Arch. Math. 92 (2009), 355–365

0003/889X/040355-11, *published online* 2009-04-10
DOI 10.1007/s00013-009-3098-1

Archiv der Mathematik

On normalized Ricci flow and smooth structures on four-manifolds with $b^+ = 1$

Masashi Ishida, Rareş Răsdeaconu, and Ioana Şuvaina

Abstract. We find an obstruction to the existence of non-singular solutions to the normalized Ricci flow on four-manifolds with $b^+ = 1$. By using this obstruction, we study the relationship between the existence or non-existence of non-singular solutions of the normalized Ricci flow and exotic smooth structures on the topological 4-manifolds $\mathbb{C}P^2 \# \ell \overline{\mathbb{C}P^2}$, where $5 \leq \ell \leq 8$.

Mathematics Subject Classification (2000). Primary 53C21; Secondary 57R55.

Keywords. Normalized Ricci flow, exotic smooth structures, Seiberg-Witten invariant.

1. Introduction. Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$. The normalized Ricci flow on X is the following evolution equation:

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}_g + \frac{2}{n}\left(\frac{\int_X s_g d\mu_g}{\int_X d\mu_g}\right) g,$$

where Ric_g, s_g are the Ricci curvature and the scalar curvature of the evolving Riemannian metric g and $d\mu_g$ is the volume measure with respect to g. Recall that a one-parameter family of metrics $\{g(t)\}$, where $t \in [0, T)$ for some $0 < T \leq \infty$, is called a solution to the normalized Ricci flow if it satisfies the above equation at all $x \in X$ and $t \in [0, T)$. A solution $\{g(t)\}$ on a time interval $[0, T)$ is said to be maximal if it cannot be extended past time T. In this paper we are interested in solutions which are particularly nice. The following definition was first introduced and studied by Hamilton [11, 6]:

Definition 1. A maximal solution $\{g(t)\}$, $t \in [0, T)$, to the normalized Ricci flow on X is called non-singular if $T = \infty$ and the Riemannian curvature tensor $\mathrm{Rm}_{g(t)}$ of $g(t)$ satisfies $\sup_{X \times [0,\infty)} |\mathrm{Rm}_{g(t)}| < \infty$.

Fang and his collaborators [9] pointed out that for a 4-manifold with negative Perelman invariant [23, 14], which is equivalent to the Yamabe invariant in this situation [2], the existence of non-singular solutions forces a topological constraint on the 4-manifold. On the other hand, the first author proved [12] that, in dimension four, the existence of non-singular solutions is fundamentally related to the smooth structure considered. An important ingredient in his theorems was the non-triviality of the Seiberg-Witten invariant. In the case when the underlying manifold has $b^+ \geq 2$, this is a diffeomorphism invariant. However, when $b^+ = 1$ the invariant depends on the choice of an orientation on $H^2(X, \mathbb{Z})$ and $H^1(X, \mathbb{R})$. The obstructions in [12, 13] are for manifolds with $b^+ \geq 2$. We extend these results to the case $b^+ = 1$ as follow:

Theorem A. *Let X be a closed oriented smooth 4-manifold with $b^+(X) = 1$ and $2\chi(X) + 3\tau(X) > 0$. Assume that X has a non-trivial Seiberg-Witten invariant. Then, there do not exist non-singular solutions to the normalized Ricci flow on $X\#k\overline{\mathbb{C}P^2}$ if*

$$k > \frac{1}{3}\left(2\chi(X) + 3\tau(X)\right).$$

By using Theorem A, we study manifolds with small topology and emphasize how the change of the smooth structure reflects on the existence or non-existence of solutions of the normalized Ricci flow:

Theorem B. *For $5 \leq \ell \leq 8$, the topological 4-manifold $M := \mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$ satisfies the following properties:*

1. *M admits a smooth structure of positive Yamabe invariant on which there exists a non-singular solution to the normalized Ricci flow.*
2. *M admits a smooth structure of negative Yamabe invariant on which there exist non-singular solutions to the normalized Ricci flow.*
3. *M admits infinitely many distinct smooth structures all of which have negative Yamabe invariant and on which there are no non-singular solutions to the normalized Ricci flow for any initial metric.*

