



## On Wallen-type formulae for integrated semigroups and sine functions

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**Abstract.** We prove Wallen-type formulae for integrated semigroups and sine functions with values in a unital Banach algebra with unit  $u$ . As the main application, we show that  $p(t) = tu, t \geq 0$  is an isolated integrated semigroup and sine function.

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**1. Introduction.** In 1966 Cox [12] proved that there is no square matrix  $A$  such that

$$\sup_{n \geq 1} \|A^n - I\| < 1, \quad (1)$$

where  $I$  is the identity matrix, except for  $A = I$ . This result has immediately been generalized: in 1967 Nakamura and Yoshida showed that (1) implies  $A = I$  in the case where  $A$  is a bounded linear operator in a Hilbert space and  $I$  is the identity operator, and one year later Hirschfeld [17] showed the same implication in the case where  $A$  is a member of a normed algebra. Later Wils [24], Chernoff [8], Nagisa and Wada [21], and Kalton et al. [18] provided further related results.

The approach of Wallen [23], who treated the case of a normed algebra  $\mathbb{A}$ , and published his result even a bit before Hirschfeld, seems to be of particular simplicity and elegance. He noted that if  $\mathbb{A}$  is an algebra with unit  $u$ , then for any  $a \in \mathbb{A}$ :

$$a - u = \frac{a^n - u}{n} + \frac{a - u}{n} \sum_{i=1}^{n-1} (u - a^i), \quad n \geq 1, \quad (2)$$

and concluded that if the algebra is normed, then the conditions

$$\lim_{n \rightarrow \infty} \frac{\|u - a^n\|}{n} = 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} \|u - a^i\| < 1 \tag{3}$$

imply  $a = u$ . In particular,  $a = u$  provided  $\sup_{n \geq 1} \|a^n - u\| < 1$ .

A renewed interest in the theory has been marked by the recent paper [6] in which, following the companion paper [5] on differences between semigroups of operators and cosine functions, it has been noted that for any Banach space  $\mathbb{X}$ , and any real  $\kappa$  the ‘scalar’ cosine function

$$C_\kappa(t) = \cos(\kappa t)I_{\mathbb{X}}, \quad t \in \mathbb{R},$$

where  $I_{\mathbb{X}}$  is the identity operator in  $\mathbb{X}$ , is isolated. More specifically, for a strongly continuous cosine function  $\{C(t), t \in \mathbb{R}\}$ , the condition

$$\sup_{t \geq 0} \|C(t) - C_\kappa(t)\| < \frac{1}{2} \tag{4}$$

implies  $C(t) = C_\kappa(t)$ . Moreover, there are no other isolated cosine functions but those that are ‘scalar’. Remarkably, the theory for semigroups is not analogous, especially if  $\mathbb{X}$  is a real Banach space: although the ‘trivial’ semigroup  $T(t) = I_{\mathbb{X}}$  is isolated, the ‘scalar’ semigroups  $T_\kappa(t) = e^{-\kappa t}I_{\mathbb{X}}, \kappa > 0$  are not.

The result of [6] has later been improved in various ways. As shown by Schwenninger and Zwart [22], in the case  $\kappa = 0$ , condition (4) with  $\frac{1}{2}$  replaced by 2 still implies that  $C(t) = C_0(t) = I_{\mathbb{X}}$ , and the constant 2 is optimal—for any non-zero  $\kappa$ ,  $\sup_{t \geq 0} \|C_\kappa(t) - I_{\mathbb{X}}\| = 2$ . In [11], Chojnacki has proved that an analogue of Schenninger and Zwart’s result holds for cosine functions in normed algebras and, remarkably, no assumption of continuity of the families is required. Also, he established a close link with the related  $0 - \frac{3}{2}$  law of Arendt [1]. Furthermore, in [7] and, independently but slightly later, in [14], it was proved that for  $\kappa \neq 0$ , the constant on the right-hand side of (4) may be enlarged to  $\frac{8}{3\sqrt{3}}$  but not to any greater number. See also, e.g., [10, 13, 14] for further results.

Remarkably, despite many efforts to make the proofs less sophisticated, the (optimal) results for cosine functions described above are proved by quite involved techniques, coming short of the simplicity of Wallen’s argument. It appears that while the Wallen formula works well in the case of discrete-parameter and continuous-parameter semigroups, an analogue for cosine functions that would work as efficiently remains unknown. In particular, Arendt’s formula [1], though simple and elegant, does not lead to the optimal result.

