

Gaps in the space of skeletal signatures

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Abstract. Skeletal signatures were introduced in Anderson and Wootton (see [1]) as a tool to describe the space of all signatures with which a group can act on a surface of genus $\sigma \geq 2$. In the present paper, we provide an essentially complete description of the regular gaps that appear in the space of skeletal signatures, together with proofs of those parts of the conjectures posed in Anderson and Wootton (see [1]) related to these regular gaps.

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1. Introduction and preliminaries. In [1], the authors introduced the notion of a *skeletal signature* for the action of a finite group on a closed Riemann surface of genus $\sigma \geq 2$, together with the resultant space \mathcal{K}_σ of skeletal signatures, as tools for understanding the space of all signatures of all actions of finite groups on closed Riemann surfaces of a fixed genus $\sigma \geq 2$. The study of automorphism groups of Riemann surfaces and the signatures of such actions has been the focus of much study since the early days of the development of Riemann surfaces, and remains a vibrant area of activity. We note that most of the current activity focuses on properties of particular groups or families of groups, rather than on the genus of the surface being acted upon.

To set notation, let X be a closed Riemann surface of genus $\sigma \geq 2$, and let G be a (necessarily finite) group acting on X by conformal homeomorphisms. Let $(h; n_1, \dots, n_r)$ be the *signature* of the action of G on X , by which we mean that the quotient surface orbifold X/G has genus h and the orbifold covering $X \rightarrow X/G$ is branched over r points with orders n_1, \dots, n_r .

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To the signature $(h; n_1, \dots, n_r)$, we associate the *skeletal signature* (h, r) , which is the ordered pair consisting of the genus of the quotient surface orbifold and the number of branch points. Define \mathcal{K}_σ to be the set of all skeletal signatures associated to all groups acting on all closed Riemann surfaces of a fixed genus $\sigma \geq 2$.

The main reason for the interest in these skeletal signatures is that they provide a reasonable summary of the information contained in the corresponding full signatures, and skeletal signatures are significantly more tractable than full signatures. One demonstration of this is contained below, in our description of the regular gaps that appear in the space of skeletal signatures for all genera. To our knowledge, no corresponding result is known for full signatures.

The main significance of these gaps is that if the point (h_0, r_0) cannot be realized as a skeletal signature for genus σ , then we are able to exclude a large number of potential signatures. Specifically, there cannot exist any group G acting on any closed Riemann surface X of genus σ for which the action of G on X has signature $(h_0; n_1, \dots, n_{r_0})$ for any n_1, \dots, n_{r_0} .

Referring the reader to Fig. 1, one curious and notable feature of \mathcal{K}_σ is the existence of regular gaps in which no skeletal signatures appear. While we present only \mathcal{K}_{48} in this figure, such gaps appear in all genera. The data from which this figure was constructed arises from the work of Breuer [2] and his classification of all signatures arising from actions of groups on all closed Riemann surfaces of genera $2 \leq \sigma \leq 48$. This data is available via the **genus** package in GAP [4]. Indeed, it was from our attempts to visualize Breuer's data that led us to consider skeletal signatures initially.

While we focus primarily on the gaps that arise for all genera, we will also discuss in Section 4 some results for sporadic points, which are those points in the (h, r) -plane which arise as skeletal signatures for infinitely many genera and which do not arise as the skeletal signatures for infinitely many genera as well. In particular, in Theorem 4.1 we resolve Conjecture 3.15 from [1] by showing that all skeletal signatures $(h, 1)$ are sporadic for all $h \geq 2$.

