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Sharp inequalities for dyadic A_1 weights

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Abstract. We show how the Bellman function method can be used to obtain sharp inequalities for the maximal operator of a dyadic A_1 weight on $\mathbb{R}^n$. Using this approach, we determine the optimal constants in the corresponding weak-type estimates. Furthermore, we provide an alternative, simpler proof of the related maximal L^p-inequalities, originally shown by Melas.

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1. Introduction. A locally integrable nonnegative function w on $\mathbb{R}^n$ is called a dyadic A_1 weight if it satisfies the condition

$$\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x \leq C \operatorname*{essinf}_{x\in Q} w(x) \tag{1.1}$$

for any dyadic cube Q in $\mathbb{R}^n$. This is equivalent to saying that

$$M_d w(x) \leq C w(x) \quad \text{for almost all } x \in \mathbb{R}^n, \tag{1.2}$$

where M_d is the dyadic maximal operator, given by

$$M_d w(x) = \sup\left\{\frac{1}{|Q|}\int_Q w(t)\mathrm{d}t \;:\; x\in Q,\, Q\subset\mathbb{R}^n \text{ a dyadic cube}\right\}.$$

The smallest C for which (1.1) (equivalently (1.2)) holds is called the dyadic A_1 constant of w and is denoted by $[w]_1$. A classical result of Coifman and

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Fefferman [2] states that any A_1 weight satisfies the reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q w(x)^p \mathrm{d}x\right)^{1/p} \leq \frac{c}{|Q|}\int_Q w(x)\mathrm{d}x \tag{1.3}$$

for certain $p > 1$ and $c \geq 1$ which depend only on the dimension n and the value of $[w]_1$. The exact information on the range of possible p's was studied by Melas [3] (see also [1] for related results in the non-dyadic case). Here is the precise statement.

Theorem 1.1. *Let w be a dyadic weight on $\mathbb{R}^n$. Then for every p such that*

$$1 \leq p < p_0(n, [w]_1) := \frac{\log(2^n)}{\log\left(2^n - \frac{2^n-1}{[w]_1}\right)}$$

and for every dyadic cube Q, we have

$$\frac{1}{|Q|}\int_Q (M_d w(x))^p\, dx \leq \frac{2^n - 1}{\left(2^n - \frac{2^n-1}{[w]_1}\right)^p - 2^n}\left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)^p. \tag{1.4}$$

Both the range of p and the corresponding constant in (1.4) are best possible.

This result implies that the range of admissible exponents p in the reverse Hölder inequality (1.3) is at least $[1, p_0(n, [w]_1))$. To prove that both intervals are actually equal, Melas [3] constructed, for any $\lambda > 1$, a dyadic weight on $[0,1]^n$ such that $[w]_1 = \lambda$ and $\int_{[0,1]^n} w(x)^{p_0(n,\lambda)}\mathrm{d}x = \infty$.

The purpose of this paper is to study the corresponding weak-type estimates. We will prove the following result.

Theorem 1.2. *Let w be a dyadic weight on $\mathbb{R}^n$, and let $1 \leq p \leq p_0(n, [w]_1)$. Then for every dyadic cube Q, we have*

$$\frac{1}{|Q|}|\{x \in Q : M_d w(x) > 1\}| \leq \left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)^p. \tag{1.5}$$

Both the range of p and the constant 1 are already best possible in the estimate

$$\frac{1}{|Q|}|\{x \in Q : w(x) > 1\}| \leq \left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)^p.$$

A few words about the proof. Using a standard dilation argument, it is enough to establish (1.5) for $Q = [0,1]^n$. In fact, we will prove the estimate in a wider context of probability spaces equipped with a tree-like structure similar to the dyadic one. Next, while Melas' proof of Theorem 1.1 is combinatorial and rests on a clever linearization of the dyadic maximal operator, our approach will be entirely different and will exploit the properties of a certain special function. In the literature, this type of argument is called the Bellman function method and has been applied recently in various settings: see e.g. [4–8], and references therein.