2. Polarized 4-manifolds and Seiberg-Witten invariants. Let X be a closed oriented smooth 4-manifold. Any Riemannian metric g on X gives rise to a decomposition $H^2(X, \mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$, where $\mathcal{H}_g^+$, $\mathcal{H}_g^-$ consists of cohomology classes for which the harmonic representative is g-self-dual or g-anti-self-dual, respectively. Notice that $b^+(X) := \dim\mathcal{H}_g^+$ is a non-negative integer which is independent of the metric g. In this article, we always assume that $b^+(X) \geq 1$ and mainly consider the case when $b^+(X) = 1$. For a fixed $b^+(X)$-dimensional subspace $H \subset H^2(M, \mathbb{R})$ on which the intersection form is positively defined, we consider the set of all Riemannian metrics g for which $\mathcal{H}_g^+ = H$ is satisfied. The Riemannian metric g satisfying this property is called a H-adapted metric. Under the assumption that there is at least one H-adapted metric, H is called a polarization of X and we call (X, H) a polarized 4-manifold following [19]. For any given element $\alpha \in H^2(X, \mathbb{R})$ and a polarization H of X, we use α^+ to denote the orthogonal projection of α, with respect to the intersection form of X, on the polarization H. For any polarized 4-manifold (X, H), we can define a differential topological invariant [32, 19] of (X, H), by using Seiberg-Witten monopole equations [32]. We briefly recall the definition, referring to [32, 19] for more details. Let $\mathfrak{s}$ be a spin^c structure of the polarized 4-manifold (X, H). Let $c_1(\mathcal{L}_\mathfrak{s}) \in H^2(X, \mathbb{R})$ be the first Chern class of the complex line bundle $\mathcal{L}_\mathfrak{s}$ associated to $\mathfrak{s}$. Suppose that $d_\mathfrak{s} := (c_1^2(\mathcal{L}_\mathfrak{s}) - 2\chi(X) - 3\tau(X))/4 = 0$, which forces the virtual dimension of the Seiberg-Witten moduli space to be zero. Let g be a H-adapted metric and assume that $c_1^+(\mathcal{L}_\mathfrak{s}) \neq 0$ with respect to $H = \mathcal{H}_g^+$ is satisfied. Then [32, 19], the Seiberg-Witten invariant $SW_X(\mathfrak{s}, H)$ is defined to be the number of solutions of a generic perturbation of the Seiberg-Witten monopole equations, modulo gauge transformation and counted with orientations. We can still define the Seiberg-Witten invariant of (X, H) for any spin^c structure $\mathfrak{s}$ for which $c_1^+(\mathcal{L}_\mathfrak{s}) \neq 0$ and $d_\mathfrak{s}$ is even and positive. In this case, $SW_X(\mathfrak{s}, H)$ is defined as the pairing $< \eta^{\frac{d_\mathfrak{s}}{2}}, [\mathcal{M}_\mathfrak{s}] >$. Here η is the first Chern class of the based moduli space as a S^1-bundle over the Seiberg-Witten moduli space $\mathcal{M}_\mathfrak{s}$ and $[\mathcal{M}_\mathfrak{s}]$ is the fundamental homology class of $\mathcal{M}_\mathfrak{s}$. Hence, the Seiberg-Witten invariant $SW_X(\mathfrak{s}, H)$ of a polarized 4-manifold (X, H) is well-defined for any spin^c structure $\mathfrak{s}$ with $c_1^+(\mathcal{L}_\mathfrak{s}) \neq 0$. Moreover, it is known [21] that $SW_X(\mathfrak{s}, H)$ is independent of the choice of the polarization H if $b^+(X) \geq 2$, or $b^+(X) = 1$ and $2\chi(X) + 3\tau(X) > 0$.

