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ERRATUM

Erratum to: A note on the Brück conjecture

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Abstract. In the article "A note on the Brück conjecture" (Arch. Math. 95 (2010), 257–268), the proofs of our results Theorem 1.4 and Theorem 1.6 mainly depend on Lemma 2.6, which is wrong. In this corrigendum, we give a correct proof of Theorem 1.4. Theorem 1.6 can be proved similarly; so we omit its proof.

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The following counterexample for Lemma 2.6 in [2] is due to Professor Igor Chyzhykov, to whom we wish to express our gratitude.

Example. Let $f(z) = e^{z^p}, p \in \mathbb{N}$, take $p = 1$ for simplicity, and $E_\theta = \{0\}$. Then the inequality $|f(r_k e^{i\theta_k})| \geq AM(r_k, f)$ is equivalent to $r_k \cos\theta_k \geq r_k + \log A$. Since A is a constant, we have $\theta_k \to 0 \in E_\theta$ as $r_k \to +\infty$.

Unfortunately, in [2], the proofs of our results Theorem 1.4 and Theorem 1.6 mainly depend on Lemma 2.6. We express regret for these mistakes. In the following, we give a correct proof of Theorem 1.4. Theorem 1.6 can be proved similarly, and its proof is hence omitted. To prove Theorem 1.4, we need some lemmas here.

Lemma A. *Let*

$$p(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_0, \quad q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_0,$$

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where n is a positive integer, $p_n = \alpha e^{i\theta}, q_n = \beta e^{i\varphi}, \alpha \geq \beta > 0, \theta, \varphi \in [0, 2\pi)$. If $p_n \neq q_n$, then for any given $\varepsilon > 0$, there exists some $r_0 > 1$, such that for all $z = re^{-i\frac{\theta}{n}}$ satisfying $r \geq r_0$, we have

$$Re\{p(re^{-i\frac{\theta}{n}})\} > \alpha(1-\varepsilon)r^n$$

and

$$Re\{p(re^{-i\frac{\theta}{n}}) - q(re^{-i\frac{\theta}{n}})\} > (\alpha - \beta\cos(\theta-\varphi))(1-\varepsilon)r^n.$$

Proof. Since $Re\{p_n(re^{-i\frac{\theta}{n}})^n\} = \alpha r^n$, we can easily prove this conclusion. □

Lemma B. ([1]) *Let $f(z)$ be a meromorphic function with $\rho(f) = \alpha < +\infty$, then for any given $\varepsilon > 0$, there exists a set $E \subset [0, +\infty)$ with finite linear measure $mE < \infty$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, and r sufficiently large,*

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\alpha+\varepsilon}\}.$$

Lemma C. ([3]) *If $f_j(z)$ $(j = 1, 2, \ldots, n)$ and $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 2)$ are entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$;
(ii) *The orders of f_j are less than that of $e^{g_k - g_h}$ for $1 \leq j \leq n$, $1 \leq h < k \leq n$, then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.*

Proof of Theorem 1.4. Without loss of generality, we can suppose that $\eta = 1$. Denote $g(z) = f(z) - a$, then $\rho(g) = \rho(f) = \rho$ and

$$\Delta^n f(z) = \Delta^n g(z) = \sum_{j=0}^{n}(-1)^{n-j}C_n^j g(z+j),$$

where $C_n^0, C_n^1, \ldots, C_n^n$ are non-zero integers. By assumption, we have

$$\frac{\Delta^n f(z) - a}{f(z) - a} = \frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j g(z+j) - a}{g(z)} = e^{p(z)}, \tag{1.1}$$

where $p(z)$ is a polynomial with $0 \leq d = \deg(p) \leq \rho$. Now set

$$p(z) = p_d z^d + p_{d-1}z^{d-1} + \cdots + p_0,$$

where $p_d \neq 0, p_{d-1}, \ldots, p_0$ are constants, $p_d = \alpha_d e^{i\theta_d}, \alpha_d > 0, \theta_d \in [0, 2\pi)$.

Firstly, we prove that $\rho \geq 1$. Otherwise, we have $\rho < 1$ and hence $p(z) \equiv C \in \mathbb{C}$. By Lemma 2.3 stated in [2], for any given ε_1 $(0 < 2\varepsilon_1 < 1 - \rho)$, there exists a set $E_1 \subset (1, \infty)$ of finite logarithmic measure, so that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{\Delta^n g(z)}{g(z)}\right| \leq |z|^{n(\rho-1)+\varepsilon_1}. \tag{1.2}$$

Chose an infinite sequence of points $\{z_k = r_k e^{i\theta_k}\}$ such that

$$|g(z_k)| = M(r_k, g) \geq \exp\{r_k^{\rho-\varepsilon_1}\}, \quad r_k \notin E_1. \tag{1.3}$$

From (1.1)–(1.3), we get a contradiction that

$$|e^C| \leq \left|\frac{\Delta^n g(z_k)}{g(z_k)}\right| + \frac{|a|}{M(r_k, g)} \leq {r_k}^{n(\rho-1)+\varepsilon_1} + o(1) = o(1).$$

Secondly, we assert that $\rho \leq \lambda(f-a)+1$. Otherwise, we have $\rho > \lambda(f-a)+1$ and hence $\rho(g) > \lambda(g)+1$. It follows from the Hadamard factorization theorem that,

$$g(z) = h(z)e^{q(z)},$$

where $q(z)$ is a polynomial such that

$$q(z) = -(q_l z^l + q_{l-1}z^{l-1} + \cdots + q_0),$$

where $q_l \neq 0, q_{l-1}, \ldots, q_0$ are constants, $q_l = \beta_l e^{i\varphi_l}, \beta_l > 0, \varphi_l \in [0, 2\pi)$, and $h(z) = z^m W(z)$, where m is the order of zero of $g(z)$ and $W(z)$ is the Weierstrass canonical product of the nonzero zeros of $g(z)$ such that $\rho(h) = \rho(W) = \lambda(g) < \rho - 1 = l - 1$.