The main technical result used in the descriptions of the gaps and of the proofs of the relevant parts of the Conjectures from [1] is the Riemann–Hurwitz formula. Remarkably, although satisfaction of the Riemann–Hurwitz formula is not sufficient to guarantee the existence of a group action yielding a given signature, current evidence suggests that satisfaction of the Riemann–Hurwitz formula is nearly always sufficient to guarantee the existence of a group action yielding a given skeletal signature. Work on this question is on-going. To close this section, let X be a compact Riemann surface of genus $\sigma \geq 2$, let G be a group acting on X by conformal homeomorphisms, and suppose that the signature of the action of G on X is $(h; n_1, \dots, n_r)$. The Riemann–Hurwitz formula then states that

$$\sigma - 1 = |G| \left(h - 1 + \frac{r}{2} - \frac{1}{2} \sum_{j=1}^r \frac{1}{n_j} \right). \quad (1)$$

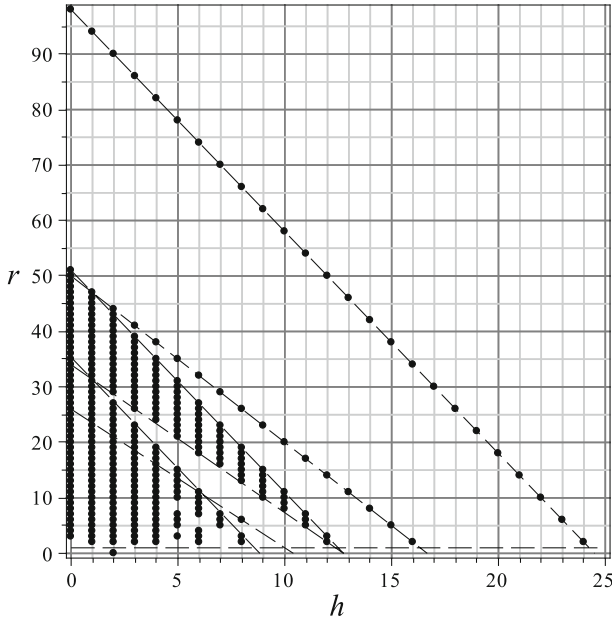


FIGURE 1. The (h, r) -plane for genus $\sigma = 48$. The top line is the *hyperelliptic line* $L_{48}(2, 1)$, containing the skeletal signatures for all C_2 actions on X . The next two lines are the lines L_3^{48} and below it U_4^{48} , bounding the gap $\mathcal{G}_{48}(3, 4)$. Below U_4^{48} is L_4^{48} , which together with U_4^{48} bounds the triangular region $\mathcal{P}_{48}(4)$, which here is completely filled by skeletal signatures. Next is the line U_6^{48} , which together with L_4^{48} bounds the gap $\mathcal{G}_{48}(4, 6)$, into which intrudes the line $L^{48}(5) = L_{48}(5, 1)$ bringing into $\mathcal{G}_{48}(4, 6)$ the single skeletal signature $(8, 6)$. We also include the line $r = 1$, to illustrate that there are no skeletal signatures of the form $(h, 1)$ in genus 48, as per the proof of Theorem 4.1. This figure was produced using MAPLE 15 and the data from the genus package in GAP [4]

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2. Triangles of skeletal signatures in the (h, r) -plane. Fix a genus $\sigma \geq 2$. In this section, we describe the regular gaps in the (h, r) -plane, showing that their existence is forced by the Riemann–Hurwitz formula. By a *gap*, we mean a subset of the (h, r) -plane, described in terms of σ , which contains no points of the space \mathcal{K}_σ of skeletal signatures for genus σ .

We begin with the following construction. Take an integer $N \geq 2$ and consider the Riemann–Hurwitz formula for the action of any group G of order N on a closed Riemann surface X of genus $\sigma \geq 2$, where G acts on X with

signature $(h; n_1, \dots, n_r)$. In particular, we do not pay particular attention to any aspect of G other than its order $|G| = N$.

The ramifications orders n_j of the covering $X \rightarrow X/G$ trivially satisfy the inequalities $2 \leq n_j \leq N = |G|$ for all $1 \leq j \leq r$. (Though we do not need to do so here, we note that it is straightforward to refine this estimate for a specific group G , for instance by bounding the ramification orders by the minimum and maximum orders of non-trivial elements of G .) Substituting this into Eq. (1) and rearranging, we see for a fixed h that

$$\frac{2(\sigma - 1 + N(1 - h))}{N - 1} \leq r \leq \frac{4(\sigma - 1 + N(1 - h))}{N}. \tag{2}$$

We interpret these inequalities as giving, for the fixed $\sigma \geq 2$ and for the given $h \geq 0$, the r -coordinates for all of the skeletal signatures (h, r) that might arise from the action of some group of order N on X . That is, we are covering the space \mathcal{K}_σ of skeletal signatures by these vertical slices.

