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Fixed point indices of iterates of a low-dimensional diffeomorphism at a fixed point which is an isolated invariant set

Grzegorz Graff

Abstract. Let f be an $\mathbb{R}^n$-diffeomorphism, where $n = 2, 3$, for which $\{0\}$ is an isolated invariant set. We determine all possible forms of the sequences of fixed point indices of iterates of f at 0, $\{\mathrm{ind}(f^n, 0)\}_n$, confirming in $\mathbb{R}^3$ the conjecture of Ruiz del Portal and Salazar (J Differ Equ 249, 989–1013, 2010).

1. Introduction. The fixed point index of a map f at a point p is a well-known topological device that is used in fixed and periodic point theory. In studying the dynamics of f, the whole sequence of all indices of iterates $\{\mathrm{ind}(f^n, p)\}_n$ often provides deep insight into the structure of periodic points as well as local behavior of f near fixed and periodic points. However, establishing the form of a possible sequence of indices for a given class of maps is usually an uneasy problem. The pioneering work in this direction for a low-dimensional case was the result of Brown [3] (cf. also [11]), who studied a planar orientation preserving homeomorphism and showed that indices can take only two values. The full description of the forms of $\{\mathrm{ind}(f^n, p)\}_n$ for planar homeomorphisms was provided in [19].

In 1997 Le Calvez and Yoccoz studied the behavior of an orientation preserving planar homeomorphism f which, near a fixed point p, satisfies the following condition: there is a neighborhood U of p such that each orbit (different from $\{p\}$) leaves U either in positive or negative time (cf. condition (2.2), also expressed as "$\{p\}$ is an isolated invariant set"). Under this assumption

they found the forms of local fixed point indices [17]. One of the most powerful application of this result (which uses the fact that indices for some iterates are non-positive) is a solution of the old Ulam problem from The Scottish Book of non-existence of minimal homeomorphisms of a punctured sphere $\mathbb{S}^2$ (cf. Problem 115 in [18]).

A higher dimensional analog of Le Calvez and Yoccoz theorem was obtained in [16] for an orientation preserving $\mathbb{R}^3$-homeomorphism f. In this case the sequence of indices must be periodic although there are no other restrictions (such as non-positivity). In the recent paper [13] the forms of $\{\mathrm{ind}(f^n, p)\}_n$ were established also for an orientation reversing $\mathbb{R}^3$-homeomorphism f.

Another line of research, started by Shub and Sullivan [22], was related to finding the forms of indices for smooth maps, and resulted in a complete description of possible forms of indices in this case (cf. [1,5,7,8,10,12]).

In this paper we consider $\mathbb{R}^2$- and $\mathbb{R}^3$-diffeomorphisms with a fixed point which is an isolated invariant set. In other words, we work with the class of maps which satisfies both above mentioned conditions: isolation of a fixed point as an invariant set and smoothness. We establish the form of indices of iterates in such situation, answering in the positive the conjecture posed for $\mathbb{R}^3$-diffeomorphisms by Ruiz del Portal and Salazar in [21, Theorem 4.4]. In Concluding Remarks we discuss also the problem of finding all possible forms of indices for diffeomorphisms and non-injective maps in higher dimensions.

2. Preliminary facts and definitions. Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map, with 0 an isolated fixed point for each f^n, $n = 1, 2, \ldots$. Then the following congruences, called Dold relations, hold (cf. [6]):

$$\sum_{k|n} \mu(n/k)\mathrm{ind}(f^k, 0) \equiv 0 \pmod{n}, \tag{2.1}$$

where μ is the arithmetic Möbius function, i.e. $\mu \colon \mathbb{N} \to \mathbb{Z}$ is defined by the following properties: $\mu(1) = 1$, $\mu(k) = (-1)^r$ if k is a product of r different primes, $\mu(k) = 0$ otherwise (cf. [4]).

Let us remark that the Dold relations are true also in more a general setting for self-maps of ENR and an isolated set of fixed points.

Using the Dold relations one may represent sequences of indices of iterations in the simple form of k-periodic expansion, i.e. by an integral combination of some basic periodic sequences described in Definition 2.1 below.

Definition 2.1. For a given k we define a basic sequence $\{\mathrm{reg}_k(n)\}_{n=1}^{\infty}$ by the formula:

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k|n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Observe that reg_k is a periodic sequence: $(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots)$, where the non-zero entries appear for indices of the sequence divisible by k.