The paper is organized as follows. Section 2 contains some preliminary definitions. The description of the Bellman method can be found in Sect. 3, and it is applied in two final parts of the paper: in Sect. 4 we present the study of the weak type estimate, while in Sect. 5 we provide an alternative proof of Melas' result.

2. Measure spaces with a tree-like structure. Assume that $(X, \mathcal{F}, \mu)$ is a given non-atomic probability space. We assume that it is equipped with an additional tree structure.

Definition 2.1. Let $\alpha \in (0, 1]$ be a fixed number. A sequence $\mathcal{T} = (\mathcal{T}_n)_{n\geq 0}$ of partitions of X is said to be α-splitting if the following conditions hold.

(i) We have $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_n \subset \mathcal{F}$ for all n.
(ii) For any $n \geq 0$ and any $E \in \mathcal{T}_n$, there are pairwise disjoint sets $E_1, E_2, \ldots, E_m \in \mathcal{T}_{n+1}$ whose union is E and such that $|E_i|/|E| \geq \alpha$ for all i.

Let us stress that the number m in (ii) may be different for different E.

Example. Assume that $X = (0, 1]^n$ is the unit cube of $\mathbb{R}^n$ with Borel subsets and Lebesgue's measure. Let $\mathcal{T}_k$ be the collection of all dyadic cubes of volume 2^{-kn} contained in X (i.e., products of intervals of the form $(a2^{-k}, (a+1)2^{-k}]$, where $a \in \{0, 1, 2, \ldots, 2^k - 1\}$). Then $\mathcal{T} = (\mathcal{T}_n)_{n\geq 0}$ is 2^{-n}-splitting.

In what follows, we will restrict ourselves to $\alpha \leq 1/2$ since for $\alpha > 1/2$ there is only one α-splitting tree: $\mathcal{T} = (\{X\}, \{X\}, \{X\}, \ldots)$. Let us define the maximal operator and A_1 class corresponding to the structure $\mathcal{T}$.

Definition 2.2. Given a probability space $(X, \mathcal{F}, \mu)$ with a sequence $\mathcal{T}$ as above, we define the corresponding maximal operator $\mathcal{M}_{\mathcal{T}}$ as

$$\mathcal{M}_{\mathcal{T}} f(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |f| \mathrm{d}\mu \ : \ x \in I \in \mathcal{T} \right\}$$

for any $f \in L^1(X, \mathcal{F}, \mu)$. We will also use the notation $\mathcal{M}^n_{\mathcal{T}}$ for the truncated maximal operator, associated with $\mathcal{T}^n = (\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{n-1}, \mathcal{T}_n, \mathcal{T}_n, \mathcal{T}_n, \ldots)$.

Definition 2.3. A nonnegative integrable function w is an A_1 weight with respect to $\mathcal{T}$ if there is a finite constant C such that

$$\frac{1}{|E|} \int_E w(x) \mathrm{d}x \leq C \operatorname*{essinf}_{x\in E} w(x)$$

for any $E \in \mathcal{T}$. This is equivalent to saying that

$$\mathcal{M}_{\mathcal{T}} w(x) \leq C w(x)$$

for almost all $x \in X$. The smallest C for which the above holds is called the A_1 constant of w and will be denoted by $[w]_1$.

3. On the method of proof. Now we will describe the technique which will be used to establish the inequalities announced in Section 1. Throughout this section, $c > 1, \alpha \in (0, 1/2]$ are fixed constants. Distinguish the following subset of $\mathbb{R}_+^3$:

$$D = D_c = \{(x, y, z) \in \mathbb{R}_+^3 : y \leq x \leq cy,\ z \leq cx\}.$$

Let Φ, $\Psi : \mathbb{R}_+ \to \mathbb{R}$ be two given functions, and assume that we want to show that

$$\int\limits_X \Phi(M_{\mathcal{T}}^n w(x))\,\mathrm{d}x \leq \Psi\left(\int\limits_X w(x)\mathrm{d}x\right), \quad n = 0,\, 1,\, 2,\, \ldots, \tag{3.1}$$

for any A_1 weight w with respect to an α-splitting tree $\mathcal{T}$, such that $[w]_1 \leq c$. The key idea in the study of this problem is to construct a special function $B = B_{c,\alpha,\Phi,\Psi} : D \to \mathbb{R}$, which satisfies the following conditions.