Equation (2) yields a pair of lines. The left-hand inequality yields the line

$$L_N^\sigma = \{2\sigma - 2 + 2N = (N - 1)r + 2Nh\},$$

and the right-hand side yields the line

$$U_N^\sigma = \{4(N + \sigma - 1) = Nr + 4Nh\}.$$

Calculating, we see that L_N^σ and U_N^σ intersect at the point $(1 + \frac{\sigma-1}{N}, 0)$.

Define $\mathcal{P}_\sigma(N)$ to be the closed triangular region in the (h, r) -plane bounded by the lines L_N^σ and U_N^σ ; that is, we define $\mathcal{P}_\sigma(N)$ to be all those points (h, r) for which

$$\frac{2(\sigma - 1 + N(1 - h))}{N - 1} \leq r \leq \frac{4(\sigma - 1 + N(1 - h))}{N}$$

for each $0 \leq h \leq 1 + \frac{\sigma-1}{N}$. We refer to L_N^σ as the *lower line* for $\mathcal{P}_\sigma(N)$ and to U_N^σ as the *upper line* for $\mathcal{P}_\sigma(N)$, for the obvious reason that $\mathcal{P}_\sigma(N)$ lies above L_N^σ and below U_N^σ . We note that

$$\mathcal{K}_\sigma \subset \bigcup_{N \geq 2} \mathcal{P}_\sigma(N).$$

For $N = 2$, observe that L_2^σ and U_2^σ coincide, as both are the line $\{2(\sigma - 1) = r + 4h\}$. This is not a surprise, given the method we used to generate the lines. In fact, this is a welcome observation, as we know from our previous work that all of the skeletal signatures arising from C_2 -actions on surfaces of genus σ lie on this line.

This observation holds more generally, for any group of the form $(C_p)^n$ for a prime $p \geq 2$ and an integer $n \geq 1$. As an illustration of how we might refine the argument given above, we give the proof of this observation.

Lemma 2.1. *For a prime $p \geq 2$ and an integer $n \geq 1$, suppose that $(C_p)^n$ acts by conformal homeomorphisms on the closed Riemann surface X of genus $\sigma \geq 2$. The skeletal signature (h, r) of the action of $(C_p)^n$ on X with signature $(h; n_1, \dots, n_r)$ then lies on the line*

$$L_\sigma(p, n) = \{2p^n h + (p - 1)p^{n-1} r = 2p^n - 2 + 2\sigma\}.$$

In particular, the triangle $\mathcal{P}_\sigma((C_p)^n)$ collapses to the line $L_\sigma(p, n)$.

Proof. The key to this observation is that all of the non-trivial elements of $(C_p)^n$ have order p , and so all of the ramification orders satisfy $n_j = p$. Equation (1) then becomes

$$\sigma - 1 = p^n \left(h - 1 + \frac{r(p - 1)}{2p} \right),$$

which is the line

$$L_\sigma(p, n) = \{2p^n h + (p - 1)p^{n-1}r = 2p^n - 2 + 2\sigma\}.$$

□

To understand how gaps arise in the (h, r) -plane for a fixed genus $\sigma \geq 2$, we need to understand how the $\mathcal{P}_\sigma(N)$ behave relative to one another as N varies. We start by noting that the upper lines U_N^σ are all parallel to one another over all values of N , as all have slope -4 .

Also, for a fixed $\sigma \geq 2$, the lower lines L_N^σ are almost parallel as N varies, as the slope of L_N^σ is $\frac{N-1}{2N} = \frac{1}{2} - \frac{1}{2N}$. Interestingly, the lower lines L_N^σ all pass through the point $(h, r) = (\sigma, 2 - 2\sigma) = (\sigma, \chi(X))$; since $\sigma \geq 2$, this immediately yields that the L_N^σ are disjoint in the (h, r) -plane as N varies. However, we have not yet seen any particular consequence of the fact that the L_N^σ all pass through the same point.