One of the crucial properties of the Seiberg-Witten invariants above is that the non-triviality of the value $SW_X(\mathfrak{s}, H)$ for a spin^c structure $\mathfrak{s}$ with $c_1^+(\mathcal{L}_\mathfrak{s}) \neq 0$ forces the existence of a non-trivial solution of the Seiberg-Witten monopole equations for any H-adapted metric. Using this, LeBrun [17, 20, 19] proved

Theorem 1 **(**[17, 20]**).** *Let (X, H) be a polarized smooth compact oriented 4-manifold and let $\mathfrak{s}$ be a spin^c structure of X and let $c_1^+ \neq 0$ be the orthogonal projection to H with respect to the intersection form of X. Assume that $SW_X(\mathfrak{s}, H) \neq 0$. Then, every H-adapted metric g satisfies the following bounds:*

$$\int_X s_g^2 d\mu_g \geq 32\pi^2 (c_1^+)^2,$$

$$\frac{1}{4\pi^2} \int_X \left(2|W_g^+|^2 + \frac{s_g^2}{24} \right) d\mu_g \geq \frac{2}{3} (c_1^+)^2.$$

s_g is the scalar curvature of g and W_g^+ is the self-dual Weyl curvature of g.

On the other hand, let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$ and $k = b_2(N)$. By the celebrated result of Donaldson [8], there are classes $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_k \in H^2(N, \mathbb{Z})$ descending to a basis of $H^2(N, \mathbb{Z})$/torsion with respect to which the intersection form is diagonal and $\mathfrak{e}_i^2 = -1$ for all i. An element $\beta \in H^2(N, \mathbb{Z})$ is called characteristic if the intersection number $\beta \cdot x \equiv x \cdot x \pmod 2$.

If β is characteristic, then $\beta \equiv w_2(N) \pmod 2$ and moreover there is a spin^c structure $\mathfrak{t}$ on N such that $c_1(\mathcal{L}_{\mathfrak{t}}) = \beta$. Then modulo torsion, β can be written as $\sum_{i=1}^{k} a_i \mathfrak{e}_i$, where a_i are integers. Let $x := \sum_{i=1}^{k} x_i \mathfrak{e}_i$, where x_i are integers. Then we have $\beta \cdot x = -\sum_{i=1}^{k} a_i x_i$ and $x \cdot x = -\sum_{i=1}^{k} x_i^2$. This tells us that β is characteristic if and only if the a_i are odd integers, where $i = 1, \cdots, k$. For example, we can obtain characteristic elements by taking $a_i = \pm 1$. The following result includes Lemma 1 of [19] as a special case.

Proposition 2. *Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 1$ and $2\chi(X) + 3\tau(X) > 0$. Moreover, suppose that the Seiberg-Witten invariant of X is non-trivial. Let N be a closed oriented smooth 4-manifold with $b_1(N) = b^+(N) = 0$. Let H be any polarization of a connected sum $M := X\#N$. Then there is a spin^c structure $\mathfrak{s}$ on M such that $SW_X(\mathfrak{s}, H) \neq 0$ and the self-dual part c_1^+ of the first Chern class of the complex line bundle associated with $\mathfrak{s}$ satisfies*

$$(c_1^+)^2 \geq 2\chi(X) + 3\tau(X). \tag{1}$$

Proof. Notice that the Seiberg-Witten invariant of X is well-defined and independent of the choice of the polarization under the assumption that $b^+(X) \geq 2$ or $b^+(X) = 1$ and $2\chi(X) + 3\tau(X) > 0$. Suppose that $\mathfrak{c}$ is the spin^c structure on X with non-trivial Seiberg-Witten invariant. Let $\alpha := c_1(\mathcal{L}_{\mathfrak{c}}) \in H^2(X, \mathbb{Z})$ be the first Chern class of the complex line bundle $\mathcal{L}_{\mathfrak{c}}$ associated to $\mathfrak{c}$. Then, the non-triviality of Seiberg-Witten invariant forces the dimension $d_{\mathfrak{c}}$ of the Seiberg-Witten moduli space to be non-negative and we therefore have $\alpha^2 \geq 2\chi(X) + 3\tau(X) > 0$. Moreover, since for any given polarization of X, we have $(\alpha^+)^2 \geq \alpha^2$, we obtain