1° We have $B(x, y, x \vee z) = B(x, y, z)$ for any $(x, y, z) \in D$.
2° We have $B(x, y, z) \geq \Phi(z)$ for any $(x, y, z) \in D$.
3° We have $B(x, y, x) \leq \Psi(x)$ for all x, y such that $(x, y, x) \in D$.
4° For any $(x, y, z) \in D$ there exists $A = A(x, y, z) \in \mathbb{R}$ such that whenever $(x', y', z) \in D$ satisfies $x' \leq \frac{c-1+\alpha}{c\alpha} x$ and $y' \geq y$, then

$$B(x', y', z) \leq B(x, y, z) + A(x, y, z)(x' - x). \tag{3.2}$$

A few remarks concerning these conditions are in order. The condition 1° is a technical assumption which enables the proper handling of the maximal operator. The conditions 2° and 3° are appropriate majorizations. The most complicated (and most mysterious) condition is the last one. To shed some light on it, observe that it yields the following concavity-type property of B.

Lemma 3.1. *Let (x, y, z) be a fixed point belonging to D, and let $n \geq 2$ be an arbitrary integer. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers which sum up to 1, such that $\alpha_i \geq \alpha$ for each i. Assume further that (x_1, y_1, z), (x_2, y_2, z), ..., $(x_n, y_n, z) \in D$ satisfy*

$$x = \sum_{i=1}^n \alpha_i x_i \quad \text{and} \quad y = \min\{y_1,\, y_2, \ldots,\, y_n\}.$$

Then

$$B(x, y, z) \geq \sum_{i=1}^n \alpha_i B(x_i, y_i, z). \tag{3.3}$$

Proof. Apply 4° to $x' = x_i, y' = y_i, i = 1,\, 2,\, \ldots,\, n$, multiply both sides by α_i, and finally sum the obtained inequalities. Then, as the result, we get (3.3). Thus, all we need is to verify whether the requirements for x', y' appearing in 4° are fulfilled. The inequality $y_i \geq y$ is assumed in the statement of the lemma. Furthermore, by the definition of D,

$$\begin{aligned} x = \sum_{i=1}^n \alpha_i x_i &\geq \alpha_i x_i + \sum_{j\neq i} \alpha_j y_j \\ &\geq \alpha_i x_i + (1-\alpha_i)y \geq \alpha x_i + (1-\alpha)y \geq \alpha x_i + (1-\alpha)x/c, \end{aligned}$$

which yields the desired bound $x_i \leq \frac{c-1+\alpha}{c\alpha}x$. □

We turn to the main result of this section.

Theorem 3.2. *Suppose that $(X, \mathcal{F}, \mu)$ is a probability space equipped with an α-splitting tree $\mathcal{T}$. If there is a function $B = B_{c,\alpha,\Phi,\Psi}$ satisfying 1°–4°, then (3.1) holds for any A_1 weight w with respect to $\mathcal{T}$ such that $[w]_1 \leq c$.*

Proof. Let w be as in the statement. Define two sequences $(w_n)_{n\geq 0}, (v_n)_{n\geq 0}$ of measurable functions on X as follows. Given an integer n, an element E of $\mathcal{T}_n$, and a point $x \in E$, set

$$w_n(x) = \frac{1}{\mu(E)} \int\limits_E w(t)\mathrm{d}\mu(t) \quad \text{and} \quad v_n(x) = \operatorname*{essinf}_{t\in E} w(t).$$