Theorem 2.2. (cf. [15]). *A sequence* $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ *can be expressed in the following form of a periodic expansion:*

$$\mathrm{ind}(f^n, 0) = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k(n),$$

where

$$a_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) \mathrm{ind}(f^k, 0).$$

Furthermore, by the Dold relations (2.1), *the coefficients* a_n *are always integers.*

Definition 2.3. Let $f \colon X \to X$ be a homeomorphism, p be a fixed point of f. We will say that $\{p\}$ is an isolated invariant set if there is a neighborhood W of p such that

$$\bigcap_{k \in \mathbb{Z}} f^k(W) = \{p\}. \tag{2.2}$$

The condition (2.2) is crucial when studying some important classes of discrete dynamical systems, as it makes it possible to apply Conley index theory to study the dynamics near p (or more generally near an invariant set of f).

3. Fixed point indices of iterates of $\mathbb{R}^2$-diffeomorphisms at a fixed point which is an isolated invariant set. Before analyzing the problem in dimension 3, for the sake of completeness in this section we deal with an easier 2-dimensional case. We start with giving the complete list of indices of iterates of C^1 maps in dimension 2.

Theorem 3.1. (cf. [1, 10]) *Let g be a planar C^1 map, then there are four possible forms of* $\{\mathrm{ind}(g^n, 0)\}_{n=1}^{\infty}$:

(α) $c_\alpha(n) = a_1 \mathrm{reg}_1(n)$,
(β) $c_\beta(n) = \mathrm{reg}_1(n) + a_d \mathrm{reg}_d(n)$,
(γ) $c_\gamma(n) = a_2 \mathrm{reg}_2(n)$,
(δ) $c_\delta(n) = -\mathrm{reg}_1(n) + a_2 \mathrm{reg}_2(n)$,

where $a_i \in \mathbb{Z}$, $d \geq 2$ in all cases.

Moreover, every sequence of integers which is of one of the forms (α)$--$(δ) *can be realized as a sequence of local indices of iterates of a C^1 self-map of $\mathbb{R}^2$.*

The additional demanding that a map is a diffeomorphism for which $\{0\}$ is an isolated invariant set leads to further restrictions on the indices.

Theorem 3.2. *Let g be an $\mathbb{R}^2$-diffeomorphism with $\{0\}$ an isolated invariant set. Then*

(1) *its indices of iterates at 0 satisfy the following restrictions with respect to the cases listed in Theorem 3.1: in the case (α) $a_1 \leq 1$, in the case (β) $a_d \leq 0$, in the case (γ) $a_2 \leq 0$, and in the case (δ) $a_2 \leq 1$.*

(2) *Each of the sequences satisfying the restrictions described in (1) can be realized as an $\mathbb{R}^2$-diffeomorphism with $\{0\}$ an isolated invariant set.*

Proof. We start with showing part (1). From the proof of Theorem 3.1 given in [1] we may identify the forms of indices in dependence on the orientation of a diffeomorphism. Namely, if f is orientation preserving, then it has the form (α) or (β), while when it is orientation reversing, it has the forms (β') $c_{\beta'}(n) = \text{reg}_1(n) + a_2\text{reg}_2(n)$, ($\gamma$), or ($\delta$).

Notice that if 0 is a sink or a source that preserves the orientation, then $\text{ind}(g^n, 0) = \text{reg}_1(n)$. If 0 is a source that reverses the orientation, then $(\text{ind}(g^n, 0))_n = (-1, 1, -1, 1, \ldots)$, i.e. $\text{ind}(g^n, 0) = -\text{reg}_1(n) + \text{reg}_2(n)$.

Now we use Le Calvez' and Yoccoz' theorem (for another argument for deducing the bounds for the coefficients, see Main Theorem in [20] which was proved by the use of Conley index methods, see also Remark 6.4).

Let g be orientation preserving. If 0 is neither a sink nor a source, then its indices may be expressed in the language of periodic expansion as $\text{ind}(g^n, 0) = \text{reg}_1(n) - s\text{reg}_q(n)$, where $s, q > 0$ (cf. [17]). We admit $q = 1$ for which we get $\text{ind}(g^n, 0) = (1 - s)\text{reg}_1(n)$, where $(1 - s) \leq 0$. Taking into account that an orientation preserving sink or source in the plane has the sequence of indices of the form $\text{reg}_1(n)$, we get the restrictions for a_1 in the case (α). If $q > 1$, then we obtain immediately the restrictions $a_d < 0$ in the case (β), while the case $a_d = 0$ reduces to the case (α).

