alpha_{\vec{c}_+} = c_1\alpha_1 + \cdots + c_{i-1}\alpha_{i-1} + \alpha_i$. Note trivially that $|\alpha_{\vec{c}_+}| = |\alpha_i|$. Then, by part (1),

$$|\Omega_i(q^n - 1)| = \left| \sum_{\vec{c}_+} \sum_c (c + \alpha_{\vec{c}_+})^{1-q^n} \right| \leq \max\{|\alpha_{\vec{c}_+}|^{q^n(1-q)}\} = |\alpha_i|^{q^n(1-q)}.$$

□

Since $j_\varepsilon(f)$ is invariant with respect to the rescaling by $\bar{\mathfrak{q}}_N$ given in (1), we may write

$$j_\varepsilon(f) = (T^q - T)^{q+2} \cdot \frac{(1 + \Omega_1(q-1) + \cdots)^{q+1}}{\Delta}, \quad \Delta := U - V,$$

where

$$\begin{aligned} U &= ((T^q - T)(1 + \Omega_1(q-1) + \cdots))^{q+1} \\ &= T^{q(q+1)} - T^{q^2+1} - T^{2q} + T^{q+1} + T^{q(q+1)} \sum_c (c + \alpha_1)^{1-q} + \text{lower} \end{aligned}$$

(in the above, we use that $(T^q - T)^{q+1} = (T^{q^2} - T^q)(T^q - T))$ and

$$\begin{aligned} V &= (T^q - T)(T^{q^2} - T)(1 + \Omega_1(q^2 - 1) + \cdots) \\ &= T^{q(q+1)} - T^{q^2+1} - T^{q+1} + T^2 + T^{q(q+1)} \sum_c (c + \alpha_1)^{1-q^2} + \text{lower}. \end{aligned}$$

Thus,

$$\Delta = -T^{2q} + 2T^{q+1} - T^2 + T^{q(q+1)} \sum_c (c + \alpha_1)^{1-q} + \text{lower}.$$

Notice that we have made use of the estimates in Lemma 1 to write "+lower" in the above lines.

Lemma 2 [3, Lemma 6]. $|\Delta| = \begin{cases} q^{q+1} & \textit{if } |\alpha_1| = q, \\ q^{2q} & \textit{otherwise.} \end{cases}$

Proof. Suppose first that $|\alpha_1| = q$. Now $\alpha_1 = T$ or $\beta_{N+1} = \bar{\mathfrak{q}}_{N+1}/\bar{\mathfrak{q}}_N$ where the (unnormalized) best approximations satisfy $\mathfrak{q}_{N+1} = a_{N+1}\mathfrak{q}_N + \mathfrak{q}_{N-1}$ for a_{N+1} linear. Since the normalized best approximations are monic, it follows that we may write $\alpha_1 = T + \delta$ where $|\delta| < q$. In particular, by Lemma 1, item (1), we have

$$\begin{aligned} &\left| \sum_c (c + \alpha_1)^{1-q} - \sum_c (c + T)^{1-q} \right| \\ &\quad = \left| \frac{(\alpha_1^q - \alpha_1) \prod_c (c + T^q) - (T^q - T) \prod_c (c + \alpha_1^q)}{\prod_c \big((c + \alpha_1^q)(c + T^q)\big)} \right| \\ &\quad < q^{q(1-q)}. \end{aligned}$$

Therefore, we may write

$$\Delta = -T^{2q} + 2T^{q+1} + T^{q(q+1)} \sum_c (c + T)^{1-q} + \text{lower}.$$

By Lemma 1, item (1),

$$\begin{aligned} &-T^{2q}+2T^{q+1}+T^{q(q+1)}\sum_c (c+T)^{1-q} \\ &\quad=\frac{(-T^{2q}+2T^{q+1})\cdot\prod_c(c+T^q)+T^{q(q+1)}\cdot(T^q-T)}{\prod_c(c+T^q)} \\ &\quad=\frac{T^{q(q+1)+1}+\text{lower}}{\prod_c(c+T^q)}. \end{aligned}$$

It follows that $\left|-T^{2q}+2T^{q+1}+T^{q(q+1)}\sum(c+T)^{1-q}\right| = q^{q+1}$, and we conclude that in this case, $|\Delta| = q^{q+1}$.

If $|\alpha_1| > q$, by Lemma 1, item (1),

$$\left|T^{q(q+1)}\sum(c+\alpha_1)^{1-q}\right| = q^{q(q+1)}\cdot|\alpha_1|^{q(1-q)} < q^{2q} = |T^{2q}|$$

hence $|\Delta| = q^{2q}$. □

Proof of Theorem 3. When $l \neq 1$, then $\alpha_1 = T$: thus $|\alpha_1| = q$, $|\Delta| = q^{q+1}$ and $|j_\varepsilon(f)| = q^{q^2+q-1}$. The same is true when $l = 1$ and $\beta_{N+1} = \alpha_1$ satisfies $|\beta_{N+1}| = q$. If $l = 1$ and $|\beta_{N+1}| = |\alpha_1| > q$, then $|\Delta| = q^{2q}$ and $|j_\varepsilon(f)| = q^{q^2}$. □

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