$$(\alpha^+)^2 \geq 2\chi(X) + 3\tau(X). \tag{2}$$

Let $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_n \in H^2(N, \mathbb{Z})$ be cohomology classes descending to a basis of $H^2(N, \mathbb{Z})$/torsion with respect to which the intersection form is diagonal, where $n = b_2(N)$. Let H be a polarization of the connected sum $M := X\#N$. Choose new generators $\hat{\mathfrak{e}}_i = \pm \mathfrak{e}_i$ for $H^2(N, \mathbb{Z})$ such that

$$\alpha^+ \cdot (\hat{\mathfrak{e}}_i)^+ \geq 0 \tag{3}$$

with respect to the polarization H. Then $\sum_{i=1}^{n} \hat{\mathfrak{e}}_i$ is a characteristic class and there is a spin^c structure $\mathfrak{t}$ on N such that $c_1(\mathcal{L}_{\mathfrak{t}}) = \sum_{i=1}^{n} \hat{\mathfrak{e}}_i$. Notice also that we have $c_1^2(\mathcal{L}_{\mathfrak{t}}) = (\sum_{i=1}^{n} \hat{\mathfrak{e}}_i)^2 = -n = -b_2(N)$.

Consider the spin^c structure $\mathfrak{s} := \mathfrak{c}\#\mathfrak{t}$ on the connected sum $M := X\#N$. The first Chern class of the complex line bundle associated with $\mathfrak{s}$ satisfies $c_1(\mathcal{L}_{\mathfrak{s}}) = \alpha + c_1(\mathcal{L}_{\mathfrak{t}}) = \alpha + \sum_{i=1}^{n} \hat{\mathfrak{e}}_i$, where we are using the same notations, $\alpha, c_1(\mathcal{L}_{\mathfrak{t}}), \hat{\mathfrak{e}}_i$, to denote the induced cohomology classes in $H^2(M, \mathbb{Z})$. The gluing construction for the solutions of the Seiberg-Witten monopole equations on $M := X\#N$, as in the proof of Theorem 3.1 in [28] (see also the proof of Proposition 2 in [16] for $b^+ \geq 2$), tells us that $SW_M(\mathfrak{s}, H) \neq 0$. On the other hand, we obtain the following bound on the square $(c_1^+)^2$ of the orthogonal projection c_1^+ of $c_1(\mathcal{L}_\mathfrak{s})$ into the polarization H:

$$\begin{aligned} (c_1^+)^2 &= \left(\alpha^+ + \sum_{i=1}^{n} (\hat{\mathfrak{e}}_i)^+\right)^2 = (\alpha^+)^2 + 2\sum_{i=1}^{n}(\alpha^+ \cdot (\hat{\mathfrak{e}}_i)^+) + \sum_{i=1}^{n}((\hat{\mathfrak{e}}_i)^+)^2 \\ &\geq (\alpha^+)^2, \end{aligned}$$

where we used (3). This bound and (2) implies the desired bound (1). □

3. Asymptotic behavior of the Ricci curvature of non-singular solutions. Let us recall the following result on the trace free part of the Ricci curvature of a long time solution of the normalized Ricci flow.

Lemma 3 ([9, 12])**.** *Let X be a closed oriented Riemannian n-manifold and assume that there is a long time solution $\{g(t)\}$, $t \in [0, \infty)$, to the normalized Ricci flow. Assume moreover that the solution satisfies $|s_{g(t)}| \leq C$ and*

$$\hat{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) \leq -c < 0, \tag{4}$$

where the constants C and c are independent of both $x \in X$ and time $t \in [0, \infty)$. Then, the trace-free part $\overset{\circ}{r}_{g(t)}$ of the Ricci curvature satisfies

$$\int_{m}^{m+1} \int_{X} |\overset{\circ}{r}_{g(t)}|^2 d\mu_{g(t)} dt \longrightarrow 0 \tag{5}$$

when $m \to \infty$.

