



A Liouville-type theorem for biharmonic maps between complete Riemannian manifolds with small energies

VOLKER BRANDING 

Abstract. We prove a Liouville-type theorem for biharmonic maps from a complete Riemannian manifold of dimension n that has a lower bound on its Ricci curvature and positive injectivity radius into a Riemannian manifold whose sectional curvature is bounded from above. Under these geometric assumptions we show that if the L^p -norm of the tension field is bounded and the n -energy of the map is sufficiently small, then every biharmonic map must be harmonic, where $2 < p < n$.

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1. Introduction and results. *Harmonic maps* belong to the most studied variational problems in differential geometry. They are defined as critical points of the Dirichlet energy

$$E_1(\phi) = \int_M |d\phi|^2 dV, \quad (1.1)$$

where $\phi: M \rightarrow N$ is a map between the two Riemannian manifolds (M, h) and (N, g) . The critical points of (1.1) are characterized by the vanishing of the so-called *tension field*, which is given by

$$0 = \tau(\phi) := \operatorname{tr}_h \nabla d\phi.$$

This is a semilinear, elliptic second order partial differential equation, for which many results on existence and qualitative behavior of its solutions have been obtained. For a recent survey on harmonic maps see [4].

A natural generalization of the harmonic map equation are the so-called *biharmonic maps*. These arise as critical points of the bienergy [6], which is

defined as

$$E_2(\phi) = \int_M |\tau(\phi)|^2 dV.$$

In contrast to the harmonic map equation, the biharmonic map equation is of fourth order and is characterized by the vanishing of the *bitension field*

$$0 = \tau_2(\phi) := -\Delta\tau(\phi) + R^N(d\phi(e_\alpha), \tau(\phi))d\phi(e_\alpha), \quad (1.2)$$

where e_α is an orthonormal basis of TM and R^N denotes the curvature tensor of the target manifold N . We make use of the Einstein summation convention, meaning that we sum over repeated indices. The connection Laplacian on ϕ^*TN is defined by $\Delta := \nabla_{e_\alpha} \nabla_{e_\alpha} - \nabla_{\nabla_{e_\alpha} e_\alpha}$. We choose the following convention for the Riemann curvature tensor $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$.

Note that the bienergy is conformally invariant in dimension $\dim M = 4$, whereas the energy for harmonic maps is conformally invariant in dimension $\dim M = 2$.

As for harmonic maps there exists a large number of results on various aspects of biharmonic maps, for a survey we refer to [10].

It is obvious that both harmonic maps and trivial maps solve the biharmonic map equation (1.2). For this reason it is interesting to find conditions under which solutions of the biharmonic map equation reduce to harmonic or trivial maps. Several results in this direction have already been established:

- (1) For biharmonic functions on flat space, a Liouville theorem was already established before the first article on biharmonic maps appeared. Let $M = \mathbb{R}^m$ and $N = \mathbb{R}$, and consider a solution $u: \mathbb{R}^m \rightarrow \mathbb{R}$ of $\Delta^2 u = 0$. If u is bounded from above and below, then u is constant [5].
- (2) Already in the first article on biharmonic maps [6] it was shown by a direct application of the maximum principle that biharmonic maps between compact Riemannian manifolds and the target having negative curvature must be harmonic.
- (3) If the domain manifold is a complete non-compact Riemannian manifold with positive Ricci curvature and the target a Riemannian manifold of non-positive sectional curvature, then every biharmonic map of finite bienergy must be harmonic [1].
- (4) This result remains true if the condition of positive Ricci curvature on the domain is replaced by either the map having finite energy or the domain manifold having finite volume [11].
- (5) This result was further generalized in [9]: it is shown that a biharmonic map from a complete Riemannian manifold to a Riemannian manifold with non-positive sectional curvature is harmonic if either the tension field is integrable in L^p with $p \geq 2$ and the volume of the domain is infinite or if the tension field is integrable in L^p with $p \geq 2$ and the Dirichlet energy of the map is finite.
- (6) Further generalizations were given in [7] and [8].

In this article we aim to establish another Liouville type theorem for biharmonic maps that does not require any assumption on the curvature of the

target. To this end let (M, h) and (N, g) be two complete Riemannian manifolds.