Nevertheless, as we would like to argue in this article, the potential in the Wallen formula has not been exploited completely yet. We will show, namely, that Wallen-type formulae for integrated semigroups and sine functions in Banach algebras (see Propositions 2.1, 3.1) may be used to prove isolation results for such functions, and the constants obtained in such analysis (in both cases equal to 1) are optimal. Our Section 2 is devoted to integrated semigroups and the main theorems there are Theorems 2.2 and 2.3; Section 3 contains results on sine functions.

**2. An isolated integrated semigroup.** A Banach algebra-valued function  $[0, \infty) \ni t \mapsto p(t) \in \mathbb{A}$  is said to be an integrated semigroup if it is integrable on any finite interval,

$$p(t)p(s) = \int_t^{s+t} p(r) \, dr - \int_0^s p(r) \, dr, \quad s, t \geq 0,$$

and  $p(0) = 0$ . A typical example of an integrated semigroup is

$$p_0(t) = 1_{[0,t)} \in L^1(\mathbb{R}^+), \tag{5}$$

where  $L^1(\mathbb{R}^+)$  is the convolution algebra of (equivalence classes of) absolutely integrable functions on  $[0, \infty)$ . In fact, as a result of Kisyński’s theorem (see, e.g., [20] or [3, 4, 9]), there is a one-to-one correspondence between Lipschitz continuous integrated semigroups (i.e. integrated semigroups such that there is an  $M$  such that  $\|p(t) - p(s)\|_{\mathbb{A}} \leq M|t - s|, t, s \geq 0$ ) and homomorphisms of the convolution algebra  $L^1(\mathbb{R}^+)$ . More specifically, each Lipschitz continuous integrated semigroup is of the form

$$p = H \circ p_0,$$

where  $H$  is the corresponding homomorphism of  $L^1(\mathbb{R}^+)$ ; the norm of  $H$  is the smallest Lipschitz constant for  $p$ .

If  $\mathbb{A}$  is a Banach algebra with unit  $u$ , then the map  $[0, \infty) \ni t \mapsto tu$  is arguably the simplest integrated semigroup. Our main goal is to prove that this integrated semigroup is isolated. We will prove two versions of this result: Theorems 2.2 and 2.3. The former theorem hinges crucially on the Wallen-type formula for integrated semigroups presented in Proposition 2.1.

Throughout the remainder of this section,  $\{p(t), t \geq 0\}$  is an integrated semigroup in a unital Banach algebra  $\mathbb{A}$  with unit  $u$ . Unless stated otherwise, the integrated semigroup need not be continuous in any sense.

**Proposition 2.1** (Wallen’s formula for integrated semigroups). *For  $s > t$ ,*

$$\begin{aligned} & (s^{-1}p(s) - u)(t^{-1}p(t) - u) \\ &= (st)^{-1} \int_s^{s+t} p(r) \, dr - (st)^{-1} \int_0^t p(r) \, dr - s^{-1}p(s) - t^{-1}p(t) + u. \end{aligned}$$

*Proof.* The left-hand side equals  $(st)^{-1}p(s)p(t) - s^{-1}p(s) - t^{-1}p(t) + u$ . By the integrated semigroup defining property, the first summand here equals

$$(st)^{-1} \left[ \int_t^{s+t} p(r) \, dr - \int_0^s p(r) \, dr \right] = (st)^{-1} \int_s^{s+t} p(r) \, dr - (st)^{-1} \int_0^t p(r) \, dr,$$

completing the proof. □

We are ready to present our main results:

**Theorem 2.2.** *Suppose that the following conditions are satisfied.*

- (a)  $\liminf_{s \rightarrow \infty} \|s^{-1}p(s) - u\|_{\mathbb{A}} =: \alpha < 1$ .

(b) For each  $t > 0$ ,  $\lim_{s \rightarrow \infty} s^{-1} \|t^{-1} \int_s^{s+t} p(r) dr - p(s)\|_{\mathbb{A}} = 0$ .

Then  $p(t) = tu, t \geq 0$ .

**Theorem 2.3.** Suppose  $p$  is continuous (but not necessarily Lipschitz continuous), and

$$\|t^{-1}p(t) - u\|_{\mathbb{A}} \leq \alpha, \quad t > 0, \tag{6}$$

holds for some  $\alpha < 1$ . Then  $p(t) = tu, t \geq 0$ .