On the other hand, there is a natural diffeomorphism invariant arising from a variational problem of the total scalar curvature of Riemannian metrics on any given closed oriented Riemannian manifold X of dimension $n \geq 3$. As was conjectured by Yamabe [33], and later proved by Trudinger, Aubin, and Schoen [4, 26, 31], every conformal class on any smooth compact manifold contains a Riemannian metric of constant scalar curvature. To be more precise, for any conformal class $[g] = \{vg \mid v : X \to \mathbb{R}^+\}$, we can consider an associated number $Y_{[g]}$, which is called the Yamabe constant of the conformal class $[g]$, and is defined by

$$Y_{[g]} = \inf_{h \in [g]} \frac{\int_X s_h \, d\mu_h}{\left(\int_X d\mu_h\right)^{\frac{n-2}{n}}},$$

where $d\mu_h$ is the volume form with respect to the metric h. It is known [4, 26, 31] that this number is realized as the constant scalar curvature of some metric in the conformal class $[g]$. Then, Kobayashi [15] and Schoen [27] independently introduced an invariant of X by considering $\mathcal{Y}(X) := \sup_{[g] \in \mathcal{C}} Y_{[g]}$, where $\mathcal{C}$ is the set of all conformal classes on X. This is now known as the Yamabe invariant of X. We have the following bound:

Lemma 4 ([12])**.** *Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$ and assume that the Yamabe invariant of X is negative, i.e., $\mathcal{Y}(X) < 0$. If there is a solution $\{g(t)\}$, $t \in [0,T)$, to the normalized Ricci flow, then the solution satisfies the bound (4). More precisely, the following is satisfied:*

$$\hat{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) \leq \frac{\mathcal{Y}(X)}{(\mathrm{vol}_{g(0)})^{2/n}} < 0.$$

We recall now the following definition:

Definition 2 ([12])**.** A maximal solution $\{g(t)\}$, $t \in [0,T)$, to the normalized Ricci flow on X is called quasi-non-singular if $T = \infty$ and the scalar curvature $s_{g(t)}$ of $g(t)$ satisfies $\sup_{X \times [0,\infty)} |s_{g(t)}| < \infty$.

Notice that any non-singular solution is quasi-non-singular.

Proposition 5. *Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 1$ and $2\chi(X) + 3\tau(X) > 0$. Assume that X has a non-trivial Seiberg-Witten invariant. Let N be a closed oriented smooth 4-manifold with $b_1(N) = b^+(N) = 0$. If there is a quasi-non-singular solution to the normalized Ricci flow on the connected sum $M := X \# N$, then the trace-free part $\overset{\circ}{r}_{g(t)}$ of the Ricci curvature satisfies (5) when $m \to +\infty$.*

Proof. First of all, notice that the connected sum M has non-trivial Seiberg-Witten invariant with respect to any polarization by Proposition 2. By Witten's vanishing theorem [32], this implies that M cannot admit any metric of positive scalar curvature. On the other hand, it is known [18] that the Yamabe invariant of any closed n-manifold Z which cannot admit metrics of positive scalar curvature is given by

$$\mathcal{Y}(Z) = -\left(\inf_{g \in \mathcal{R}_Z} \int_Z |s_g|^{n/2} d\mu_g \right)^{2/n}, \tag{6}$$

where $\mathcal{R}_Z$ is the set of all Riemannian metrics on Z. Combining the first inequality in Theorem 1 with the inequality in Proposition 2 we get the following bound:

$$\int_M s_g^2 d\mu_g \geq 32\pi^2 \left(2\chi(X) + 3\tau(X)\right). \tag{7}$$

Note that this bound holds for any H-adapted metric on M, where H is any polarization of M. In particular, it holds for any metric g. Therefore, (6) and (7) imply $\mathcal{Y}(M) \leq -4\pi\sqrt{2(2\chi(X) + 3\tau(X))} < 0$. This bound and Lemma 4 tell us that any solution to the normalized Ricci flow on M satisfies

$$\hat{s}_{g(t)} \leq -4\pi \sqrt{\frac{2(2\chi(X) + 3\tau(X))}{\mathrm{vol}_{g(0)}}} < 0.$$

