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Some tight contact foliations can be approximated by overtwisted ones

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Abstract. A contact foliation is a foliation endowed with a leafwise contact structure. In this remark we explain a turbulisation procedure that allows us to prove that tightness is not a homotopy invariant property for contact foliations.

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1. Statement of the results. Let M^{2n+1+q} be a closed smooth manifold. Let $\mathcal{F}^{2n+1}$ be a smooth codimension–q foliation on M. We say that $(M, \mathcal{F})$ can be endowed with the structure of a *contact foliation* if there is a hyperplane field $\xi^{2n} \subset \mathcal{F}$ such that, for every leaf $\mathcal{L}$ of $\mathcal{F}$, $(\mathcal{L}, \xi|_{\mathcal{L}})$ is a contact manifold.

In [2, Theorem 1.1] it was shown that $(M^4, \mathcal{F}^3)$ admits a leafwise contact structure if there exists a 2–plane field tangent to $\mathcal{F}$. This was later extended in [1] to foliations of any dimension, of any codimension, and admitting a leafwise formal contact structure. In both cases, the foliations produced have all leaves overtwisted; therefore, the meaningful question is whether one can construct and classify contact foliations with tight leaves.

1.1. Instability of tightness under homotopies. If the foliation $\mathcal{F}$ is fixed, the parametric Moser trick [2, Lemma 2.8] implies that any two homotopic contact foliations $(\mathcal{F}, \xi_0)$ and $(\mathcal{F}, \xi_1)$ are actually isotopic by a flow tangent to the leaves. In particular, if $\mathcal{F}$ is fixed, tightness is preserved under homotopies. Our main result states that this is not the case anymore if $\mathcal{F}$ is allowed to move:

Theorem 1. *Let N be a closed orientable 3–manifold. There is a path of contact foliations $(N \times \mathbb{S}^1, \mathcal{F}_s, \xi_s)_{s\in[0,1]}$, satisfying:*

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- *the leaves of* $(N \times \mathbb{S}^1, \mathcal{F}_0, \xi_0)$ *are tight,*
- *the leaves of* $(N \times \mathbb{S}^1, \mathcal{F}_s, \xi_s)$ *are overtwisted for all* $s > 0$.

1.2. Foliations transverse to even–contact structures. Given a codimension–1 contact foliation $(M^{2n+2}, \mathcal{F}^{2n+1}, \xi^{2n})$ and a line field $\mathfrak{X}$ transverse to $\mathcal{F}$, it is immediate that the codimension–1 distribution $\mathcal{E} = \xi \oplus \mathfrak{X}$ is maximally non–integrable. Such distributions are called *even–contact structures*.

The *kernel* or *characteristic foliation* of $\mathcal{E}$ is a line field $\mathcal{W} \subset \mathcal{E}$ uniquely defined by the expression $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. Given an even–contact structure $\mathcal{E}$, any codimension–1 foliation transverse to its kernel is imprinted with a leafwise contact structure. It is natural to study the moduli of contact foliations arising in this manner from $\mathcal{E}$. Our second result states:

Theorem 2. *Let* N *be a closed orientable* 3*–manifold. There are foliations* $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *and an even–contact structure* $\mathcal{E}$ *such that:*

- *the leaves of* $(N \times \mathbb{S}^1, \mathcal{F}_0, \xi_0 = \mathcal{E} \cap \mathcal{F}_0)$ *are tight,*
- *the leaves of* $(N \times \mathbb{S}^1, \mathcal{F}_1, \xi_1 = \mathcal{E} \cap \mathcal{F}_1)$ *are overtwisted.*

Proof. During the proof of Theorem 1 we shall see that the contact foliations $(\mathcal{F}_s, \xi_s)_{s \in [0,1]}$ are imprinted by the same even–contact structure $\mathcal{E}$. □

This result is in line with the theorem of McDuff [6] stating that even–contact structures satisfy the complete h–principle: one should expect this flexibility to manifest in other ways.

2. Turbulisation of contact foliations. In this section we explain how to turbulise a contact foliation along a loop of legendrian knots.