In case g is orientation reversing we apply Le Calvez' and Yoccoz' result for g^2. Then $\text{ind}(g^2, 0) = 1 - sq \leq 0$, or $\text{ind}(g^2, 0) = 1$, thus $\text{ind}(g^2, 0) \leq 1$. On the other hand, for an orientation reversing homeomorphism g, always $\text{ind}(g, 0) \in \{-1, 0, 1\}$ (cf. [2]). By Dold relations (2.1) for $n = 2$, we get $a_2 = \frac{1}{2}(\text{ind}(g^2, 0) - \text{ind}(g, 0)) \in \mathbb{Z}$. Finally, the above bounds for $\text{ind}(g, 0)$ and $\text{ind}(g^2, 0)$ lead to the restrictions for a_2 in the cases (β'), (γ), and (δ).

We prove now part (2) showing the realizations of each case listed in Theorem 3.2 by a planar diffeomorphism f with $\text{Fix}(f) = \text{Per}(f) = \{0\}$ an isolated invariant set.

Let us consider a planar smooth flow $h_q \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that its phase portraits consist of $2|q - 1| = 2(|q| + 1)$ hyperbolic regions for $q < 1$, or with 0 as a source for $q = 1$. The exact formulas for such flows are given in [10].

The classical Poincaré-Bendixson formula states that the index of the discretization of each of the above flows is equal to $1 - h/2$ where h is the number of hyperbolic regions (and there are no elliptic regions). Thus, taking $H_q = h_q(\cdot, \cdot, 1) \colon \mathbb{R}^2 \to \mathbb{R}^2$, we get that H_q has $\{0\}$ as an isolated invariant set and $\text{ind}(H_q, 0) = q$. Furthermore, $\text{ind}(H_q^n, 0) = q\text{reg}_1(n)$. Thus for $q := a_1$ we get the realization of the sequences of the form (α).

Now, for a given natural number $d \geq 2$ and integer $a_d < 0$, we define the map O as the $\frac{2\pi}{d}$ rotation around 0 and put $q := a_d d + 1$. Then O commutes with H_p and

$$\text{ind}((O \circ H_q)^n, 0) = \begin{cases} 1 & \text{if } d \nmid n, \\ a_d d + 1 & \text{if } d \mid n \end{cases} = \text{reg}_1(n) + a_d\text{reg}_d(n), \quad (3.1)$$

which gives the realization of the sequences of the form (β).

To realize sequences of the form (γ), let us consider the discretization G_l of a flow having one hyperbolic region which coincides with the first quadrant
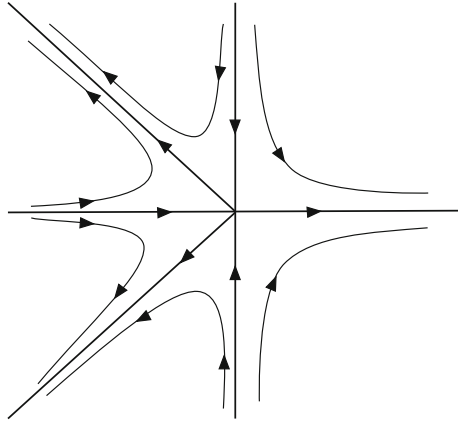
FIGURE 1. The flow used in the realization of the sequence of the form (γ)

(or with first and second quadrant if $l = 0$), $2l$ hyperbolic regions in the second quadrant, symmetric about the x-axis, and such that on the x-axis its phase portrait is conjugated to the one given by the equation $\dot{x} = 1$ (see an example of such a flow for $l = 1$ in Figure 1).

Define $\overline{G}_l = s \circ G_l$, where s is the symmetry about the x-axis. Observe that $\overline{G}_l$ is homotopic on a small ball B centered at 0, by a linear homotopy without fixed points on the ∂B, to a map $G_l \circ r$, where r is retraction onto the x-axis $r(x, y) = (x, 0)$.

By the property of homotopical invariance of the fixed point index and the definition of the index we get: $\operatorname{ind}(\overline{G}_l, 0) = \operatorname{ind}(G_l \circ r, 0) = \operatorname{ind}(G_{l|\mathbb{R}\times\{0\}}, 0) = 0$ and the same is true for odd iterates of $\overline{G}_l$.