Before continuing, let us comment on the relation between these two theorems. Certainly, condition (a) in Theorem 2.2 is similar, but weaker than the main assumption of Theorem 2.3. Condition (b) in Theorem 2.2 then corresponds to continuity assumption in Theorem 2.3. Indeed, it may be proved (compare Corollary 2.6, later on) that if the integrated semigroup is Lipschitz continuous, (b) is automatically satisfied. (Hence, for Lipschitz continuous integrated semigroups, (a) alone implies the thesis.) On the other hand, this condition is an analogue of the first condition in (3), and so in a sense restricts the growth of the integrated semigroup (or rather of the semigroup involved, if the latter exists, see Corollary 2.7).

*Proof of Theorem 2.2.* By assumption (b), there exists a sequence  $(s_n)_{n \geq 1}$  converging to infinity such that

$$\|s_n^{-1}p(s_n) - u\| \leq \frac{1 + \alpha}{2} < 1, \quad n \geq 1.$$

Replacing  $s$  by  $s_n$  in Proposition 2.1, we obtain the following estimate, valid for all  $t > 0$ , and almost all  $n \geq 1$ ,

$$\begin{aligned} \|t^{-1}p(t) - u\| &\leq \frac{1 + \alpha}{2} \|t^{-1}p(t) - u\| \\ &+ s_n^{-1} \left\| t^{-1} \int_{s_n}^{s_n+t} p(r) dr - p(s_n) \right\| + (s_n t)^{-1} \left\| \int_0^t p(r) dr \right\|. \end{aligned}$$

By assumption (b), the second summand on the right-hand side converges to 0, and so does the third. Thus, letting  $n \rightarrow \infty$  yields

$$\|t^{-1}p(t) - u\| \leq \frac{1 + \alpha}{2} \|t^{-1}p(t) - u\|.$$

Since  $\frac{1+\alpha}{2} < 1$ ,  $p(t) = tu, t > 0$ , as desired. □

*Proof of Theorem 2.3.* Assumption (6) implies

$$\|p(t)\|_{\mathbb{A}} \leq (1 + \alpha)t, \quad t \geq 0,$$

and so  $r_\lambda := \lambda \int_0^\infty e^{-\lambda t} p(t) dt, \lambda > 0$ , are well-defined.

Now, a calculation presented in [2, proof of Proposition 3.2.4] reveals that, since  $p$  is an integrated semigroup,  $r_\lambda, \lambda > 0$ , is a pseudo-resolvent, i.e. the Hilbert equation is satisfied:

$$(\lambda - \mu)r_\lambda r_\mu = r_\mu - r_\lambda, \quad \lambda, \mu > 0.$$

On the other hand, since  $\lambda \int_0^\infty e^{-\lambda t} dt = \lambda^2 \int_0^\infty e^{-\lambda t} t dt = 1$ ,

$$\begin{aligned} \|\lambda r_\lambda - u\| &= \left\| \lambda^2 \int_0^\infty e^{-\lambda t} p(t) dt - \lambda^2 \int_0^\infty e^{-\lambda t} t u dt \right\| \\ &\leq \lambda^2 \int_0^\infty e^{-\lambda t} \|p(t) - tu\| dt \\ &\leq \alpha \lambda^2 \int_0^\infty e^{-\lambda t} t dt = \alpha. \end{aligned}$$

Therefore,  $\lambda r_\lambda$  is invertible and so is  $r_\lambda, \lambda > 0$ .

Multiplying the Hilbert equation by  $r_\lambda^{-1}$  from the left, and by  $r_\mu^{-1}$  from the right yields  $\lambda u - \mu u = r_\lambda^{-1} - r_\mu^{-1}$ . It follows that

$$a := \lambda u - r_\lambda^{-1}$$

does not depend on  $\lambda > 0$ . Let

$$p^\sharp(t) = \int_0^t e^{sa} ds, \quad t \geq 0.$$

Then, for  $\lambda > \|a\|_\mathbb{A}$ ,

$$\lambda \int_0^\infty e^{-\lambda t} p^\sharp(t) dt = \int_0^\infty e^{-\lambda t} e^{ta} dt = (\lambda u - a)^{-1} = r_\lambda, \quad \lambda > 0,$$