2.1. Local model around a loop of legendrian knots. Let (N, ξ) be a contact 3–manifold. Any legendrian knot $K \subset (N, \xi)$ has a tubular neighbourhood with the following normal form:

$$(\mathcal{O}p(K) \subset N, \xi) \cong (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\text{leg}} = \ker(\cos(z)dx + \sin(z)dy)),$$

where (x, y, z) are the coordinates in $\mathbb{D}^2 \times \mathbb{S}^1$. A convenient way of thinking about the model is that it is the space of oriented contact elements of the disc. In particular, any diffeomorphism ϕ of $\mathbb{D}^2$ relative to the boundary induces a contactomorphism $C(\phi)$ of the model, also relative to the boundary, as follows:

$$C(\phi)(x, y, z) = (\phi(x, y), d\phi(z)).$$

Here we think of z as an oriented line in $T_{(x,y)}\mathbb{D}^2$ and we make $d\phi$ act by pushforward.

We can now define a contact foliation in the product with $\mathbb{S}^1$:

$$\left(M_{\text{leg}} = \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{S}^1, \mathcal{F}_{\text{leg}} = \coprod_t \mathbb{D}^2 \times \mathbb{S}^1 \times \{t\}, \xi_{\text{leg}} \right).$$

Given a contact foliation $(M, \mathcal{F}, \xi)$ and an embedded torus $K : \mathbb{S}^1 \times \mathbb{S}^1 \to M$ such that each $K_t = K(t, -)$ is a legendrian knot in a leaf of $\mathcal{F}$, it follows that there is an embedding $(M_{\text{leg}}, \mathcal{F}_{\text{leg}}, \xi_{\text{leg}}) \to (M, \mathcal{F}, \xi)$ providing a local model around K. It is therefore sufficient to describe the turbulisation process in $(M_{\text{leg}}, \mathcal{F}_{\text{leg}}, \xi_{\text{leg}})$.

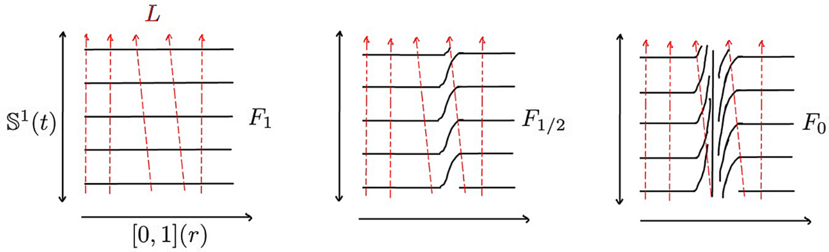

Figure 1. The solid lines represent (the foliations induced by) the path of line fields $(F_s)_{s\in[0,1]}$. The dotted ones with arrows on top represent the line field L

2.2. Fixing the even–contact structure. Our aim now is to fix an even–contact structure $\mathcal{E}_{\text{leg}}$ in $(M_{\text{leg}}, \mathcal{F}_{\text{leg}})$ imprinting ξ_{leg}. The reason why we do not simply choose $\xi_{\text{leg}} \oplus \langle\partial_t\rangle$ is that $\mathcal{E}_{\text{leg}}$ should allow us to turbulise.

Take polar coordinates (r, θ) on $\mathbb{D}^2$. Fix a vector field $h(r)\partial_r$ with $h(r) < 0$ in the region $r \in (1/2, 2/3)$ and $h(r) = 0$ everywhere else. Its flow $(\phi_t)_{t\in\mathbb{R}}$ is a 1–parameter subgroup of $\text{Diff}(\mathbb{D}^2)$; as explained above, it can be lifted to a 1–parameter subgroup $(\Phi_t)_{t\in\mathbb{R}}$ of contactomorphisms of $\mathbb{D}^2 \times \mathbb{S}^1$. We denote by X the (unique) contact vector field in $\mathbb{D}^2 \times \mathbb{S}^1$ that generates $(\Phi_t)_{t\in\mathbb{R}}$. By construction X is a lift of $h(r)\partial_r$ and, in particular, it has a negative radial component in the region $r \in (1/2, 2/3)$.

Since X is a contact vector field, the 3–distribution $\mathcal{E}_{\text{leg}}(x, y, z, t) = \xi_{\text{leg}} \oplus \langle\partial_t + X(x, y, z)\rangle$ is an even–contact structure whose kernel is $\mathcal{W}_{\text{leg}} = \langle\partial_t + X(x, y, z)\rangle$ and whose imprint on $(M_{\text{leg}}, \mathcal{F}_{\text{leg}})$ is precisely ξ_{leg}.