By the Poincaré-Bendixson formula $\operatorname{ind}(\overline{G}_l^2, 0) = 1 - \frac{2 \cdot 2l + 2}{2} = -2l$ and the same holds for even iterates. Taking $l := -a_2$ we obtain the realization of the sequences of the form (γ).

Finally, consider K_l as the discretization of a flow having in the upper half-plane $2l$ hyperbolic regions, symmetric about the x-axis and such that on the x-axis it is a source at 0. We define $\overline{K}_l = s \circ K_l$. Then $\operatorname{ind}(\overline{K}_l^n, 0) = \operatorname{ind}(K_{l|\mathbb{R}\times\{0\}}^n, 0) = -1$ for odd n and, again by the Poincaré-Bendixson formula, $\operatorname{ind}(\overline{K}_l^2, 0) = 1 - \frac{2 \cdot 2l}{2} = 1 - 2l = -1 - 2(l-1)$ and the same holds for even n. As a consequence, $\operatorname{ind}(\overline{K}_l^n, 0) = -\operatorname{reg}_1(n) - (l-1)\operatorname{reg}_2(n)$ which provides the realization of the sequences of the form (δ) for $l := -a_2 + 1 \leq 0$ if $a_2 \leq 0$. A source that changes the orientation gives the indices $-\operatorname{reg}_1(n) + \operatorname{reg}_2(n)$, realizing (δ) with $a_2 = 1$. $\qquad\square$

4. Fixed point indices of iterates of $\mathbb{R}^3$-diffeomorphisms at a fixed point which is an isolated invariant set.
The following theorem provides a complete description of sequences of local indices of iterates for C^1 self-maps of $\mathbb{R}^3$ [10].

Theorem 4.1. *(1) Let f be a C^1 self-map of $\mathbb{R}^3$. Then the sequence of local indices of iterates $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$ has one of the following forms:*

(A) $c_A(n) = a_1 \mathrm{reg}_1(n) + a_2 \mathrm{reg}_2(n)$,
(B) $c_B(n) = \mathrm{reg}_1(n) + a_d \mathrm{reg}_d(n)$,
(C) $c_C(n) = -\mathrm{reg}_1(n) + a_d \mathrm{reg}_d(n)$,
(D) $c_D(n) = a_d \mathrm{reg}_d(n)$,
(E) $c_E(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_d \mathrm{reg}_d(n)$,
(F) $c_F(n) = \mathrm{reg}_1(n) + a_d \mathrm{reg}_d(n) + a_{2d} \mathrm{reg}_{2d}(n)$, *where d is odd,*
(G) $c_G(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_d \mathrm{reg}_d(n) + a_{2d} \mathrm{reg}_{2d}(n)$, *where d is odd.*

In all cases $d \geq 3$ and $a_i \in \mathbb{Z}$.

(2) Every sequence of integers which is of one of the forms (A)–(G) can be realized as a sequence of local indices of iterates of a C^1 self-map of $\mathbb{R}^3$.

In [21] Ruiz del Portal and Salazar studied diffeomorphisms of $\mathbb{R}^3$ and asked which are all the forms of $\{\mathrm{ind}(f^n,0)\}_n$ under the additional assumption that $\{0\}$ is an isolated invariant set. Of course the necessary conditions are given by Theorem 4.1. On the other hand, these authors were able to find the realizations only for some forms of sequences listed in (A)-(G) of Theorem 4.1, namely:

Theorem 4.2. ([21, Proposition 1]). *There exist diffeomorphisms $f \colon \mathbb{R}^3 \to \mathbb{R}^3$, with $\mathrm{Fix}(f) = \mathrm{Per}(f) = \{0\}$ an isolated invariant set, for the following cases of the sequences of Theorem 4.1: The case (A) if $a_1 \leq 1$ or $a_2 = 0$. The cases (B), (C), and (D). The case (E) if $a_d \leq 0$ or d even. The cases (F) and (G) if $a_d \leq 0$.*

Thus, some forms of indices remain unrealized in the considered class of diffeomorphisms. This provokes the question whether Theorem 4.2 is optimal, which was expressed in the following way:

Conjecture 4.3. ([21, Question 1]). *Is it true that if the sequence of fixed point indices of the iterates of an $\mathbb{R}^3$-diffeomorphism f, at an isolated fixed point 0 such that $\mathrm{Fix}(f) = \mathrm{Per}(f) = \{0\}$, does not follow one of the patterns of Theorem 4.2, then $\{0\}$ is not an isolated invariant set?*