The following interplay between these objects will be important to us. Let n, E be as above, and let $E_1, E_2, \dots, E_m$ be the elements of $\mathcal{T}_{n+1}$ whose union is E. Then we easily check that

$$\frac{1}{\mu(E)} \int\limits_E w_n(t)\mathrm{d}\mu(t) = \sum_{i=1}^m \frac{\mu(E_i)}{\mu(E)} \cdot \frac{1}{\mu(E_i)} \int\limits_{E_i} w_{n+1}(t)\mathrm{d}\mu(t)$$

and

$$v_n|_E = \min\{v_{n+1}|_{E_1}, v_{n+1}|_{E_2}, \dots, v_{n+1}|_{E_m}\}.$$

Furthermore, the inequality $[w]_1 \leq c$ implies that the triple $(w_n, v_n, \mathcal{M}^n_{\mathcal{T}}w)$ takes values in D. These conditions, combined with Lemma 3.1, yield the inequality

$$\int\limits_E B(w_n(t), v_n(t), \mathcal{M}^n_{\mathcal{T}}w(t))\mathrm{d}\mu(t) \geq \int\limits_E B(w_{n+1}(t), v_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}}w(t))\mathrm{d}\mu(t).$$

Indeed, we have $\mathcal{M}^{n+1}_{\mathcal{T}}w = \mathcal{M}^n_{\mathcal{T}}w \vee w_{n+1}$, so by 1°,

$$B(w_{n+1}(t), v_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}}w(t)) = B(w_{n+1}(t), v_{n+1}(t), \mathcal{M}^n_{\mathcal{T}}w(t)), \quad t \in E.$$

It remains to use (3.3) with $x = w_n|_E$, $y = v_n|_E$, $z = \mathcal{M}^n_{\mathcal{T}}w|E$, $x_i = w_{n+1}|_{E_i}$, $y_i = v_{n+1}|_{E_i}$, and $\alpha_i = \mu(E_i)/\mu(E)$. Summing over all $E \in \mathcal{T}_n$, we get

$$\int\limits_X B(w_n(t), v_n(t), \mathcal{M}^n_{\mathcal{T}}w(t))\mathrm{d}\mu(t) \geq \int\limits_X B(w_{n+1}(t), v_{n+1}(t), \mathcal{M}^{n+1}_{\mathcal{T}}w(t))\mathrm{d}\mu(t)$$

and therefore, by induction,

$$\int\limits_X B(w_0(t), v_0(t), \mathcal{M}^0_{\mathcal{T}}w(t))\mathrm{d}\mu(t) \geq \int\limits_X B(w_n(t), v_n(t), \mathcal{M}^n_{\mathcal{T}}w(t))\mathrm{d}\mu(t).$$

However, the left-hand side equals

$$B\left(\int_X w(t)\mathrm{d}t,\ \operatorname*{essinf}_{t\in X} w(t),\ \int_X w(t)\mathrm{d}t\right),$$

and hence the application of 2° and 3° completes the proof of (3.1). □

4. A sharp weak-type estimate. The principal result of this section is the following.

Theorem 4.1. *Suppose that $(X,\mathcal{F},\mu)$ is a probability space equipped with an α-splitting tree $\mathcal{T}$. Then for any A_1 weight w with respect to $\mathcal{T}$ and any p satisfying*

$$1 \le p \le p_0(\alpha,[w]_1) := -\frac{\log\alpha}{\log\left(\frac{[w]_1+\alpha-1}{[w]_1\alpha}\right)},$$

we have

$$\mu(\{x\in X : \mathcal{M}_{\mathcal{T}}w(x) > 1\}) \le \left(\int_X w(x)\,dx\right)^p. \tag{4.1}$$

The range of p and the constant 1 are already the best possible in

$$\mu(\{x\in X : w(x) > 1\}) \le \left(\int_X w(x)\,dx\right)^p. \tag{4.2}$$

Before we proceed, let us establish the following technical fact.