2.3. Turbulisation. Consider the surface $S = [0, 1] \times \mathbb{S}^1$ with coordinates (r, t); the manifold M_{leg} projects onto S in the obvious way. Under this projection the kernel $\mathcal{W}_{\text{leg}}$ is mapped to the line field $L = \langle\partial_t + h(r)\partial_r\rangle$. Similarly, the foliation $\mathcal{F}_{\text{leg}}$ is the pullback of $F_1 = \langle\partial_r\rangle$. The line fields F_1 and L are transverse to one another. We can find a homotopy $(F_s)_{s\in[0,1]}$ in S satisfying:

- $F_1 = \langle\partial_r\rangle$,
- F_s is transverse to L, for all s,
- F_s is isotopic to F_1 for every $s > 0$,
- F_0 is as in the last frame of Figure 1: it has a closed orbit bounding a (half) Reeb component.

This path of line fields lifts to a path of codimension–1 foliations $\mathcal{F}_{\text{leg},s}$ in M_{leg}. The foliation $\mathcal{F}_{\text{leg},1}$ is precisely $\mathcal{F}_{\text{leg}}$ and $\mathcal{F}_{\text{leg},s}$ is isotopic to it for every positive s. The foliation $\mathcal{F}_{\text{leg},0}$ has a single compact leaf, which is diffeomorphic to T^3; this leaf bounds a Reeb component whose interior leaves are diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$. Transversality of L with respect to F_s implies that $\mathcal{E}_{\text{leg}}$ imprints a contact foliation $\xi_{\text{leg},s}$ on each $\mathcal{F}_{\text{leg},s}$.

Let us package this construction:

Definition 3. Let $(M, \mathcal{F}, \xi)$ be a contact foliation. Suppose there is a region $U \subset M$ such that $(U, \mathcal{F}, \xi)$ is diffeomorphic to the model $(M_{\text{leg}}, \mathcal{F}_{\text{leg}}, \xi_{\text{leg}})$. We say that the homotopy $(M, \mathcal{F}_s, \xi_s)_{s\in[0,1]}$ given by the procedure just described is the **turbulisation** of $(M, \mathcal{F}, \xi)$ along U.

2.4. Turbulisation preserves tightness.

Lemma 4. *Let $(M, \mathcal{F}, \xi)$ be a contact foliation with tight leaves. Denote its turbulisation along $U \subset M$ by $(M, \mathcal{F}_s, \xi_s)_{s\in[0,1]}$. Then all the leaves of $(M, \mathcal{F}_s, \xi_s)$ are tight.*

This lemma can be proven by elementary means, but it is convenient for us to introduce at this point the following result of V. Colin [3, Théorème 4.2]:

Proposition 5. *Let (N, ξ) be a 3–dimensional contact manifold. Let $S \subset N$ be a quasi pre-lagrangian incompressible torus. Then (N, ξ) is universally tight if and only if $(N \setminus S, \xi)$ is universally tight.*

As explained in [3], any torus having a *linear* characteristic foliation $TS \cap \xi$ is quasi pre-lagrangian. Incompressibility means that $\pi_1(S)$ injects into $\pi_1(N)$. Finally, we say that (N, ξ) is universally tight if the lift of ξ to the universal cover $\tilde{N}$ is tight.

Proof of Lemma 4. Let us write $\mathcal{E}$ for the even–contact structure imprinting ξ_s and $\mathcal{W}$ for its characteristic foliation. $\mathcal{E}$ is chosen arbitrarily away from U, but in the model it is given by $\mathcal{E}_{\mathrm{leg}}$.

Consider first the case $s > 0$. The foliations $(\mathcal{F}_s)_{s\in(0,1]}$ are all isotopic to $\mathcal{F}_1 = \mathcal{F}$. This isotopy, by construction, can be realised by a flow tangent to $\mathcal{W}$ (which in particular preserves $\mathcal{E}$). This immediately induces a contactomorphism between any leaf $(\mathcal{L}_s, \xi_s)$ of $\mathcal{F}_s$ and the corresponding leaf $(\mathcal{L}, \xi)$ of $\mathcal{F}$ to which it is isotopic.

Assume now $s = 0$. We can argue similarly to show that the open leaves of $\mathcal{F}_0$ can be identified, as contact manifolds, with open subsets of leaves of $\mathcal{F}$, proving tightness. We claim that the remaining leaf (T^3, ξ_0), bounding the Reeb component, is also tight. Consider the 2–torus $S = T^3 \cap \{t = t_0\} \subset T^3$. Due to the rotational symmetry of the model, S is quasi pre-lagrangian and also incompressible. Using a flow along $\mathcal{W}$ again, we can identify the contact manifold $(T^3 \setminus S, \xi)$ with an open subset in $\mathcal{L} \cap U$, where $\mathcal{L}$ is a leaf of $\mathcal{F}$ and U is the region where the turbulisation takes place. In particular, we are identifying it with a subset of the local model $(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{leg}})$, which is universally tight. An application of Proposition 5 concludes the claim. □

In particular, the model $(M_{\mathrm{leg}}, \mathcal{F}_{\mathrm{leg}}, \xi_{\mathrm{leg}})$ and its turbulisation are tight.