We confirm Conjecture 4.3, i.e. we will prove the following statement:

Theorem 4.4. *The only forms of local indices of iterates of an $\mathbb{R}^3$-diffeomorphism for which $\{0\}$ is an isolated invariant set are given in Theorem 4.2.*

5. Proof of Theorem 4.4. In this section we will give the proof of Theorem 4.4, which is based on two ingredients: the result of Chow, Mallet-Paret, and Yorke on the forms of indices of smooth maps (Theorems 5.1 and 5.2) and a recent result of Hernández-Corbato, Le Calvez, and Ruiz del Portal [13] (Theorem 5.3).

5.1. Restrictions on indices of iterates of smooth maps. In 1983 Chow, Mallet-Paret, and Yorke showed that for a C^1 map there are strong restrictions on indices of iterates. Namely, they proved that there could be only a finite number

of basic sequences $(\mathrm{reg}_k)_n$ which may appear in the periodic expansion of $(\mathrm{ind}(f^n, 0))_n$ and that one may identify all possible k by a use of the derivative $Df(0)$ of f at 0.

Let us denote by Δ the set of degrees of all primitive roots of unity which are contained in $\sigma(Df(0))$, the spectrum of $Df(0)$. By σ_+ we denote the number of real eigenvalues of $Df(0)$ greater than 1 and by σ_- - the number of real eigenvalues of $Df(0)$ less than -1, in both cases counting with multiplicity. Let

$$\overline{O} = \{\mathrm{LCM}(K)\colon K \subset \Delta\} \cup \{1\},$$

where by $\mathrm{LCM}(K)$ we mean the least common multiple of all elements in the set K. Finally let us define $\overline{O}_{odd} = \overline{O} \cap \{n \in \mathbb{N}\colon n = 2k - 1, k = 1, 2, \ldots\}$.

We present now the result of Chow, Mallet-Paret, and Yorke in two theorems below, using the language of k-periodic expansion (cf. [9]).

Theorem 5.1. *Let $U \subset \mathbb{R}^m$ be an open neighborhood of 0, $f\colon U \to \mathbb{R}^m$ be a C^1 map such that 0 is an isolated fixed point for each of its iterates. Then:*

$$\mathrm{ind}(f^n, 0) = \sum_{k \in O} a_k \mathrm{reg}_k(n), \quad where$$

$$O = \begin{cases} \overline{O} & \text{if } \sigma_- \text{ is even,} \\ \overline{O} \cup 2\overline{O}_{odd} & \text{if } \sigma_- \text{ is odd.} \end{cases} \tag{5.1}$$

$$\text{If } \sigma_- \text{ is odd and } k \in 2\overline{O}_{odd}\backslash\overline{O}, \text{ then } a_k = -a_{k/2}. \tag{5.2}$$

Theorem 5.2. *The coefficients a_1 and a_2 in Theorem 5.1 satisfy the following conditions:*

(1) $a_1 = (-1)^{\sigma_+}$ *if $1 \notin \sigma(Df(0))$.*
(2) $a_1 \in \{-1, 0, 1\}$ *if 1 is the eigenvalue of $Df(0)$ with multiplicity 1.*
(3) $a_2 \in \{0, (-1)^{\sigma_+ + 1}\}$ *if $1 \notin \sigma(Df(0))$ and -1 is the eigenvalue of $Df(0)$ with multiplicity 1.*

5.2. Restrictions on indices of iterates of orientation reversing homeomorphisms in dimension 3 with a fixed point being an isolated invariant set.

Theorem 5.3. *([13, Theorem C]). Given a sequence $\{I_n\}_{n \geq 1}$ such that $I_n = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k(n)$, there exists an orientation reversing homeomorphism $f\colon U \to f(U) \subset \mathbb{R}^3$, where $U \subset \mathbb{R}^3$ is open, with a fixed point p isolated as an invariant set and such that $(I_n)_{n \geq 1} = (\mathrm{ind}(f^n, p))_{n \geq 1}$ if and only if*

(1) *the coefficients a_k are integers,*
(2) *there are finitely many nonzero a_k,*
(3) *$a_1 \leq 1$ and $a_k \leq 0$ for all odd $k > 1$.*

5.3. Proof of Theorem 4.4.

We will show that there are not any more forms of sequences of indices except for those listed in Theorem 4.2. As all sequences mentioned in the cases (B), (C), and (D) could be realized in the class of $\mathbb{R}^3$-diffeomorphisms with $\{0\}$ an isolated invariant set, we consider the remaining situations.