We will not have to assume that the target manifold has negative curvature, instead we will demand that the differential of the map is small in a certain L^p -norm.

In the following the positive constant A will denote an upper bound on the sectional curvature K^N of N . We will denote the dimension of M by $\dim M = n$. The main result of this note is the following

Theorem 1.1. *Let $\phi: M \rightarrow N$ be a smooth biharmonic map. Suppose that the Ricci curvature of M is bounded from below and that the injectivity radius of M is positive. Assume that the sectional curvature K^N of N satisfies $K^N \leq A$. If*

$$\int_M |\tau(\phi)|^p dV < \infty$$

and

$$\int_M |d\phi|^n dV < \varepsilon$$

for $2 < p < n$ and $\varepsilon > 0$ sufficiently small, then ϕ must be harmonic.

If $\phi: M \rightarrow N$ is an isometric immersion, we have $\tau(\phi) = nH$, where H denotes the mean curvature vector of the immersion. The last theorem implies the following geometric result:

Corollary 1.2. *Let $\phi: M \rightarrow N$ be a smooth biharmonic isometric immersion. Suppose that the Ricci curvature of M is bounded from below and that the injectivity radius of M is positive. Assume that the sectional curvature K^N of N satisfies $K^N \leq A$.*

If

$$\int_M |H|^p dV < \infty$$

and

$$\int_M |d\phi|^n dV < \varepsilon$$

for $2 < p < n$ and $\varepsilon > 0$ sufficiently small, then ϕ must be a minimal immersion.

2. Proof of the Theorem. In order to prove Theorem 1.1 we extend the ideas developed in [9] and [11]. First, we need to establish some technical lemmas.

We choose a cutoff function $0 \leq \eta \leq 1$ on M that satisfies

$$\eta(x) = 1 \text{ for } x \in B_r(x_0), \quad \eta(x) = 0 \text{ for } x \in B_{2r}(x_0), \quad |\nabla\eta| \leq \frac{C}{r} \text{ for } x \in M,$$

where $B_r(x_0)$ denotes the geodesic ball around x_0 with radius r .

Lemma 2.1. *Let $\phi: M \rightarrow N$ be a smooth biharmonic map. Suppose that the sectional curvature K^N of N satisfies $K^N \leq A$. Then the following inequality holds*

$$\begin{aligned} \frac{1}{2} \int_M \eta^2 |\nabla \tau(\phi)|^2 |\tau(\phi)|^{p-2} dV &\leq A \int_M \eta^2 |\tau(\phi)|^p |d\phi|^2 dV + \frac{C}{R^2} \int_M |\tau(\phi)|^p dV \\ &\quad - (p-2) \int_M \eta^2 |\langle \nabla \tau(\phi), \tau(\phi) \rangle|^2 |\tau(\phi)|^{p-4} dV. \end{aligned} \tag{2.1}$$

Proof. We test the biharmonic map equation (1.2) with $\eta^2 \tau(\phi) |\tau(\phi)|^{p-2}$ and find

$$\eta^2 |\tau(\phi)|^{p-2} \langle \Delta \tau(\phi), \tau(\phi) \rangle = \eta^2 |\tau(\phi)|^{p-2} \langle R^N(d\phi(e_\alpha), \tau(\phi)) d\phi(e_\alpha), \tau(\phi) \rangle.$$

Integrating over M and using integration by parts, we obtain

$$\begin{aligned} \int_M \eta^2 |\tau(\phi)|^{p-2} \langle \Delta \tau(\phi), \tau(\phi) \rangle dV &= -2 \int_M \langle \nabla \tau(\phi), \tau(\phi) \rangle |\tau(\phi)|^{p-2} \eta \nabla \eta dV \\ &\quad - (p-2) \int_M \eta^2 |\langle \nabla \tau(\phi), \tau(\phi) \rangle|^2 |\tau(\phi)|^{p-4} dV \\ &\quad - \int_M \eta^2 |\nabla \tau(\phi)|^2 |\tau(\phi)|^{p-2} dV \\ &\leq \frac{C}{R^2} \int_M |\tau(\phi)|^p dV - \frac{1}{2} \int_M \eta^2 |\nabla \tau(\phi)|^2 |\tau(\phi)|^{p-2} dV \\ &\quad - (p-2) \int_M \eta^2 |\langle \nabla \tau(\phi), \tau(\phi) \rangle|^2 |\tau(\phi)|^{p-4} dV, \end{aligned}$$