Lemma 4.2. *For any $c \ge 1$ and $\alpha\in(0,1/2]$, we have $(p_0(\alpha,c)-1)c \le p_0(\alpha,c)$.*

Proof. The claim is equivalent to $1/p_0(\alpha,c) \ge 1-1/c$, or

$$\log\left(\frac{c\alpha}{c+\alpha-1}\right) \le \log\alpha^{1-1/c}.$$

Substituting $x = 1/c \in [0,1]$ and working a little bit turns this bound into

$$\alpha^x \le 1+(\alpha-1)x,$$

which is evident: the left-hand side is convex as a function of x, and both sides are equal when $x\in\{0,1\}$. □

4.1. Proof of (4.1). We may assume that $[w]_1 > 1$, since otherwise w is constant and the assertion holds true. If the average $\int_X w$ is at least 1, then the inequality is trivial. So, suppose that $\int_X w < 1$; then it suffices to prove the weak-type estimate for $p = p_0(\alpha,[w]_1)$. In view of Theorem 3.2, all we need is to construct an appropriate special function corresponding to $c = [w]_1 > 1$, $\alpha\in(0,1/2]$, $\Phi(z) = \chi_{\{z\ge 1\}}$ and $\Psi(x) = x^p$. Indeed, this will yield (3.1), and letting n go to ∞ will complete the proof. Introduce $B = B_{c,\alpha,\Phi,\Psi} : D \to \mathbb{R}_+$ by

$$B(x,y,z) = \begin{cases} c^p y^{p-1}(x-y)/(c-1) & \text{if } y \le c^{-1} \text{ and } x \vee z < 1, \\ (x-y)/(1-y) & \text{if } y > c^{-1} \text{ and } x \vee z < 1, \\ 1 & \text{if } x \vee z \ge 1. \end{cases}$$

We will exploit the following auxiliary property of B.

Lemma 4.3. *For fixed x, $z > 0$, the function $B(x,\cdot,z) : y \mapsto B(x,y,z)$ is nonincreasing on $[x/c, x]$.*

Proof. We may assume that x, $z < 1$, since otherwise the claim is obvious. Note that for $y \ge x/c$ we have

$$\frac{\partial}{\partial y}\left[y^{p-1}(x-y)\right] = y^{p-2}((p-1)x - py) \le y^{p-1}((p-1)c - p) \le 0$$

in light of Lemma 4.2. Furthermore, for any $y < 1$,

$$\frac{\partial}{\partial y}\left[\frac{x-y}{1-y}\right] = \frac{x-1}{(1-y)^2} < 0.$$

Since B is continuous, this gives the desired monotonicity. $\square$

Now we turn to the verification that B satisfies the conditions $1^\circ - 4^\circ$. The first two of them are obvious, so let us look at 3°. By Lemma 4.3, it suffices to prove the majorization for $y = x/c$. But then the estimate is clear: both sides are equal when $x < 1$, and for $x \ge 1$ the inequality takes the form $1 \le x^p$. Finally, we will check 4° with

$$A(x,y,z) = \begin{cases} c^p y^{p-1}(c-1) & \text{if } y \le c^{-1} \text{ and } x \vee z < 1, \\ 1/(1-y) & \text{if } y > c^{-1} \text{ and } x \vee z < 1, \\ 0 & \text{if } x \vee z \ge 1. \end{cases}$$

We may and do assume that $x \vee z < 1$ since otherwise the right-hand side of (3.2) is equal to 1 and there is nothing to prove. By the preceding lemma, it suffices to show (3.2) under the assumption that

$$y' = (x'/c) \vee y. \tag{4.3}$$

Suppose first that $y > c^{-1}$; then (3.2) becomes

$$B(x', y', z) \le \frac{x' - y}{1 - y}.$$

If $x' \ge 1$, then this bound is clear; if $x' < 1$, then $y' = y$ (see (4.3)) and thus both sides are equal. Finally, assuming that $y \le c^{-1}$, we see that (3.2) reads

$$B(x', y', z) \le \frac{c^p}{c-1} y^{p-1}(x' - y).$$

If $x' \le cy$, then $y' = y$ (see (4.3)) and hence both sides are equal. If $x' > cy$, then (4.3) implies that $y' = x'/c$ and the inequality becomes

$$(x')^p \le \frac{c^p}{c-1} y^{p-1}(x' - y),$$

or, after the substitution $t = x'/y$,

$$t^p \le \frac{c^p}{c-1}(t-1). \tag{4.4}$$

We have $t > c$ by the assumption we have just made above. On the other hand, exploiting the requirements appearing in 4°, we get

$$t = \frac{x'}{y} \le \frac{cx'}{x} \le \frac{c-1+\alpha}{\alpha}.$$

It suffices to note that the left-hand side of (4.4) is a convex function of t and both sides are equal for the extremal values of t: $t = c$ and $t = (c-1+\alpha)/\alpha$ (the equality for the latter value of t is just the definition of $p_0(\alpha, c)$).

