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On pair correlation and discrepancy

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Abstract. We say that a sequence $(x_n)_{n\geq 1}$ in $[0,1)$ has Poissonian pair correlations if

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N \colon \|x_l - x_m\| < \frac{s}{N} \right\} = 2s$$

for all $s > 0$. In this note we show that if the convergence in the above expression is—in a certain sense—fast, then this implies a small discrepancy for the sequence $(x_n)_{n\geq 1}$. As an easy consequence it follows that every sequence with Poissonian pair correlations is uniformly distributed in $[0,1)$.

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1. Introduction. The concept of Poissonian pair correlations for a sequence $(x_n)_{n\geq 1}$ in $[0,1)$ was introduced by Rudnick and Sarnak [5], and has been intensively studied by several authors over the last years (see, for instance, [2, 3, 6–8]). Let $\|\cdot\|$ denote distance to the nearest integer. We say that a sequence $(x_n)_{n\geq 1}$ of real numbers in $[0,1)$ has *Poissonian pair correlations* if

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N \colon \|x_l - x_m\| < \frac{s}{N} \right\} = 2s \tag{1.1}$$

for every $s > 0$.

In this note we are concerned with the relation between the Poissonian pair correlation property and the notion of uniform distribution. We say that the sequence $(x_n)_{n\geq 1}$ is *uniformly distributed*, or *equidistributed*, in $[0,1)$ if

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$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \le n \le N : x_n \in [a,b)\} = b - a$$

for all $0 \le a \le b \le 1$. It is well known that uniform distribution does not necessarily imply Poissonian pair correlations. One example confirming this is the Kronecker sequence $(\{n\alpha\})_{n\ge1}$, which is uniformly distributed for every irrational α, but does *not* have Poissonian pair correlations for any value of α. Whether the converse implication holds has until recently remained an open question: is every sequence in $[0,1)$ with Poissonian pair correlations uniformly distributed? We answer this question in the affirmative by establishing a quantitative result connecting the speed of convergence in (1.1) to the star-discrepancy D_N^* of the sequence. We recall that the star-discrepancy D_N^* of $(x_n)_{n\ge1}$ is defined as

$$D_N^* = \sup_{0\le a\le 1} \left| \frac{1}{N} \cdot A_N([0,a)) - a \right|,$$

where $A_N([0,a)) := \#\{1 \le n \le N : x_n \in [0,a)\}$, and that $(x_n)_{n\ge1}$ is uniformly distributed in $[0,1)$ if and only if $\lim_{N\to\infty} D_N^* = 0$ (see, for example, [4]).

The main result of this paper is the following.

Theorem 1.1. *Let $(x_n)_{n\ge1}$ be a sequence in $[0,1)$, and suppose that there exists a function $F : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^+$ which is monotonically increasing in its first argument, and which satisfies*

$$\max_{s=1,\dots,K} \left| \frac{1}{2s} \# \left\{ 1 \le l \ne m \le N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \right| \le F(K,N) \qquad (1.2)$$

for all $N \in \mathbb{N}$ and all $K \le N/2$. One can then find an integer $N_0 > 0$ such that for $N \in \mathbb{N}$, $N \ge N_0$, and arbitrary K satisfying

$$\min\left(\frac{1}{2} N^{2/5}, \frac{N}{F(K^2,N)} \right) \le K \le N^{2/5}, \qquad (1.3)$$

we have

$$ND_N^* \le 5 \cdot \max\left(N^{4/5}, \sqrt{N \cdot F(K^2,N)} \right)$$

where D_N^* is the star-discrepancy of $(x_n)_{n\ge1}$.

The next result is an easy consequence of Theorem 1.1.

Corollary 1.2. *If the sequence $(x_n)_{n\ge1}$ in $[0,1)$ has Poissonian pair correlations, then it is uniformly distributed.*[1]

Proof. Suppose that $(x_n)_{n\ge1}$ has Poissonian pair correlations, and fix any $\varepsilon > 0$. We then have

$$\max_{s=1,\dots,\lfloor 1/\varepsilon^5 \rfloor} \left| \frac{1}{2s} \# \left\{ 1 \le l \ne m \le N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \right| \le \varepsilon N,$$

for all sufficiently large $N \ge N(\varepsilon)$. Hence, we may construct a function F satisfying (1.2) where $F(L,N) = \varepsilon N$ for $N \ge N(\varepsilon)$ and $L \le 1/\varepsilon^5$. Without

[1] Simultaneously with our proof, another elegant proof of this result was given by Aistleitner et al. [1]. However, their approach is less elementary and does not provide the quantitative bound on the star discrepancy given by Theorem 1.1.