where we used Young’s inequality and the properties of the cutoff function η . Combining both equations we find

$$\begin{aligned} \frac{1}{2} \int_M \eta^2 |\nabla \tau(\phi)|^2 |\tau(\phi)|^{p-2} dV &\leq \frac{C}{R^2} \int_M |\tau(\phi)|^p dV - (p-2) \int_M \eta^2 |\langle \nabla \tau(\phi), \tau(\phi) \rangle|^2 |\tau(\phi)|^p dV \\ &\quad - \int_M \eta^2 |\tau(\phi)|^{p-2} \langle R^N(d\phi(e_\alpha), \tau(\phi)) d\phi(e_\alpha), \tau(\phi) \rangle dV. \end{aligned}$$

The result follows by estimating the last term on the right-hand side. □

Now we recall the following fact: let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded from below and with positive injectivity radius. Then for $f \in W^{1,s}(M)$ with compact support the following Gagliardo-Nirenberg type inequality holds

$$\|f\|_{L^r} \leq C \|df\|_{L^s}, \quad \frac{1}{r} = \frac{1}{s} - \frac{1}{n}, \quad 1 \leq s < n, \tag{2.2}$$

see [3, Corollary 3.19]. This inequality allows us to give the following

Lemma 2.2. *Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded from below and with positive injectivity radius. For $p < n$ the following inequality holds*

$$\int_M \eta^2 |\tau(\phi)|^p |d\phi|^2 dV \leq C \left(\int_M |d\phi|^n dV \right)^{\frac{2}{n}} \times \left(\frac{1}{R^2} \int_M |\tau(\phi)|^p dV + \int_M \eta^2 |d|\tau(\phi)|^{\frac{p}{2}}|^2 dV \right), \tag{2.3}$$

where the positive constant C depends on n, p , and the geometry of M .

Proof. We set $f := |\tau(\phi)|^{\frac{p}{2}}$, which allows us to write

$$\int_M \eta^2 |\tau(\phi)|^p |d\phi|^2 dV = \int_M \eta^2 f^2 |d\phi|^2 dV.$$

By Hölder’s inequality we find

$$\int_M \eta^2 f^2 |d\phi|^2 dV \leq \left(\int_M (\eta f)^{2r} dV \right)^{\frac{1}{r}} \left(\int_M |d\phi|^{\frac{2r}{r-1}} dV \right)^{\frac{r-1}{r}}.$$

Note that we may now apply the Gagliardo-Nirenberg type inequality (2.2) since ηf has compact support. Consequently, we find

$$\left(\int_M |\eta f|^{2r} dV \right)^{\frac{1}{2r}} \leq C \left(\int_M |d(\eta f)|^{\frac{2rn}{2r+n}} dV \right)^{\frac{2r+n}{2rn}}.$$

Now, we choose $r = \frac{n}{n-2}$ and obtain

$$\left(\int_M (\eta f)^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} \leq C \int_M |d(\eta f)|^2 dV.$$

Using the properties of the cutoff function η , we find

$$\int_M \eta^2 f^2 |d\phi|^2 dV \leq C \left(\int_M |d\phi|^n dV \right)^{\frac{2}{n}} \left(\int_M |d\eta|^2 f^2 dV + \int_M \eta^2 |df|^2 dV \right).$$

The result follows by using that $f = |\tau(\phi)|^{\frac{p}{2}}$. □

Lemma 2.3. *Let $\phi: M \rightarrow N$ be a smooth biharmonic map. Suppose that the Ricci curvature of M is bounded from below and that the injectivity radius of M is positive. Moreover, assume that the sectional curvature K^N of N satisfies $K^N \leq A$.*

If

$$\int_M |\tau(\phi)|^p dV < \infty$$

and

$$\int_M |d\phi|^n dV < \varepsilon$$

for $2 < p < n$ and $\varepsilon > 0$ sufficiently small, then $|\tau(\phi)|$ is constant.