showing that the Laplace transforms of the continuous functions  $p$  and  $p^\sharp$  coincide. Thus

$$p(t) = \int_0^t e^{sa} ds, \quad t \geq 0. \tag{7}$$

Finally, we recall that for a Banach algebra element  $b$ , the inequality  $\|b - u\|_\mathbb{A} < 1$  implies not only existence of  $b^{-1}$  but also the estimate  $\|b^{-1} - u\|_\mathbb{A} \leq \frac{\|b - u\|_\mathbb{A}}{1 - \|b - u\|_\mathbb{A}}$ . Therefore, for  $b = \lambda r_\lambda$ , we obtain (note  $b^{-1} = \lambda^{-1}(\lambda u - a)$ )

$$\left\| \frac{\lambda u - a}{\lambda} - u \right\|_\mathbb{A} \leq \frac{\|\lambda r_\lambda - u\|_\mathbb{A}}{1 - \|\lambda r_\lambda - u\|_\mathbb{A}} \leq \frac{\alpha}{1 - \alpha},$$

or

$$\|a\|_\mathbb{A} \leq \frac{\alpha}{1 - \alpha} \lambda.$$

Since this holds for all  $\lambda > 0$ , we must have  $\|a\|_\mathbb{A} = 0$ . Formula (7) now yields  $p(t) = tu, t \geq 0$ . □

**Example 2.4.** This example shows that the requirement  $\alpha < 1$  in Theorems 2.2 and 2.3 is optimal: for  $\alpha = 1$  the result is false. For, if the integrated semigroup  $p$  is given by  $p(t) = \int_0^t e^{-\lambda s} u \, ds$ , where  $\lambda > 0$ , then on the one hand,

$$\sup_{t>0} \|t^{-1}p(t) - u\|_{\mathbb{A}} \leq \sup_{t>0} \frac{1}{t} \int_0^t (1 - e^{-\lambda s}) \, ds \leq 1,$$

and on the other hand

$$\sup_{t>0} \|t^{-1}p(t) - u\|_{\mathbb{A}} \geq \lim_{t \rightarrow \infty} \left| \frac{1 - e^{-\lambda t}}{\lambda t} - 1 \right| = 1.$$

Therefore,  $\sup_{t>0} \|t^{-1}p(t) - u\|_{\mathbb{A}} = 1$  (and similarly  $\liminf_{t \rightarrow \infty} \|t^{-1}p(t) - u\|_{\mathbb{A}} = 1$ ) while  $p(t) \neq tu$ .

Before completing this section, we note - without proof - the following strong topology companion to Theorem 2.3, and two of its corollaries.

**Theorem 2.5.** *Let  $\{P(t), t \geq 0\}$  be an integrated semigroup of operators in the Banach algebra  $\mathcal{L}(\mathbb{X})$  of linear operators on a Banach space  $\mathbb{X}$ . Suppose that the following conditions are satisfied.*

- (a)  $\liminf_{s \rightarrow \infty} \|s^{-1}P(s) - I_{\mathbb{X}}\|_{\mathcal{L}(\mathbb{X})} =: \alpha < 1$ .
- (b) For each  $t > 0$ ,  $\lim_{s \rightarrow \infty} s^{-1}[t^{-1} \int_s^{s+t} P(r) \, dr - P(s)] = 0$  (strongly).

Then  $P(t) = tI_{\mathbb{X}}, t \geq 0$ .

**Corollary 2.6.** *Suppose  $\{P(t), t \geq 0\}$  is an integrated semigroup of operators. If condition (a) of Theorem 2.5 is satisfied, and for each  $x \in \mathbb{X}$  there exists  $M(x) > 0$  such that  $\|P(t)x - P(s)x\| \leq M(x)|t - s|$ , then  $P(t) = tI_{\mathbb{X}}, t \geq 0$ .*

*Proof.* (By the uniform boundedness principle, our assumption implies existence of  $M > 0$  such that  $\|P(t) - P(s)\|_{\mathcal{L}(\mathbb{X})} \leq M|t - s|$ , but we will not use this information.) It suffices to show that our assumptions imply (b) in Theorem 2.5. To this end we note that

$$\begin{aligned} \left\| t^{-1} \int_s^{s+t} P(r)x \, dr - P(s)x \right\| &= \left\| t^{-1} \int_s^{s+t} [P(r)x - P(s)x] \, dr \right\| \\ &\leq t^{-1} \int_s^{s+t} \|P(r)x - P(s)x\| \, dr \\ &\leq M(x)t^{-1} \int_s^{s+t} (r - s) \, dr = M(x)t/2. \end{aligned}$$