Moreover, as N increases, we have that the U_N^σ and the L_N^σ move to the left. Combining these facts, we have the following observation regarding the behaviours of $\mathcal{P}_\sigma(M)$ and $\mathcal{P}_\sigma(N)$ for $M < N$.

Lemma 2.2. Fix a genus $\sigma \geq 2$, and let $3 \leq M < N$ be integers.

1. $\mathcal{P}_\sigma(M)$ lies strictly above the lower line L_N^σ of $\mathcal{P}_\sigma(N)$.
2. $\mathcal{P}_\sigma(N)$ lies strictly below the upper line U_M^σ of $\mathcal{P}_\sigma(M)$.

Proof. The proof of both parts of this Lemma follows immediately from the analysis that led to the definition of $\mathcal{P}_\sigma(N)$.

1. The lower lines of $\mathcal{P}_\sigma(M)$ and $\mathcal{P}_\sigma(N)$, while not disjoint, intersect at the point $(h, r) = (\sigma, 2 - 2\sigma)$, which lies below the h -axis in the (h, r) -plane. This forces $\mathcal{P}_\sigma(M)$ to lie in the complement of the lower line of $\mathcal{P}_\sigma(N)$. The statement then follows from the observation that the lower line for $\mathcal{P}_\sigma(M)$ lies above the lower line for $\mathcal{P}_\sigma(N)$.
2. The upper lines of $\mathcal{P}_\sigma(M)$ and $\mathcal{P}_\sigma(N)$ are parallel, as both have slope -4 , and hence are disjoint. This forces $\mathcal{P}_\sigma(N)$ to lie in the complement of the upper line of $\mathcal{P}_\sigma(M)$. The statement then follows from the observation that the upper line for $\mathcal{P}_\sigma(N)$ lies below the upper line for $\mathcal{P}_\sigma(M)$. □

As a consequence of this, we see that the $\mathcal{P}_\sigma(N)$ overlap in a saw-tooth pattern, see Fig. 1. With this in mind, define $\mathcal{G}_\sigma(N, N + 1)$ to be the open triangular region in the (h, r) -plane bounded by the lines U_{N+1}^σ and L_N^σ , lying to the right of their point $Q_{N, N+1}^\sigma$ of intersection, which we calculate to be

$$Q_{N,N+1}^\sigma = \left(\frac{(N-1)^2 + \sigma(N-3)}{(N-2)(N+1)}, \frac{4(\sigma-1)}{(N-2)(N+1)} \right).$$

Similarly, define $\mathcal{G}_\sigma(N, N+2)$ to be the open triangular region in the (h, r) -plane bounded by the lines U_{N+2}^σ and L_N^σ , lying to the right of their point $Q_{N,N+2}^\sigma$ of intersection, which we calculate to be

$$Q_{N,N+2}^\sigma = \left(\frac{N^2 - N + \sigma(N-4)}{N^2 - 4}, \frac{8(\sigma-1)}{N^2 - 4} \right).$$

Theorem 2.3. *Fix a genus $\sigma \geq 2$, and let $N \geq 3$ be an integer. In the case that $N+1$ is not prime, the region $\mathcal{G}_\sigma(N, N+1)$ is a gap in the (h, r) -plane, so that the intersection $\mathcal{G}_\sigma(N, N+1) \cap \mathcal{K}_\sigma$ is empty.*

In the case that $N+1$ is prime, the region $\mathcal{G}_\sigma(N, N+2)$ is a gap in the (h, r) -plane, with the exception of the points within this region lying on the line $L_\sigma(N+1, 1)$ corresponding to the cyclic group of order $N+1$.

Proof. The proof follows immediately by considering the relationship between the three triangular regions $\mathcal{P}_\sigma(N)$, $\mathcal{P}_\sigma(N+1)$ and $\mathcal{P}_\sigma(N+2)$. For both statements, the equations of the point of intersection follows immediately from the equations of the relevant lines, while the existence of the gaps follows from Lemma 2.2.