Remark 6. There is an alternate way to describe the turbulisation process. The contact foliation $(M_{\mathrm{leg}}, \mathcal{F}_{\mathrm{leg}}, \xi_{\mathrm{leg}})$ is the space of oriented contact elements of the foliation $(\mathbb{D}^2 \times \mathbb{S}^1, \coprod_{t\in\mathbb{S}^1} \mathbb{D}^2 \times \{t\})$. Then, the turbulisation process upstairs amounts to turbulising $(\mathbb{D}^2 \times \mathbb{S}^1, \coprod_{t\in\mathbb{S}^1} \mathbb{D}^2 \times \{t\})$ and applying the contact elements construction. This construction also works for higher dimensional contact foliations and highlights the fact that the resulting leaves (in the model) are tight.

3. Applications.

3.1. Proof of Theorems 1 and 2. In [4] K. Dymara proved that there are legendrian links in overtwisted contact manifolds that intersect *every* overtwisted disc; that is, their complement is tight. Such a link is said to be *non–loose*. Let (N, ξ) be an overtwisted contact manifold with K a non–loose legendrian link. Consider the contact foliation

$$(M, \mathcal{F}_1, \xi_1) = \left(N \times \mathbb{S}^1, \coprod_{t \in \mathbb{S}^1} N \times \{t\}, \xi \right),$$

where we abuse notation and write ξ for the leafwise contact structure lifting (N, ξ). Take U to be the tubular neighbourhood of $K \times \mathbb{S}^1 \subset M$ and apply the turbulisation process to $(M, \mathcal{F}_1, \xi_1)$ (on each component) to yield a path of contact foliations $(M, \mathcal{F}_s, \xi_s)_{s \in [0,1]}$. It is immediate that $(M, \mathcal{F}_s, \xi_s)$ is diffeomorphic to $(M, \mathcal{F}_1, \xi_1)$ if s is positive, because the foliations themselves are isotopic and Gray's stability applies. In particular, the leaves of all of them are overtwisted. We claim that $(M, \mathcal{F}_0, \xi_0)$ has all leaves tight. This is clear for the leaves in the Reeb components we have introduced, as shown in Lemma 4. Similarly, the leaves outside of the Reeb components are tight because a neighbourhood of the non–loose legendrian link has been removed. We conclude by recalling that every closed overtwisted 3–manifold admits a non–loose legendrian link: the legendrian push–off of the binding of a supporting open book [5]. □

Remark 7. The foliation $(M, \mathcal{F}_1)$ is taut, since it admits a transverse $\mathbb{S}^1$. As pointed out by V. Shende during a talk of the author: we are trading tautness of the foliation to achieve tightness of the leaves.

3.2. A more general statement. A slightly more involved argument shows:

Theorem 8. *Let M be a 4–manifold. Suppose that M admits a contact foliation $(\mathcal{F}, \xi)$ with tight leaves. Then M admits a contact foliation $(\mathcal{F}_0, \xi_0)$ with tight leaves that can be approximated by contact foliations $(\mathcal{F}_s, \xi_s)_{s \in (0,1]}$ with overtwisted leaves.*

Proof. Fix $\gamma : \mathbb{S}^1 \to M$ an embedded curve transverse to $\mathcal{F}$. Such a curve always exists in a neighbourhood of an open leaf. If $\mathcal{F}$ has no open leaves, then it is immediate that it must be the foliation by fibres of some submersion $M \to \mathbb{S}^1$, so a transverse curve exists as well. We can find an $\mathbb{S}^1$–family of Darboux balls $(\mathbb{D}^3, \xi_{\text{std}})$ along γ:

$$(M_{\text{std}}, \mathcal{F}_{\text{std}}, \xi_{\text{std}}) = \left(\mathbb{D}^3 \times \mathbb{S}^1, \coprod_{t \in \mathbb{S}^1} \mathbb{D}^3 \times \{t\}, \xi_{\text{std}} \right) \to (M, \mathcal{F}, \xi).$$

This allows us to choose a legendrian knot $K \subset (\mathbb{D}^3, \xi_{\text{std}})$ and lift it to $K \times \mathbb{S}^1 \subset (M_{\text{std}}, \mathcal{F}_{\text{std}}, \xi_{\text{std}}) \subset (M, \mathcal{F}, \xi)$. Turbulisation in a neighbourhood of $K \times \mathbb{S}^1$ yields a contact foliation $(M, \mathcal{F}', \xi')$ whose leaves are still tight due to Lemma 4.