Let us start from the cases (E), (F), and (G), which we consider together in Case I. Notice that in each of these cases there are present, in the periodic

expansion of $(\mathrm{ind}(f^n, 0))_n$, both basic sequences reg_2 and reg_d or reg_d and reg_{2d} ($d \geq 3$). Using the terminology introduced in Section 5.1, we get that necessarily $\{2, d\} \subset O$. Now consider two subcases.

Case I (i) σ_- is even. Then either $\sigma_- = 2$ or $\sigma_- = 0$. In the first case, there could be only one root of unity in $\sigma(Df(0))$ (of degree 1 or 2) thus $d \notin O$, contradiction. In the second case $\Delta = \{2, d\}$, which implies that f is a local diffeomorphism and that it changes the orientation. By the formula (5.1) we can obtain the case (E) here. Applying Theorem 5.3 (3) we get that $a_k \leq 0$ for all odd $k > 1$, which gives all the restrictions listed in Theorem 4.2 for the case (E).

Case I (ii) σ_- is odd. The case $\sigma_- = 3$ must be rejected by the same dimension reasoning as used in Case I (i) (otherwise $d \notin O$). Let us consider the case $\sigma_- = 1$.

Then the following equality is satisfied:

$$O = \overline{O} \cup 2\overline{O}_{odd} = \{1, 2, d, 2d\}, \qquad (5.3)$$

where $\overline{O} = \{\mathrm{LCM}(K) \colon K \subset \Delta\} \cup \{1\}$. This implies that d must belong to Δ, i.e. there is a pair of primitive roots of unity of degree d in $\sigma(Df(0))$. Thus also in this situation f is an orientation reversing local diffeomorphism and $a_k \leq 0$ for all odd $k > 1$ by Theorem 5.3 (3).

Now let us analyze the Case II, which encompasses the case (A) of Theorem 4.1. We will show that the form of indices:

$$\mathrm{ind}(f^n, 0) = a_1 \mathrm{reg}_1(n) + a_2 \mathrm{reg}_2(n)$$

with $a_1 > 1$ and an arbitrary $a_2 \neq 0$ is not possible, obtaining the restrictions from the case (A) of Theorem 4.2. Again this part of the proof will be partitioned into two subcases.

Case II (i) σ_- is even. Notice that then $O = \overline{O} \supset \{1, 2\}$, which means that necessarily $2 \in \Delta$, i.e. $-1 \in \sigma(Df(0))$. Now, if $\sigma_- = 2$, then f is an orientation reversing local diffeomorphism and by Theorem 5.3 (3) $a_1 \leq 1$. In case $\sigma_- = 0$ we have in addition to -1 two eigenvalues in $\sigma(Df(0))$. If 1 is not one of these eigenvalues then by Theorem 5.2 (1) $a_1 \in \{-1, 1\}$. If $1 \in \sigma(Df(0))$ with multiplicity 1, then by Theorem 5.2 (2) $a_1 \in \{-1, 0, 1\}$. Finally, if $1 \in \sigma(Df(0))$ with multiplicity 2, then $a_1 \leq 1$ by Theorem 5.3 (3).

Case II (ii) σ_- is odd. Here, in case $\sigma_- = 3$, we apply Theorem 5.3 (3). If $\sigma_- = 1$, we repeat the reasoning from Case II (i) for $\sigma_- = 0$ considering the cases with $1 \in \sigma(Df(0))$ with different multiplicities and obtain the needed restrictions for a_1. This completes the proof of Theorem 4.4.

6. Concluding remarks. The result obtained in Theorem 4.4 may be generalized for the class of local maps which are not necessarily injective. Then instead of an orbit of a point x, we can consider *solutions through x*, which enables us to define the notion of isolated invariant sets in a more general context and formulate the counterpart of Theorem 4.4.