For the second statement, the only additional point is that the middle triangle $\mathcal{P}_\sigma(N+1)$ collapses to the line $L_\sigma(N+1, 1)$, which passes through the region $\mathcal{G}_\sigma(N, N+2)$ and so determines points of \mathcal{K}_σ that lie in $\mathcal{G}_\sigma(N, N+2)$. \square

The proof of Theorem 2.3 makes use of the region bounded by the lower line of $\mathcal{P}_\sigma(N)$ and the upper line of $\mathcal{P}_\sigma(N+1)$, which intersect at the point $Q_{N,N+1}^\sigma$, and so determine a pair of regions. The gap consists of those points strictly between the two lines and to the right of the point of intersection, while the points between the two lines and to the left of the point of intersection are skeletal signatures (h, r) that correspond signatures for both order N and order $N+1$ actions on some closed Riemann surface of genus σ .

We close this section by expanding the discussion given in the proof of Proposition 3.7 of [1]. When N divides $\sigma-1$, the point $(\frac{\sigma-1}{N} + 1, 0)$ at which $\mathcal{P}_\sigma(N)$ meets $\{r=0\}$ is always the skeletal signature of some group action of order N on a surface of genus σ (indeed, a cyclic group action, see [6]). In fact, the point $(\frac{\sigma-1}{N} + 1, 0)$ is a skeletal signature for **all** groups of order N provided $(\frac{\sigma-1}{N} + 1) \geq n+1$ where n is the largest power of any prime dividing N . This follows using an argument similar to Hurwitz’s original argument that there exists a group action of every finite group on some surface, see [2, Corollary 3.15] and from the fact that the cardinality of a generating set for such a group is bounded by $n+1$, see [5].

3. Gaps in the (h, r) -plane. The purpose of this section is to consider the parts of the Conjectures from [1] related to the existence of regular gaps in the (h, r) -plane for various values of σ . We recall some notation. Let $[x]$ be the result of rounding x to the nearest integer. (This is a minor change from the notation in [1], where we originally used (x) .) A *persistently missing point*

(h_0, r_0) in the (h, r) -plane is a point satisfying $(h_0, r_0) \notin \mathcal{K}_\sigma$ for all $\sigma \geq \sigma_0$ for some constant σ_0 .

[1, Conjecture 3.12] *For $\sigma \geq 9$, let E_σ be the line with slope -3 passing through $(1, \sigma - 1)$ and let D_σ be the line with slope -4 passing through $(1, \sigma - 1)$. Then no point strictly between E_σ and D_σ lies in \mathcal{K}_σ .*

Proof. Note that $E_\sigma = L_3^\sigma$ and $D_\sigma = U_4^\sigma$ in our current terminology. Conjecture 3.12 follows immediately from Theorem 2.3, since the points strictly between $E_\sigma = L_3^\sigma$ and $D_\sigma = U_4^\sigma$ form the gap $\mathcal{G}_\sigma(3, 4)$. \square

[1, Conjecture 3.13] *The point $(2, [\frac{2}{3}\sigma - 4])$ is persistently missing for all $\sigma \geq 7$.*

[1, Conjecture 3.14] *The points $(3, [\frac{2}{3}\sigma - 7])$ and $(3, [\frac{2}{3}\sigma - 8])$ are persistently missing for all $\sigma \geq 18$. For $\sigma \equiv 2 \pmod{3}$, the point $(3, [\frac{2}{3}\sigma - 6])$ is persistently missing for all $\sigma \geq 18$.*

Proof. For both Conjectures 3.13 and 3.14, we consider the gap $\mathcal{G}_\sigma(4, 6)$, which consists of the points strictly between the lines L_4^σ and U_6^σ , with the exception of those points that lie on the line $L_\sigma(5, 1)$. The equation for L_4^σ is $3r + 8h = 2\sigma + 6$ and the equation for U_6^σ is $3r + 12h = 2\sigma + 10$.