loss of generality, we may assume that $N(\varepsilon) \geq 1/\varepsilon^5$. If we fix $K := \lfloor 1/\varepsilon^2 \rfloor$, then for $N \geq N(\varepsilon)$ we have

$$\frac{N}{F(K^2, N)} = \frac{N}{\varepsilon N} = \frac{1}{\varepsilon} \leq K \leq N^{2/5},$$

and accordingly K satisfies (1.3). By Theorem 1.1 it thus follows that

$$D_N^* \leq \frac{5}{N} \cdot \max\left(N^{4/5}, N\varepsilon\right) = 5\sqrt{\varepsilon}$$

for $N \geq N_0$ (where in particular $N_0 \geq N(\varepsilon)$). □

2. Proof of Theorem 1.1. For a fixed pair of integers (N, K), where K satisfies (1.3), we introduce the notation

$$H(N, K) := 5 \cdot \max\left(N^{4/5}, \sqrt{N \cdot F(K^2, N)}\right).$$

Aiming for a proof by contradiction, we assume that $ND_N^* > H(N, K)$ for infinitely many pairs (N, K). That is, there exist integers $1 < N_1 < N_2 < \cdots$ and corresponding integers $K_1, K_2, \ldots$ satisfying (1.3), as well as real numbers $B_1, B_2, \ldots \in (0, 1)$, such that either

$$\#\{1 \leq n \leq N_j : x_n \in [0, B_j)\} - N_j B_j > H(N_j, K_j) \tag{2.1}$$

for every j, or

$$\#\{1 \leq n \leq N_j : x_n \in [0, B_j)\} - N_j B_j < -H(N_j, K_j) \tag{2.2}$$

for every j. We assume in what follows that (2.1) holds (the case when (2.2) holds is treated analogously). Note that (2.1) implies

$$N_j - N_j B_j - H(N_j, K_j) > 0. \tag{2.3}$$

Let $N := N_j$, $K := K_j$, $B := B_j$, and $H := H(N_j, K_j)$ for some fixed j. We now consider the distribution of the points x_n into subintervals of $[0, 1)$ of length K/N. Let

$$A_i := \#\left\{1 \leq n \leq N : x_n \in \left[i \cdot \frac{K}{N}, (i+1) \cdot \frac{K}{N}\right)\right\}$$

for $i = 0, 1, \ldots, \lfloor N/K \rfloor - 1$, and let

$$A_{\lfloor N/K \rfloor} := \#\left\{1 \leq n \leq N : x_n \in \left[\left\lfloor \frac{N}{K} \right\rfloor \cdot \frac{K}{N}, 1\right)\right\}.$$

Moreover, for arbitrary positive integers l, let

$$A_l := A_{l \bmod (\lfloor N/K \rfloor + 1)}.$$

If we introduce the notation

$$\mathcal{H}_L := \#\left\{1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{LK}{N}\right\}$$

for $L = 1, 2, \ldots, K$, then

$$\left|\frac{1}{2LK}\mathcal{H}_L - N\right| \leq F(K^2, N). \tag{2.4}$$

We have that

$$\begin{aligned}\mathcal{H}_L &\geq \sum_{i=0}^{\lfloor N/K \rfloor} \left(A_i(A_i-1)+2A_i\left(A_{i+1}+\cdots+A_{i+L-1}\right)\right)\\ &= \sum_{i=0}^{\lfloor N/K \rfloor} \left(\left(A_i+\cdots+A_{i+L-1}\right)^2-\left(A_{i+1}+\cdots+A_{i+L-1}\right)^2\right)-N\\ &=: 2LKN\cdot\gamma_L - N,\end{aligned}$$

where

$$\gamma_L = \frac{1}{2LKN}\sum_{i=0}^{\lfloor N/K \rfloor} \left(\left(A_i+\cdots+A_{i+L-1}\right)^2-\left(A_{i+1}+\cdots+A_{i+L-1}\right)^2\right).$$

Thus, we get

$$\frac{1}{2LKN}\cdot\mathcal{H}_L \geq \gamma_L - \frac{1}{2LK}. \tag{2.5}$$

Now consider

$$\Gamma_K := \min_{x_1,\ldots,x_N}\max_{L=1,2\ldots,K}\gamma_L, \tag{2.6}$$

where by $\min_{x_1,\ldots,x_N}$ we mean the minimum over all configurations of the points $x_1,\ldots,x_N$ satisfying (2.1). If we define

$$Z_L := \frac{1}{2LKN}\sum_{i=0}^{\lfloor N/K \rfloor}\left(A_i+A_{i+1}+\cdots+A_{i+L-1}\right)^2,$$

then

$$\gamma_L = Z_L - \frac{L-1}{L}\cdot Z_{L-1},$$

and thus

$$\Gamma_K = \min_{x_1,\ldots,x_N}\max\left(Z_1, Z_2-\frac{1}{2}Z_1,\ldots,Z_K-\frac{K-1}{K}Z_{K-1}\right).$$

We have

$$\max\left(Z_1, Z_2-\frac{1}{2}Z_1,\ldots,Z_K-\frac{K-1}{K}Z_{K-1}\right) \geq \frac{2}{K+1}Z_K.$$