The last quantity does not depend on  $s$ , and so (b) holds. □

**Corollary 2.7.** *Suppose  $\{T(t), t \geq 0\}$  is a strongly measurable semigroup of operators in a Banach space  $\mathbb{X}$ , such that*

- (a)  $T(0) = I_{\mathbb{X}}$  and the (strong) integral  $\int_0^t T(s) \, ds$  exists for all  $t > 0$ ,
- (b)  $\liminf_{t \rightarrow \infty} \|t^{-1} \int_0^t T(s) \, ds - I_{\mathbb{X}}\|_{\mathcal{L}(\mathbb{X})} =: \alpha < 1$ ,

(c)  $\lim_{t \rightarrow \infty} t^{-1}T(t) = 0$  in the strong topology.

Then  $T(t) = I_{\mathbb{X}}, t \geq 0$ .

*Proof.* This theorem may be deduced from the following Wallen-type identity: for  $s, t > 0$ ,

$$\begin{aligned} & \left( s^{-1} \int_0^s T(r) \, dr - I_{\mathbb{X}} \right) (T(t) - I_{\mathbb{X}}) \\ &= s^{-1} \int_t^{t+s} T(r) \, dr - s^{-1} \int_0^s T(r) \, dr - T(t) + I_{\mathbb{X}}. \end{aligned}$$

However, we will deduce it from Theorem 2.5, by introducing

$$P(t) = \int_0^t T(s) \, ds, \quad t > 0; \tag{8}$$

these operators are well-defined by assumption (a). By assumption (b) in turn, condition (a) in Theorem 2.5 is fulfilled, and all we need to show is that condition (b) in the latter theorem holds.

Changing the order of integration yields

$$\begin{aligned} \int_s^{s+t} P(r) \, dr &= \int_s^{s+t} \int_0^r T(u) \, du \, dr = \int_0^{s+t} \int_{s \vee u}^{s+t} T(u) \, dr \, du \\ &= \int_0^s \int_s^{s+t} T(u) \, dr \, du + \int_s^{s+t} \int_u^{s+t} T(u) \, dr \, du \\ &= tP(s) + \int_s^{s+t} (s+t-u)T(u) \, du. \end{aligned}$$

Hence, it remains to show that for any  $y \in \mathbb{X}$ ,  $y(s) := s^{-1} \int_s^{s+t} (s+t-u)T(u)y \, du$ , converges to 0, as  $s \rightarrow \infty$ . However, given  $\epsilon > 0$ , for sufficiently large  $s$ ,  $\|T(u)y\| \leq \epsilon u$  for  $u > s$ . Therefore,  $\|y(s)\|$  does not exceed

$$\epsilon s^{-1} \int_s^{s+t} (s+t-u)u \, du = \epsilon \frac{3st^2 + t^3}{6s}.$$

We see that  $\limsup_{s \rightarrow \infty} \|y(s)\| \leq \frac{1}{2}\epsilon t^2$ , implying the claim, since  $\epsilon$  is arbitrary. □

**3. An isolated sine function.** A Banach algebra-valued function  $[0, \infty) \ni t \mapsto p(t) \in \mathbb{A}$  is said to be a sine function if it is integrable on any finite interval,

$$2p(t)p(s) = \int_0^{s+t} p(r) \, dr - \int_0^{|s-t|} p(r) \, dr, \quad s, t \geq 0,$$

and  $p(0) = 0$ . A typical example of a sine function is

$$p_0(t) = \frac{1}{2}1_{(-t,t)} \in L_e^1(\mathbb{R}), \tag{9}$$

where  $L_e^1(\mathbb{R})$  is the convolution algebra of even, absolutely integrable functions on  $\mathbb{R}$ . In fact (see [3, Section 5]), each Lipschitz continuous sine function is of the form

$$p = H \circ p_0$$

where  $H$  is a related homomorphism of  $L_e^1(\mathbb{R})$ .

We start with the following Wallen-type formula for sine functions, while omitting its proof since it is very similar to the proof of Proposition 2.1.

