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# Continuous rational maps into the unit 2-sphere

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**Abstract.** Investigated are continuous rational maps from a compact nonsingular real algebraic set into unit spheres. Special attention is devoted to such maps with values in the unit 2-sphere.

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**1. Introduction.** Let $X$ and $Y$ be nonsingular real algebraic sets. A map $f\colon X \to Y$ is said to be *continuous rational* if it is continuous and there exists a regular map $\varphi\colon U \to Y$, defined on a Zariski open and dense subset $U$ of $X$, with $f|_U = \varphi$. Let $P(f)$ denote the smallest algebraic subset of $X$ for which the restriction map $f|_{X\backslash P(f)}\colon X\backslash P(f) \to Y$ is regular. Thus $P(f)$ is the indeterminacy locus of the rational map from $X$ into $Y$ determined by $\varphi$. Maps with $f(P(f)) \neq Y$ will be called *nice*. There exist continuous rational maps which are not nice, cf. [6, Example 2.2]. Continuous rational maps form a natural intermediate class between regular and continuous semi-algebraic maps, with many specific properties, cf. [3,5–7].

In [6,7], continuous rational maps with values in the unit $p$-sphere

$$\mathbb{S}^p = \{(u_1, \dots, u_{p+1}) \in \mathbb{R}^{p+1} \mid u_1^2 + \cdots + u_{p+1}^2 = 1\}$$

are investigated. They behave like regular maps in problems involving homotopy or approximation, provided that $p = 1$, cf. [6, Corollary 1.4]. However, one encounters new phenomena for $p \geq 2$, cf. [6, p. 518] and [7, Example 1.8]. The main results of the present paper, Theorems 1.2 and 1.4, concern approximation of continuous maps from $X$ into $\mathbb{S}^2$ by continuous rational maps. Here approximation refers to the compact open topology.

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First some preparation is necessary. Let $M$ be a smooth (of class $\mathcal{C}^\infty$) manifold, and let $N$ be a smooth submanifold of $M$ of codimension $p$. By convention, submanifolds are assumed to be closed subsets of the ambient manifold. Assume that the normal bundle of $N$ in $M$ is, oriented, and denote by $\tau_N^M$ the Thom class of $N$ in the cohomology group $H^p(M, M\backslash N; \mathbb{Z})$, cf. [8, p. 118]. The image of $\tau_N^M$ by the restriction homomorphism $H^p(M, M\backslash N; \mathbb{Z}) \to H^p(M; \mathbb{Z})$, induced by the inclusion map $M \hookrightarrow (M, M\backslash N)$, will be denoted by $[N]^M$ and called the *cohomology class represented by* $N$. If $M$ is oriented and $N$ is endowed with the compatible orientation, then $[N]^M$ is up to sign Poincaré dual to the homology class in $H_*(M; \mathbb{Z})$ represented by $N$, cf. [8, p. 136].

Let $P$ be a smooth manifold, and let $Q$ be a smooth submanifold of $P$. A continuous map $f\colon M \to P$ is said to be *transverse* to $Q$ if it is smooth in an open neighborhood $U$ of $f^{-1}(Q)$ in $M$ and the restriction map $f|_U\colon U \to P$ is transverse to $Q$ in the usual sense. In that case, if the normal bundle of $Q$ in $P$ is oriented and the normal bundle of the smooth submanifold $N := f^{-1}(Q)$ of $M$ is endowed with the orientation induced by $f$, then $\tau_N^M = f^*(\tau_Q^P)$, where $f$ is regarded as a map from $(M, M\backslash N)$ into $(P, P\backslash Q)$ (this follows from [4, p. 117, Theorem 6.7]). In particular, $[N]^M = f^*([Q]^P)$.

The unit sphere $\mathbb{S}^p$ is considered with a fixed orientation. Hence any point $z$ in $\mathbb{S}^p$ can be regarded as a smooth submanifold of $\mathbb{S}^p$ with oriented trivial normal bundle. Let $\sigma_p$ denote the generator of the cohomology group $H^p(\mathbb{S}^p; \mathbb{Z}) \cong \mathbb{Z}$ determined by the orientation of $\mathbb{S}^p$. In other words, $\sigma_p = [z]^{\mathbb{S}^p}$.

**Assumption.** In the rest of this section, the algebraic set $X$ is assumed to be compact.

A cohomology class $u$ in $H^p(X; \mathbb{Z})$ is said to be *adapted* if there exists an algebraic codimension $p$ subset $V$ of $X$ such that $\operatorname{Reg}(V)$ is a compact smooth submanifold of $X$ with trivial normal bundle and $u = [\operatorname{Reg}(V)]^X$ when the normal bundle of $\operatorname{Reg}(V)$ is suitably oriented. Here $\operatorname{Reg}(V)$ stands for the set of nonsingular points of $V$.