To see this, assume to the contrary that Z_1 and $Z_L-(L-1)Z_{L-1}/L$ are all less than $2Z_K/(K+1)$. Then by successive insertions we get the contradiction $Z_K < Z_K$. Hence, we have

$$\Gamma_K \geq \min_{x_1,\ldots,x_N}\frac{2}{K+1}\cdot Z_K. \tag{2.7}$$

Let us now estimate

$$\min_{x_1,\ldots,x_N} Z_K = \frac{1}{2K^2N}\min_{A_0,A_1,\ldots,A_{\lfloor N/K \rfloor}}\sum_{i=0}^{\lfloor N/K \rfloor}\left(A_i+A_{i+1}+\cdots+A_{i+K-1}\right)^2,$$

where the minimum on the right-hand side is taken over all possible values of $A_0,A_1,\ldots,A_{\lfloor N/K \rfloor}$ provided that the points $x_1,\ldots,x_N$ satisfy (2.1). By definition, we have $A_0 + \cdots + A_{\lfloor N/K \rfloor} = N$. Introducing the notation $G_i = A_i + A_{i+1} + \cdots + A_{i+K-1}$, we thus get

$$\sum_{i=0}^{\lfloor N/K \rfloor} G_i = K \cdot \sum_{i=0}^{\lfloor N/K \rfloor} A_i = KN. \tag{2.8}$$

Moreover, by invoking condition (2.1) on the distribution of $x_1, \ldots, x_N$, we have

$$\sum_{i=-K+1}^{\lfloor NB/K \rfloor} G_i \geq K \sum_{i=0}^{\lfloor NB/K \rfloor} A_i \geq K(NB + H), \tag{2.9}$$

and consequently

$$\sum_{i=\lfloor NB/K \rfloor+1}^{\lfloor N/K \rfloor - K} G_i \leq K\left(N(1-B) - H\right). \tag{2.10}$$

We get

$$\min_{x_1,\ldots,x_N} Z_K \geq \frac{1}{2K^2N} \min_{G_0,G_1,\ldots,G_{\lfloor N/K \rfloor}} \sum_{i=0}^{\lfloor N/K \rfloor} G_i^2, \tag{2.11}$$

where the minimum on the right-hand side is taken over all positive reals $G_0, G_1, \ldots, G_{\lfloor N/K \rfloor}$ satisfying (2.8)–(2.10). It is an easy exercise to verify that this minimum is attained when

$$G_i = \frac{K(NB+H)}{K + \lfloor NB/K \rfloor} \qquad \text{for } i = -K+1, \ldots, \left\lfloor \frac{NB}{K} \right\rfloor,$$

and

$$G_i = \frac{K\left(N(1-B) - H\right)}{\lfloor N/K \rfloor - K - \lfloor NB/K \rfloor} \qquad \text{for } i = \left\lfloor \frac{NB}{K} \right\rfloor + 1, \ldots, \left\lfloor \frac{N}{K} \right\rfloor - K.$$

Note that since $K \leq N^{2/5}$ and $H \geq 5N^{4/5}$, we have $K^2 \leq H/5$, and hence by (2.3) both the numerator and the denominator of these G_i are positive. Thus, we get

$$\begin{aligned} &\frac{1}{2K^2N} \min_{G_0,G_1,\ldots,G_{\lfloor N/K \rfloor}} \sum_{i=0}^{\lfloor N/K \rfloor} G_i^2 \\ &\geq \frac{1}{2K^2N} \left(\frac{K^2(NB+H)^2}{K + \lfloor NB/K \rfloor} + \frac{K^2\left(N(1-B) - H\right)^2}{\lfloor N/K \rfloor - K - \lfloor NB/K \rfloor} \right) \\ &\geq \frac{K}{2}\left(1 + \frac{H^2}{2N^2}\right) \end{aligned} \tag{2.12}$$

for all $N > N_0$. For the final inequality in (2.12), we have again used that $H \geq 5N^{5/4}$ and $K^2 \leq H/5$.

Finally, by combining (2.12), (2.11), and (2.7), we find the lower bound

$$\Gamma_K \geq \frac{K}{K+1}\left(1 + \frac{H^2}{2N^2}\right).$$

From the definition (2.6) of Γ_K and (2.5), it follows that

$$\max_{L=1,\ldots,K} \frac{1}{2LKN}\mathcal{H}_L > \Gamma_K - \frac{1}{2K} \geq 1 + \frac{H^2}{4N^2} - \frac{2}{K},$$

and recalling (2.4), we get

$$\frac{1}{N}F(K^2,N) + 1 \geq \max_{L=1,\ldots,K} \frac{1}{2LKN}\mathcal{H}_L > 1 + \frac{H^2}{4N^2} - \frac{2}{K}.$$

This implies that

$$\begin{aligned} H^2 &< \frac{8N^2}{K} + 4NF(K^2,N) \\ &\leq 12\max\left(\frac{N^2}{K}, NF\left(K^2,N\right)\right) \\ &< 25\max\left(N^{8/5}, NF\left(K^2,N\right)\right) = H^2, \end{aligned}$$

which is a contradiction. Thus, our assumption (2.1) must be incorrect, and the proof of Theorem 1.1 is complete. (Note that the last inequality above is trivially true if $N^2/K \leq NF(K^2,N)$; in the opposite case we have $K < N/F(K^2,N)$, and by the condition (1.3) imposed on K, we then get $K \geq N^{2/5}/2$, and consequently $N^2/K \leq 2N^{8/5}$.)

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