Setting $h = 2$ (which is the case of interest in Conjecture 3.13) and solving for r , we see that $(2, r)$ lies in the gap $\mathcal{G}_\sigma(4, 6)$ for all

$$\frac{2}{3}\sigma - \frac{14}{3} < r < \frac{2}{3}\sigma - \frac{10}{3}.$$

Since

$$\frac{2}{3}\sigma - \frac{14}{3} < \left[\frac{2}{3}\sigma - 4 \right] = \left[\frac{2}{3}\sigma - \frac{12}{3} \right] < \frac{2}{3}\sigma - \frac{10}{3}$$

for all $\sigma \geq 7$, we see that $(2, [\frac{2}{3}\sigma - 4])$ lies in $\mathcal{G}_\sigma(4, 6)$ for all $\sigma \geq 7$.

Setting $h = 3$ (which is the case of interest in Conjecture 3.14) and solving for r , we see that $(3, r)$ lies in the gap $\mathcal{G}_\sigma(4, 6)$ for all

$$\frac{2}{3}\sigma - \frac{26}{3} < r < \frac{2}{3}\sigma - 6.$$

Since

$$\frac{2}{3}\sigma - \frac{26}{3} < \left[\frac{2}{3}\sigma - 8 \right] < \left[\frac{2}{3}\sigma - 7 \right] < \frac{2}{3}\sigma - 6$$

for all $\sigma \geq 18$, we see that $(3, [\frac{2}{3}\sigma - 8])$ and $(3, [\frac{2}{3}\sigma - 7])$ both lie in $\mathcal{G}_\sigma(4, 6)$ for all $\sigma \geq 18$.

This leaves the case in which $\sigma \equiv 2 \pmod{3}$, where we see that

$$\frac{2}{3}\sigma - \frac{26}{3} < \left[\frac{2}{3}\sigma - 6 \right] < \frac{2}{3}\sigma - 6,$$

so that $(3, [\frac{2}{3}\sigma - 6])$ lies in $\mathcal{G}_\sigma(4, 6)$ for all $\sigma \geq 18$ satisfying $\sigma \equiv 2 \pmod{3}$. \square

We note that this approach to the description of gaps extends naturally to higher values of h and provides an essentially complete description of the regular gaps that we have seen from the available data.

4. The Line $r = 1$. The purpose of this section is to extend Theorem 3.8 from [1], in which we show that the skeletal signature $(1, 1)$ is sporadic, to all points of the form $(h, 1)$ for $h \geq 1$. This resolves Conjecture 3.15 from [1]. By *sporadic*, we mean a skeletal signature (h_0, r_0) for which there are infinitely many genera σ with $(h_0, r_0) \in \mathcal{K}_\sigma$ and infinitely many genera σ with $(h_0, r_0) \notin \mathcal{K}_\sigma$.

Let X be a closed Riemann surface of genus $\sigma \geq 2$ and let G be a finite group acting on X by conformal homeomorphisms where the action of G on X has signature $(h; n_1, \dots, n_r)$. A vector $(a_1, b_1, a_2, b_2, \dots, a_h, b_h, c_1, \dots, c_r)$ of elements of G is an $(h; n_1, \dots, n_r)$ -generating vector for G if the following hold:

1. $G = \langle a_1, b_1, a_2, b_2, \dots, a_h, b_h, c_1, \dots, c_r \rangle$;
2. The order of c_i is n_i for $1 \leq i \leq r$;
3. $\prod_{i=1}^h [a_i, b_i] \prod_{j=1}^r c_j = e_G$.

In [3], an adapted version of Riemann’s existence theorem provides necessary and sufficient conditions for existence of group actions in terms of generating vectors. Specifically, there exists an action of G on X with signature $(h; n_1, \dots, n_r)$ if and only if both the Riemann–Hurwitz formula is satisfied and there exists an $(h; n_1, \dots, n_r)$ -generating vector for G .

We are now ready to prove the main result of this section.

Theorem 4.1. *For $\sigma \geq 2$, every point $(h, 1)$ for $h > 1$ is a sporadic point.*

Proof. We shall first show that there exists an infinite sequence of genera for which there does not exist a group action with skeletal signature $(h, 1)$ on a closed Riemann surface of any genus in this sequence.