**Proposition 1.1.** *If $f\colon X \to \mathbb{S}^p$ is a nice continuous rational map, then the cohomology class $f^*(\sigma_p)$ in $H^p(X; \mathbb{Z})$ is adapted.*

*Proof.* Since $f(P(f))$ is a proper compact subset of $\mathbb{S}^p$, it follows from Sard's theorem that the regular map $f|_{X\backslash P(f)}\colon X\backslash P(f) \to \mathbb{S}^p$ is transverse to some point $z$ in $\mathbb{S}^p\backslash f(P(f))$. Hence $N := f^{-1}(z)$ is a nonsingular Zariski closed subset of $X\backslash P(f)$. If $V$ is the Zariski closure of $N$ in $X$, then $V\backslash N$ is a Zariski closed subset of $X$ contained in $P(f)$, with $\dim(V\backslash N) < \dim V$. Thus $N = \operatorname{Reg}(V)$, the set $N$ being compact. Since the continuous map $f\colon X \to \mathbb{S}^p$ is transverse to $z$, one has $f^*(\sigma_p) = [\operatorname{Reg}(V)]^X$, provided that $\operatorname{Reg}(V)$ is endowed with the orientation induced by $f|_{X\backslash P(f)}$. □

Denote by $A^p(X; \mathbb{Z})$ the subgroup of $H^p(X; \mathbb{Z})$ generated by all adapted cohomology classes. It is an open problem whether or not for a continuous rational map $f\colon X \to \mathbb{S}^p$, the cohomology class $f^*(\sigma_p)$ is in $A^p(X; \mathbb{Z})$. This problem is of particular interest for $p = 2$ in view of the following result.

**Theorem 1.2.** *If $h\colon X \to \mathbb{S}^2$ is a continuous map such that the cohomology class $h^*(\sigma_2)$ is in $A^2(X;\mathbb{Z})$, then $h$ can be approximated by continuous rational maps.*

The proof is postponed until Sect. 2.

**Corollary 1.3.** *Let $C_1, \ldots, C_n$ be compact connected nonsingular real algebraic sets of dimension 1. Then any continuous map from $C_1 \times \cdots \times C_n$ into $\mathbb{S}^2$ can be approximated by continuous rational maps.*

*Proof.* It suffices to observe that

$$A^2(C_1 \times \cdots \times C_n;\mathbb{Z}) = H^2(C_1 \times \cdots C_n;\mathbb{Z})$$

and apply Theorem 1.2. □

Denote by $A^p(X;\mathbb{Z}/2)$ the image of $A^p(X;\mathbb{Z})$ by the reduction modulo 2 homomorphism

$$H^p(X;\mathbb{Z}) \to H^p(X;\mathbb{Z}/2).$$

The reduction modulo 2 of $\sigma_p$, denoted by $\bar{\sigma}_p$, is a generator of the cohomology group $H^p(\mathbb{S}^p;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

**Theorem 1.4.** *Assume that the cohomology group $H^2(X;\mathbb{Z})$ is generated by the cohomology classes of the form $[N]^X$, where $N$ is a codimension 2 compact smooth submanifold of $X$ with oriented trivial normal bundle. If $h\colon X \to \mathbb{S}^2$ is a continuous map such that the cohomology class $h^*(\bar{\sigma}_2)$ is in $A^2(X;\mathbb{Z}/2)$, then $h$ can be approximated by continuous rational maps.*

The proof will be given in the next section. It is not clear whether Theorems 1.2 and 1.4 can be extended to maps with values in $\mathbb{S}^p$ for $p \geq 3$.

**2. Proofs.** For any topological space $T$, denote by $\varepsilon^k_T(\mathbb{C})$ the standard trivial $\mathbb{C}$-vector bundle on $T$ with total space $T \times \mathbb{C}^k$.

A topological $\mathbb{C}$-vector bundle $\xi$ on a compact nonsingular real algebraic set $X$ is said to *admit a rational structure* if there exist a topological $\mathbb{C}$-vector subbundle $\eta$ of $\varepsilon^k_X(\mathbb{C})$, for some $k$, and a Zariski open and dense subset $U$ of $X$ such that $\xi$ is isomorphic to $\eta$ and the restriction $\eta|_U$ is an algebraic $\mathbb{C}$-vector subbundle of $\varepsilon^k_U(\mathbb{C})$, cf. [6, Definition 3.2].