Definition 6.1. Let $f \colon U \to \mathbb{R}^m$ be a continuous map, where U is an open subset of $\mathbb{R}^m$. Let $\sigma \colon \mathbb{Z} \to N$ be given by $\sigma(n) = x_n$ and $x_0 = x$ and $f(x_n) = x_{n+1}$ for all n. Then we will call σ a solution through x. We define the *maximal*

invariant subset, Inv$N$, to be the set of all $x \in N$ such that there exists a solution σ with $\{\sigma(n)\}_{-\infty}^{\infty} \subset N$ and with $\sigma(0) = x$.

Definition 6.2. A compact set N is called an *isolating neighborhood* if Inv$N \subset$ Int$N$. A set S is called an *isolated invariant set* if there exists an isolating neighborhood N with $S = $ Inv$N$.

Proposition 6.3. *Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map near a fixed point 0, which is not a diffeomorphism, i.e. for which 0 is an eigenvalue of $Df(0)$. For each such a map, by Theorem 5.1 $\mathrm{ind}(f^n, 0) = a_1 \mathrm{reg}_1 + a_2 \mathrm{reg}_2$ and by Theorem 5.2 (1) and (2), $\mathrm{ind}(f, 0), \mathrm{ind}(f^2, 0) \in \{-1, 0, 1\}$. This fact together with Dold relations implies that $(\mathrm{ind}(f^n, 0))_n$ may be only among the following sequences: zero sequence, $\mathrm{reg}_1(n)$, $-\mathrm{reg}_1(n)$, $\mathrm{reg}_1(n) - \mathrm{reg}_2(n)$, $-\mathrm{reg}_1(n) + \mathrm{reg}_2(n)$.*

Remark 6.4. By Proposition 6.3 we get that indices for planar C^1 maps (non-necessarily injective) with a fixed point 0 isolated as an invariant set have exactly the same forms as for diffeomorphisms (listed in Theorem 3.2 (1)). This observation is in agreement with (and could be deduced from) the recent result of Hernández-Corbato and Ruiz del Portal (cf. [14, Theorem 1]) which states that indices of iterates of continuous planar maps with a fixed point 0 isolated as an invariant set which is neither a source nor a sink have the form

$$\mathrm{ind}(f^n, 0) = \mathrm{reg}_1(n) + \sum_{k \in F} b_k \mathrm{reg}_k(n), \tag{6.1}$$

where $b_k < 0$ and $F \neq \emptyset$ is finite.

It turns out that the same result is valid in dimension 3. Namely, Theorem 4.4 remains true also for non-injective 3-dimensional maps:

Theorem 6.5. *The only possible forms of local indices of iterates of a C^1 map $f \colon U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^3$ for which $\{0\}$ is an isolated invariant set in the sense of Definition 6.2 are given by the restrictions in Theorem 4.2.*

Proof. We can literally repeat the reasoning from the proof of Theorem 4.4, obtaining in some cases that f must be an orientation reversing local diffeomorphism and then applying Theorem 5.3 (3) or in other cases using the restrictions from Theorem 5.1 and Theorem 5.2, which are valid also for non-injective smooth maps. □

Remark 6.6. The natural question is to ask about the forms of indices for the considered class of maps in higher dimensions. Theorems 5.1 and 5.2 give some restrictions on indices of iterates at 0 for C^1 maps in $\mathbb{R}^n$. The explicit list of possible sequences of indices was given in [8, Theorem 3.1]. Additionally, in the same paper there were constructed realizations by diffeomorphisms (Theorem 3.2) with $\{0\}$ being an isolated invariant set for sequences having all coefficients a_k non-negative ($a_1 \leq 1$). Roughly speaking, this was done by realizing each $a_k \mathrm{reg}_k(n)$ on different 2-dimensional subspaces of $\mathbb{R}^n$ as a discretization of some hyperbolic flows h_q described in Section 3 and extending the obtained map to $\mathbb{R}^n$ without changing the indices.

As a consequence of Remark 6.6 the following problem remains open in $\mathbb{R}^n$, where $n \geq 4$.

Problem 6.7. *What forms of* $\{\mathrm{ind}(f^n, 0)\}_n$ *which are admitted by Theorems 5.1 and 5.2 and have positive coefficients at some terms of periodic expansion could be realized by an* $\mathbb{R}^n$-*diffeomorphism* f *(or more generally, by a* C^1 *map) having* $\{0\}$ *as an isolated invariant set?*

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Grzegorz Graff
Faculty of Applied Physics and Mathematics,
Gdańsk University of Technology,
Narutowicza 11/12,
80-233 Gdańsk,
Poland
e-mail: `grzegorz.graff@pg.edu.pl`