**Proposition 3.1** (Wallen’s formula for sines). *For  $s > t$ ,*

$$(s^{-1}p(s) - u)(t^{-1}p(t) - u) = s^{-1} \left[ \frac{1}{2t} \int_{s-t}^{s+t} p(r) dr - p(s) \right] - [t^{-1}p(t) - u].$$

This proposition combined with the argument presented in Theorem 2.2 leads to the following result.

**Theorem 3.2.** *Let  $\{p(t), t \geq 0\}$  be a sine function in a unital Banach algebra  $\mathbb{A}$  with unit  $u$ . Suppose that the following conditions are satisfied:*

- (a)  $\liminf_{s \rightarrow \infty} \|s^{-1}p(s) - u\|_{\mathbb{A}} = \alpha$ , where  $\alpha < 1$ .
- (b) For each  $t > 0$ ,  $\lim_{s \rightarrow \infty} s^{-1} \left\| \frac{1}{2t} \int_{s-t}^{s+t} p(r) dr - p(s) \right\|_{\mathbb{A}} = 0$ .

*Then  $p(t) = tu, t \geq 0$ .*

**Remark 3.3.** If  $\mathbb{A}$  possesses non-trivial idempotents of norm 1 (in particular, if  $\mathbb{A}$  is the Banach algebra of operators on a Banach space of dimension  $\geq 2$ ), then the constant in condition (a) in Theorem 3.2 is optimal in the following sense. Let  $j \in \mathbb{A}$  be a non-trivial idempotent with norm 1. Then,  $i = u - j$  is also an idempotent, and  $p(t) = ti$  is a sine function satisfying (b). Moreover, for any  $t > 0$ ,  $t^{-1}p(t) - u = j$  so that  $\|t^{-1}p(t) - u\| = 1$ . Therefore (a) holds with  $\alpha = 1$ , and yet  $p(t) = tu$  for no  $t > 0$ .

Theorem 3.2 combined with the remark given above should be compared with the main result of [22] where it was proved that for a cosine function  $\{C(t), t \in \mathbb{R}\}$  on a Banach space  $\mathbb{X}$ , condition  $\sup_{t \geq 0} \|C(t) - I_{\mathbb{X}}\| < 2$  implies  $C(t) = I_{\mathbb{X}}$ , and the constant 2 is optimal. The fact that, as our results show, for sine functions the constant involved is smaller (1 as opposed to 2) agrees with the intuition that there are many more sine functions than there are cosine functions. The same intuition is supported by Kéyantuo’s paper [19, Section 5], where it is shown that on  $L^1(\mathbb{R}^d)$  the Laplacian generates a cosine function in one dimension, and merely a sine function in dimensions two and three, see also [16] cited in [19] as Theorem 5.1. (This is a reflection of the classical result of Hadamard, see [15, Section 2.4.1 c].) On  $L^2(\mathbb{R}^d)$  one has always a cosine function in any dimension. These results show that at least outside of a Hilbert space there are few cosine functions only, and more sine functions, indeed. (I am grateful to the referee for pointing these last references to me.)



In the case of sine functions in the algebra of operators, condition (b) in Theorem 3.2 may be a bit relaxed, as in Theorem 2.5. We omit the proof of the following result.

**Theorem 3.4.** *Let  $\{P(t), t \geq 0\}$  be a sine function of operators in the Banach algebra  $\mathcal{L}(\mathbb{X})$  of linear operators on a Banach space  $\mathbb{X}$ . Suppose that the following conditions are satisfied.*

- (a)  $\liminf_{s \rightarrow \infty} \|s^{-1}P(s) - I_{\mathbb{X}}\|_{\mathcal{L}(\mathbb{X})} < 1$ .
- (b) For each  $t > 0$ ,  $\lim_{s \rightarrow \infty} s^{-1}[\frac{1}{2t} \int_{s-t}^{s+t} P(r) dr - P(s)] = 0$  (strongly).

Then  $P(t) = tI_{\mathbb{X}}, t \geq 0$ .

**Remark 3.5.** Condition (b) is automatically satisfied if for each  $x \in \mathbb{X}$  there is  $M(x) > 0$  such that  $\|P(t)x - P(s)x\| \leq M(x)|s - t|$ , for  $s, t \geq 0$ .

We refrain from stating sine function analogues of Theorem 2.3 and Corollary 2.7.

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