Identify $\mathbb{S}^2$ with the complex projective line $\mathbb{P}^1(\mathbb{C})$, and denote by $\gamma_2$ the $\mathbb{C}$-line bundle on $\mathbb{S}^2$ corresponding to the universal $\mathbb{C}$-line bundle $\lambda$ on $\mathbb{P}^1(\mathbb{C})$. Explicitly, let $a = (0,1)$ be a point in $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$, and let $\rho\colon \mathbb{S}^2 \backslash \{a\} \to \mathbb{C}$ be the stereographic projection. Then $\alpha\colon \mathbb{S}^2 \to \mathbb{P}^1(\mathbb{C})$,

$$\alpha(x) = \begin{cases} (\rho(x) : 1) & \text{for } x \text{ in } \mathbb{S}^2 \backslash \{a\} \\ (1:0) & \text{for } x = a, \end{cases}$$

is a smooth diffeomorphism and $\gamma_2 := \alpha^*\lambda$. In particular, $\gamma_2$ is an algebraic $\mathbb{C}$-vector subbundle of $\varepsilon^2_{\mathbb{S}^2}(\mathbb{C})$. The first Chern class $c_1(\gamma_2)$ of $\gamma_2$ is a generator of the cohomology group $H^2(\mathbb{S}^2;\mathbb{Z})$. It can be assumed that $\mathbb{S}^2$ is oriented in such a way that $\sigma_2 = c_1(\gamma_2)$.

*Proof of Theorem* 1.2. If $u$ is an adapted cohomology class in $H^2(X;\mathbb{Z})$, then so is $-u$. Since the cohomology class $h^*(\sigma_2)$ is in $A^2(X;\mathbb{Z})$, one can find nonnegative integers $k_1,\dots,k_r$ and adapted cohomology classes $u_1,\dots,u_r$ in $H^2(X;\mathbb{Z})$ such that

$$h^*(\sigma_2) = k_1u_1 + \cdots + k_ru_r.$$

According to [6, Theorem 2.5], for each integer $i$ satisfying $1 \le i \le r$, there exists a continuous rational map $f_i\colon X \to \mathbb{S}^2$ with $u_i = f_i^*(\sigma_2)$. The $\mathbb{C}$-line bundle $\xi_i := f_i^*\gamma_2$ on $X$ admits a rational structure. Consequently, the $\mathbb{C}$-line bundle

$$\xi := \xi_1^{\otimes k_1} \otimes \cdots \otimes \xi_r^{\otimes k_r}$$

on $X$ admits a rational structure, cf. [6, Proposition 3.3]. One has

$$c_1(\xi_i) = f_i^*(c_1(\gamma_2)) = f_i^*(\sigma_2) = u_i,$$

and hence $c_1(\xi) = h^*(\sigma_2)$. Furthermore,

$$h^*(\sigma_2) = h^*(c_1(\gamma_2)) = c_1(h^*\gamma_2).$$

Thus $c_1(\xi) = c_1(h^*\gamma_2)$, which implies that the $\mathbb{C}$-vector bundles $\xi$ and $h^*\gamma_2$ are isomorphic. It follows that $h^*\gamma_2$ admits a rational structure and hence according to [6, Corollary 3.8], the map $h$ can be approximated by continuous rational maps. □

*Proof of Theorem* 1.4. Since the cohomology class $h^*(\bar{\sigma}_2)$ is in $A^2(X;\mathbb{Z}/2)$, by the universal coefficient theorem, the cohomology class $h^*(\sigma_2)$ can be expressed as $h^*(\sigma_2) = u+2v$, where $u$ is in $A^2(X;\mathbb{Z})$ and $v$ is in $H^2(X;\mathbb{Z})$. By assumption, the cohomology class $v$ is a linear combination with integer coefficients of cohomology classes of the form $[N]^X$, where $N$ is a codimension 2 compact smooth submanifold of $X$ with oriented trivial normal bundle. According to Theorem 1.2, it suffices to prove that the cohomology class $2[N]^X$ is adapted.

Since the normal bundle of $N$ in $X$ is trivial, there exists an isotopic copy $N'$ of $N$ such that $N \cap N' = \varnothing$ and $N \cup N'$ is the boundary of a compact smooth manifold with boundary, embedded in $X$ with trivial normal bundle. Hence $N \cup N'$ can be represented as $N \cup N' = f^{-1}(0)$, where $f\colon X \to \mathbb{R}^2$ is a smooth map transverse to 0 in $\mathbb{R}^2$ (cf. for example [2, Theorem 1.12]). By the Weierstrass approximation theorem, there exists a regular map $g\colon X \to \mathbb{R}^2$ arbitrarily close to $f$ in the $\mathcal{C}^\infty$ topology. If $g$ is sufficiently close to $f$, then $g$ is transverse to 0 and the smooth submanifold $N \cup N'$ is isotopic in $X$ to the nonsingular algebraic subset $Z := g^{-1}(0)$, cf. [1, p. 51]. By construction, $2[N]^X = [Z]^X$, provided that the normal bundle of $Z$ in $X$ is suitably oriented. Hence the cohomology class $2[N]^X$ is adapted, as required. □

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