Suppose then that a group G acts with skeletal signature $(h, 1)$ on some Riemann surface of genus σ , so that the action of G on X has signature $(h; n)$ for some $n > 1$ and the Riemann–Hurwitz formula (1) is satisfied. Solving, we see that

$$\frac{2n(\sigma - 1)}{n(2h - 1) - 1} = |G|.$$

Since $n(2h - 1) - 1$ and n are relatively prime, it follows that $n(2h - 1) - 1$ must divide $2(\sigma - 1)$.

Now, set $\sigma = p + 1$ where p is an odd prime. Since $n(2h - 1) - 1$ must divide $2(\sigma - 1) = 2p$, it follows that one of following four cases must hold, so that either $n(2h - 1) - 1 = 1, n(2h - 1) - 1 = 2, n(2h - 1) - 1 = p$ or $n(2h - 1) - 1 = 2p$.

In the first case, we see that $n = 2$ and $h = 1$, while in the second case we see that $n = 3$ and $h = 1$. In particular, we have that $h = 1$ for both cases, contrary to our initial assumption that $h > 1$.

For the remaining two cases we modify the proof given in [1] to prove no such action can exist. First note that if G acts on X with signature $(h; n)$ for some $n \geq 2$, then there exists an $(h; n)$ -generating vector for G as defined above, for which c_1 of order n is in the commutator subgroup of G (since $(\prod_{i=1}^h a_i b_i a_i^{-1} b_i^{-1}) c_1 = e_G$). Since $n \geq 2$, it follows that G cannot be Abelian.

For $n(2h - 1) - 1 = p$, we have

$$|G| = \frac{2np}{n(2h - 1) - 1} = \frac{2p\left(\frac{p+1}{2h-1}\right)}{\left(\frac{p+1}{2h-1}\right)(2h - 1) - 1} = 2 \left(\frac{p + 1}{2h - 1} \right) = 2n.$$

Hence, we see that G has order $2n$ and acts on X with signature $(h; n)$, from which it follows that G has an index 2 cyclic subgroup H which contains c_1 . Since $n(2h - 1) - 1 = p$ for p an odd prime, it follows that n is even and so H has a characteristic subgroup K of index two which is thus normal in G . Therefore, the factor group G/K has order four and hence is Abelian, so it follows that $H = \langle c_1 \rangle$ cannot be contained in the commutator subgroup of G , a contradiction. Hence $(h, 1)$ is not a skeletal signature for G .

Finally, for the last case, we have

$$n = \frac{2p + 1}{2h - 1}$$

and consequently

$$|G| = \frac{2np}{n(2h - 1) - 1} = \frac{2p\left(\frac{2p+1}{2h-1}\right)}{\left(\frac{2p+1}{2h-1}\right)(2h - 1) - 1} = \frac{2p\left(\frac{2p+1}{2h-1}\right)}{2p} = \frac{2p + 1}{2h - 1} = n.$$

Since G contains an element equal to its order, it must be cyclic and hence Abelian, which is impossible.

We have now shown that $(h, 1)$ is not the skeletal signature of any group action for any genus of the form $\sigma = p + 1$ for any odd prime p , of which there are infinitely many.

To complete the proof, we need to construct an infinite sequence of genera for which $(h, 1)$ is a skeletal signature. Again we can generalize the proof for the case $(1, 1)$ given in [1]. For $n \geq 2$, let $G_n = \langle x, y \mid x^n = y^2, y^{-1}xy = x^{-1} \rangle$ denote the generalized quaternion group. Taking $a_i = b_i = e_G$ for $i > 1$, the vector $(x, y, e_G, e_G, \dots, e_G, yx^{-2}y^{-1})$ is then an $(h; n)$ -generating vector for G_n (since $(x, y, yx^{-2}y^{-1})$ is a $(1, n)$ -generating vector for G_n , as was proved in [1]). Applying the Riemann–Hurwitz formula (1), it follows that G_n acts on a surface of genus $\sigma = 2n(2(h - 1) + 1) - 1$. In particular, $(h, 1)$ is a skeletal signature for $\sigma = 2n(2(h - 1) + 1) - 1$ for all $n \geq 2$. \square

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