Proof. First of all, we note that

$$\begin{aligned} |\langle \nabla\tau(\phi), \tau(\phi) \rangle|^2 |\tau(\phi)|^{p-4} &= \frac{1}{4} |d|\tau(\phi)|^2|^2 |\tau(\phi)|^{p-4} \\ &= |d|\tau(\phi)||^2 |\tau(\phi)|^{p-2} = \frac{4}{p^2} |d|\tau(\phi)|^{\frac{p}{2}}|^2. \end{aligned}$$

Combining (2.1) with (2.3) and making use of the smallness assumption of the energy of the map, we find

$$\begin{aligned} \frac{1}{2} \int_M \eta^2 |\nabla\tau(\phi)|^2 |\tau(\phi)|^{p-2} dV &\leq \frac{C}{R^2} \int_M |\tau(\phi)|^p dV \\ &\quad + \left(C\varepsilon - \frac{4(p-2)}{p^2} \right) \int_M \eta^2 |d|\tau(\phi)|^{\frac{p}{2}}|^2 dV. \end{aligned}$$

Choosing ε sufficiently small, making use of the finiteness assumption of the tension field and taking the limit $R \rightarrow \infty$ we get

$$\int_M |\nabla\tau(\phi)|^2 |\tau(\phi)|^{p-2} dV = 0,$$

which implies that $\tau(\phi)$ is a parallel vector field and thus has constant norm. □

In the following we will make use of the following result due to Gaffney [2]:

Theorem 2.4. *Let (M, h) be a complete Riemannian manifold. If a C^1 one-form ω satisfies*

$$\int_M |\omega| dV < \infty \quad \text{and} \quad \int_M |\delta\omega| dV < \infty$$

or, equivalently, a C^1 vector field X defined by $\omega(Y) = h(X, Y)$, satisfies

$$\int_M |X| dV < \infty \quad \text{and} \quad \int_M \operatorname{div}(X) dV < \infty,$$

then

$$\int_M (\delta\omega) dV = \int_M \operatorname{div}(X) dV = 0.$$

Making use of Gaffney’s result we can now prove Theorem 1.1.

Proof of Theorem 1.1. We define a one-form ω as follows

$$\omega(X) := |\tau(\phi)|^{\frac{p(n-1)}{n}-1} \langle d\phi(X), \tau(\phi) \rangle.$$

Note that

$$\int_M |\omega| dV \leq \int_M |\tau(\phi)|^{\frac{p(n-1)}{n}} |d\phi| dV \leq \left(\int_M |d\phi|^n dV \right)^{\frac{1}{n}} \left(\int_M |\tau(\phi)|^p dV \right)^{\frac{n-1}{n}} < \infty,$$

where we made use of the finiteness assumptions of both energy and bienergy of the map ϕ . We fix an orthonormal basis of TM denoted by $e_\alpha, \alpha = 1, \dots, \dim M$ and compute

$$\begin{aligned} -\delta\omega &= \nabla_{e_\alpha} \omega(e_\alpha) \\ &= \nabla_{e_\alpha} (|\tau(\phi)|^{\frac{p(n-1)}{n}-1} \langle d\phi(e_\alpha), \tau(\phi) \rangle) \\ &= |\tau(\phi)|^{\frac{p(n-1)}{n}-1} \langle \nabla_{e_\alpha} d\phi(e_\alpha), \tau(\phi) \rangle \\ &= |\tau(\phi)|^{\frac{p(n-1)}{n}+1}, \end{aligned}$$

where we used that $\nabla_X \tau(\phi) = 0$ for all vector fields X and that $|\tau(\phi)|$ is constant. Due to the assumptions we have that $\frac{p(n-1)}{n} + 1 > 0$. Again, since $|\tau(\phi)|$ is constant and $\int_M |\tau(\phi)|^p dV$ is bounded by assumption we can apply Theorem 2.4 and conclude that $\tau(\phi) = 0$. \square

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VOLKER BRANDING
Faculty of Mathematics,
University of Vienna,
Oskar-Morgenstern-Platz 1,
1090 Vienna,
Austria
e-mail: volker.branding@univie.ac.at

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