Rewrite (1.1) as

$$\frac{a}{h(z)e^{q(z)}} = e^{p(z)} - \frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{q(z+j)-q(z)}}{h(z)}.$$

Observe that for each $j \in \{0, \ldots, n\}, \deg(q(z+j) - q(z)) = l-1$, and $\rho(\frac{a}{h(z)e^{q(z)}}) = l$, then by the equation above, we can easily get $l \leq d$. Thus we have $d = l$.

We claim that $p_d = q_d$. Otherwise, $p_d \neq q_d$, and we may assume that $\alpha_d \geq \beta_d > 0$. In what follows, set $q_j^*(z) = q(z+j) + q_d z^d, p^*(z) = p(z) - p_d z^d$. Then we obtain from (1.1) that

$$e^{p(z)} - \frac{ae^{-q(z)}}{h(z)} = \frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{q_j^*(z)}}{h(z)e^{q_0^*(z)}}. \tag{1.4}$$

Set $\rho_1 = \rho(h)$, then by Lemma B, for any given ε_2 $(0 < 2\varepsilon_2 < \min\{\rho - \rho_1, 1\})$, there exists a set $E_2 \subset [0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, and r sufficiently large,

$$\exp\{-r^{\rho_1+\varepsilon_2}\} \leq |h(z)| \leq \exp\{r^{\rho_1+\varepsilon_2}\}. \tag{1.5}$$

From Lemma A and (1.5), for $r \notin [0, 1] \cup E_2, r \to \infty$, we have

$$\left|\frac{ae^{-q(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})}\right| = o(|e^{p(re^{-i\frac{\theta_d}{d}})}|). \tag{1.6}$$

Notice that for $1 \leq j \leq n, q_j^*(z) - q_0^*(z) = q(z+j) - q(z)$ and hence $\deg(q_j^*(z) - q_0^*(z)) = d-1$. Applying Lemma A again, for $r \notin [0, 1] \cup E_2, r \to \infty$, we get from (1.4)–(1.6) that

$$\frac{1}{2}\exp\{(1-\varepsilon_2)\alpha_d r^d\} \le \frac{1}{2}\left|e^{p(re^{-i\frac{\theta_d}{d}})}\right| < \left|e^{p(re^{-i\frac{\theta_d}{d}})}\right| - \left|\frac{ae^{-q(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})}\right|$$

$$< \frac{1}{2}\left|e^{p(re^{-i\frac{\theta_d}{d}})}\right| = \frac{1}{2}\left|\frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(re^{-i\frac{\theta_d}{d}}+j)e^{q_j^*(re^{-i\frac{\theta_d}{d}})}}{h(re^{-i\frac{\theta_d}{d}})e^{q_0^*(re^{-i\frac{\theta_d}{d}})}}\right|$$

$$\le \frac{1}{2}\exp\{r^{\rho_1+\varepsilon_2}\}\sum_{j=0}^{n} C_n^j \left|h(re^{-i\frac{\theta_d}{d}}+j)e^{q_j^*(re^{-i\frac{\theta_d}{d}})-q_0^*(re^{-i\frac{\theta_d}{d}})}\right|$$

$$\le 2^{n-1}\exp\{2r^{\rho_1+\varepsilon_2}\}\exp\{r^{d-1+\varepsilon_2}\} < \exp\{r^{d-\frac{1}{2}}\},$$

which is impossible.

Now we prove that $p_d = q_d$. By (1.1), we obtain

$$\frac{\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{q_j^*(z)}}{h(z)e^{q_0^*(z)}} = e^{p_d z^d}\left(e^{p^*(z)} - \frac{a}{h(z)e^{q_0^*(z)}}\right).$$

Considering the order of each side in the equation above, we get

$$e^{p^*(z)} - \frac{a}{h(z)e^{q_0^*(z)}} \equiv 0. \tag{1.7}$$

This gives

$$\sum_{j=0}^{n}(-1)^{n-j}C_n^j h(z+j)e^{q_j^*(z)} = 0. \tag{1.8}$$

From (1.7), $h(z)$ has no zeros, and thus it must be a constant function. Since $d = l = \rho(g) > \lambda(g) + 1$, we have $d \ge 2$. Therefore, for $0 \le j < k \le n, \deg(q_j^*(z) - q_k^*(z)) \ge 1$. Applying Lemma C to (1.8), we get a contradiction that $(-1)^{n-j}C_n^j = 0,\ j = 0, 1, \ldots, n$. Hence, we prove that $1 \le \rho(f) \le \lambda(f-a) + 1$.

Finally, applying the Hadamard factorization theorem, we can complete the proof of Theorem 1.4.

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References

[1] Z. X. Chen, The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$ where the order of $Q = 1$, Science in China (Ser.A) **45** (2002), 290–300.

[2] S. Li and Z. S. Gao, A note on the Brück Conjecture, Arch. Math. **95** (2010), 257–268.

[3] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Kluwer Academic publishers, The Newtherlands, 2003.

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