4.2. Sharpness. It is obvious that the constant 1 cannot be improved in (4.2): consider a constant weight $w \equiv \lambda > 1$, and let $\lambda \downarrow 1$. To show that the weak-type estimate cannot hold with exponents larger than $p_0(\alpha, c)$, we will construct an appropriate example; a related object can be found in [3].

Suppose that $(X, \mathcal{F}, \mu)$ is a probability space equipped with an α-splitting tree $\mathcal{T}$, such that there is a monotone sequence $X = E_0 \supset E_1 \supset E_2 \supset \ldots$, with $E_n \in \mathcal{T}_n$ and $\mu(E_n) = \alpha^n$. For $x \in X$, put $N(x) = \sup\{n \ge 0 : x \in E_n\}$; this is well-defined since $E_0 = X$. Moreover, $N(x) < \infty$ almost everywhere, because the sets E_i shrink to a set of measure zero. Define a weight w by

$$w(x) = \left(\frac{c-1+\alpha}{c\alpha}\right)^{N(x)}, \quad x \in X.$$

In other words, we have $w(x) = \big[(c-1+\alpha)/(c\alpha)\big]^n$, where n is the unique number such that $x \in E_n \backslash E_{n+1}$. Then w is in the A_1 class and $[w]_1 = c$. To see this, pick $x \in X$ and let n be the unique integer such that $x \in E_n \backslash E_{n+1}$. The only elements of $\mathcal{T}$ which contain x are E_0, E_1, …, E_n, so

$$\mathcal{M}_{\mathcal{T}} w(x) = \max_{0 \le k \le n} \left\{ \frac{1}{\mu(E_k)} \int_{E_k} w(t) \mathrm{d}\mu(t) \right\}.$$

However, by the definition of w, we easily compute that

$$\begin{aligned} \frac{1}{\mu(E_k)} \int_{E_k} w(t) \mathrm{d}\mu(t) &= \frac{1}{\mu(E_k)} \sum_{\ell \ge k} \mu(E_\ell \backslash E_{\ell+1}) \cdot \left(\frac{c-1+\alpha}{c\alpha}\right)^k \\ &= c \left(\frac{c-1+\alpha}{c\alpha}\right)^k \end{aligned}$$

and hence

$$\mathcal{M}_{\mathcal{T}} w(x) = c \left(\frac{c-1+\alpha}{c\alpha}\right)^n = cw(x).$$

Putting $k = 0$ in the above calculation gives $\int_X w = c$. Fix $q \ge 1, \lambda > 1$, and consider the weight $\tilde{w} = \lambda \left[c\alpha/(c-1+\alpha)\right]^n w$. It satisfies $[\tilde{w}]_1 = c$ and

$$\mu\big(\{x \in X : \tilde{w}(x) > 1\}\big) \ge \mu(E_n) = \tfrac{1}{c\lambda} \left[\left(\tfrac{c-1+\alpha}{c\alpha}\right)^q \alpha\right]^n \left(\int_X w(x)\mathrm{d}x\right)^q.$$

Now, if we put $q = p_0(\alpha, c)$, then the expression in the square brackets is equal to 1. Therefore, if q is larger than $p_0(\alpha, c)$, then the constant on the right-hand side explodes as $n \to \infty$. This shows that the threshold $p_0(\alpha, c)$ in the weak-type estimate cannot be improved.

5. Melas' theorem revisited. Now we use the method developed in Section 3 to obtain the following version of Theorem 1.1. For a fixed $c \geq 1, \alpha \in (0, 1/2]$, and $1 \leq p < p_0(\alpha, c)$, let

$$C = C_{c,\alpha,p} = \frac{1-\alpha}{1-\alpha\left(\frac{c-1+\alpha}{c\alpha}\right)^p}$$

(when $\alpha = 2^{-n}$ and $c = [w]_1$, this is exactly the constant appearing in (1.4)).