The interior of the Reeb component we just inserted is diffeomorphic, as a contact foliation, to the model $(M_{\text{leg}}, \mathcal{F}_{\text{leg}}, \xi_{\text{leg}})$. Given a homotopically essential transverse knot $\eta \subset (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\text{leg}})$, we may perform a Lutz twist along η

to yield an overtwisted contact structure ξ_{OT} in $\mathbb{D}^2 \times \mathbb{S}^1$. The resulting local model along η reads:

$$\left(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{Lutz}} = \ker(f(r)dz + g(r)d\theta)\right)$$

where (r, θ, z) are the coordinates in a neighbourhood of $\eta = \{r = 0\}$ and $r \to (f(r), g(r))/|f, g|$ is an immersion of $[0, 1]$ onto $\mathbb{S}^1$ that is injective for $r \in [0, 1 - \delta)$ and satisfies:

$$\begin{aligned} (f(r), g(r)) &= (1, r^2) && \text{if } r \in [0, \delta] \\ f(r) &= 0 && \text{if } r \in \{1/4, 3/4\} \\ g(r) &= 0 && \text{if } r \in \{0, 1/2, 1 - \delta\} \\ (f(r), g(r)) &= (1, (r - 1 + \delta)^2) && \text{if } r \in [1 - \delta/2, 1]. \end{aligned}$$

The Lutz twist can be introduced parametrically in the t–coordinate [2] to replace $(M_{\mathrm{leg}}, \mathcal{F}_{\mathrm{leg}}, \xi_{\mathrm{leg}}) \subset (M, \mathcal{F}', \xi')$ by a contact foliation $(M_{\mathrm{leg}}, \mathcal{F}_{\mathrm{leg}}, \xi_{\mathrm{OT}})$ which has all leaves overtwisted. This produces a new contact foliation $(M, \mathcal{F}_1, \xi_1)$ from $(M, \mathcal{F}', \xi')$.

We claim that $K'(z) = (1/4, 0, z) \in (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{Lutz}}) \subset (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{OT}})$ is non–loose. The quasi–prelagrangian tori

$$\{r = r_0 > 1/4\} \subset (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{Lutz}}) \subset (\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{OT}})$$

are incompressible in $(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{OT}}) \setminus K'$ due to our choice of η and K'. Choose the tori at radii $r = 1/2, 1 - \delta$. The reader can check that the pieces $\{r < 1/2\}$, $\{1/2 < r < 1 - \delta\}$ have standard tight $\mathbb{R}^3$ as their universal cover. The remaining piece, which intersects the Lutz twist model $(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{Lutz}})$ in $\{r > 1 - \delta\}$, is contactomorphic to the complement of η in $(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{leg}})$ and is therefore universally tight as well.

Proposition 5 shows that K' is non–loose in $(\mathbb{D}^2 \times \mathbb{S}^1, \xi_{\mathrm{Lutz}})$. We then turbulise in a neighbourhood of

$$K' \times \mathbb{S}^1 \subset (M_{\mathrm{leg}}, \mathcal{F}_{\mathrm{leg}}, \xi_{\mathrm{OT}}) \subset (M, \mathcal{F}_1, \xi_1)$$

to produce the claimed family $(M, \mathcal{F}_s, \xi_s)_{s \in [0,1]}$ and conclude the proof. □

The reader can check that the resulting foliation $(M, \mathcal{F}_0, \xi_0)$ is in the same formal class as $(M, \mathcal{F}, \xi)$, since $\mathcal{F}_0$ is obtained from $\mathcal{F}$ by turbulising twice and the even–contact structures inducing ξ and ξ_0 differ from one another by a parametric (full) Lutz–twist.

A natural question to pose in light of Theorem 8 is whether any M^4 admitting a formal contact foliation admits a foliation with tight leaves; the fundamental geometric issue towards achieving this is that it seems extremely delicate to ensure that no overtwisted disc is really present. For Theorem 8 the main idea was to introduce the overtwisted discs in a controlled fashion so that they could later be destroyed.

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