Theorem 5.1. *Suppose that $(X, \mathcal{F}, \mu)$ is a probability space equipped with an α-splitting tree $\mathcal{T}$. Then for any A_1 weight w with respect to $\mathcal{T}$, we have*

$$\int_X (\mathcal{M}_{\mathcal{T}} w(x))^p \, dx \leq C_{[w]_1,\alpha,p} \left(\int_X w(x) \, dx \right)^p. \tag{5.1}$$

Both the range of p and the constant $C_{c,\alpha,p}$ are best possible.

Proof. We only show (5.1), for the construction of the extremal examples, the reader is referred to [3]. We may assume $c = [w]_1 > 1$. Define the functions $\Phi(z) = z^p$, $\Psi(x) = Cx^p$, and consider $B = B_{c,\alpha,\Phi,\Psi} : D \to \mathbb{R}$, given by

$$B(x, y, z) = (c-1)^{-1}(x \vee z)^{p-1} \left[(C-1)cx + (c-C)(x \vee z)\right].$$

It is easy to show that this function enjoys the conditions 1°, 2°, and 3°; we leave the details to the reader. Finally, we will prove 4° with

$$A(x, y, z) = (c-1)^{-1}(x \vee z)^{p-1}(C-1)c.$$

The estimate (3.2) can be rewritten in the form

$$\begin{aligned}(x' \vee z)^{p-1}&[(C-1)cx' + (c-C)(x' \vee z)] \\ &\leq (x \vee z)^{p-1}[(C-1)cx' + (c-C)(x \vee z)].\end{aligned}$$

If $x' \leq (x \vee z)$, then both sides are equal; if $x' > (x \vee z)$, then the bound becomes

$$C(c-1)(x')^p \leq (x \vee z)^{p-1}((C-1)cx' + (c-C)(x \vee z)),$$

or, after the substitution $t = x'/(x \vee z)$,

$$C(c-1)t^p \leq (C-1)ct + c - C. \tag{5.2}$$

However, we have $t > 1$ and $t \leq (c-1+\alpha)/(c\alpha)$ (see the assumptions appearing in 4°). It suffices to note that the left-hand side of (5.2) is a convex function, and that both sides are equal for $t \in \{1, (c-1+\alpha)/(c\alpha)\}$. Thus, (3.1) gives the claim for truncated maximal operator, and letting $n \to \infty$ completes the proof, by the use of Lebesgue's monotone convergence theorem. □

References

[1] B. Bojarski, C. Sbordone, and I. Wik, The Muckenhoupt class $A_1(\mathbb{R})$, Studia Math. **101** (1992), 155–163.

[2] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. **51** (1974), 241–250.

[3] A. D. Melas, A sharp inequality for dyadic A_1 weights in $\mathbb{R}^n$, Bull. London Math. Soc. **37** (2005), 919–926.

[4] F. L. Nazarov and S. R. Treil, The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis. (Russian) Algebra i Analiz **8** (1996), 32–162; translation in St. Petersburg Math. J. **8** (1997), 721–824.

[5] F. L. Nazarov, S. R. Treil, and A. Volberg, Bellman function in stochastic control and harmonic analysis, Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 393–423, Oper. Theory Adv. Appl., **129**, Birkhäuser, Basel, 2001.

[6] L. Slavin and A. Volberg, Bellman function and the $H^1 - BMO$ duality, Harmonic analysis, partial differential equations, and related topics, 113–126, Contemp. Math., 428, Amer. Math. Soc., Providence, RI, 2007.

[7] V. Vasyunin, The sharp constant in the reverse Holder inequality for Muckenhoupt weights, St. Petersburg Math. J., **15** (2004), 49–79.

[8] V. Vasyunin and A. Volberg, Monge-Ampr̀e equation and Bellman optimization of Carleson embedding theorems, Linear and complex analysis, 195–238, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

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