This inequality combined with Lemma 3 shows that any quasi-non-singular solution to the normalized Ricci flow on M must satisfy (5) when $m \to +\infty$. □

4. Proof of Theorem A. We are now in the position to prove Theorem A, which is a special case of the following result:

Theorem 6. *Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 1$ and $2\chi(X)+3\tau(X) > 0$. Assume that X has a non-trivial Seiberg-Witten invariant. Let N be a closed oriented smooth 4-manifold with $b_1(N) = b^+(N) = 0$. Then, there do not exist quasi-non-singular solutions to the normalized Ricci flow on $X\#N$ if $3b_2(N) > 2\chi(X) + 3\tau(X)$ holds. In particular, there is no non-singular solution to the normalized Ricci flow.*

Proof. Suppose that there would be a quasi-non-singular solution $\{g(t)\}$ to the normalized Ricci flow on the connected sum $M := X\#N$. Then the second inequality in Theorem 1 tells us that, for any time t, $g(t)$ must satisfy

$$\frac{1}{4\pi^2}\int_X \left(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24}\right) d\mu_{g(t)} \quad \geq \quad \frac{2}{3}(c_1^+)^2.$$

for any spin^c structure $\mathfrak{s}$ with $SW_M(\mathfrak{s}, H) \neq 0$, where $H := \mathcal{H}^+_{g(t)}$. However, Proposition 2 now asserts that the connected sum $M := X\#N$ has a spin^c structure with $(c_1^+)^2 \geq 2\chi(X) + 3\tau(X)$. We therefore conclude that

$$\frac{1}{4\pi^2}\int_M \left(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24}\right) d\mu_{g(t)} \geq \frac{2}{3}\left(2\chi(X) + 3\tau(X)\right). \tag{8}$$

On the other hand, we have the following Gauss-Bonnet like formula:

$$2\chi(M) + 3\tau(M) = \frac{1}{4\pi^2}\int_M \left(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24} - \frac{|\overset{\circ}{r}_{g(t)}|^2}{2}\right) d\mu_{g(t)}.$$

In particular, we obtain

$$\begin{aligned} 2\chi(M) + 3\tau(M) &= \int_m^{m+1} \left(2\chi(M) + 3\tau(M)\right) dt \\ &= \frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24} - \frac{|\overset{\circ}{r}_{g(t)}|^2}{2}\right) d\mu_{g(t)}dt. \end{aligned}$$

Since Proposition 5 tells us that a quasi-non-singular solution $\{g(t)\}$ must satisfy (5), by taking $m \longrightarrow \infty$ in the above inequality, we obtain

$$2\chi(M) + 3\tau(M) = \lim_{m \longrightarrow \infty} \frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24}\right) d\mu_{g(t)}dt. \tag{9}$$

Moreover, by the inequality (8), we get

$$\begin{aligned} \frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(2|W^+_{g(t)}|^2+\frac{s^2_{g(t)}}{24}\right) d\mu_{g(t)}dt &\geq \frac{2}{3}\int_m^{m+1}(2\chi(X)+3\tau(X))\,dt \\ &= \frac{2}{3}\left(2\chi(X)+3\tau(X)\right). \end{aligned}$$

This bound and (9) tell us that the following holds:

$$3\left(2\chi(M)+3\tau(M)\right)\geq 2\left(2\chi(X)+3\tau(X)\right).$$

Since we have $2\chi(M)+3\tau(M)=2\chi(X)+3\tau(X)-b_2(N)$, we get $3(2\chi(X)+3\tau(X)-b_2(N))\geq 2(2\chi(X)+3\tau(X))$, which is equivalent to $3b_2(N)\leq 2\chi(X)+3\tau(X)$. By contraposition, we obtain the desired result. □

5. Proof of Theorem B. Part of the proof of Theorem B is based on the existence results for simply connected manifolds with $b^+=1$ and ample canonical line bundle. Such examples have only been recently found, and we state the results we need for convenience:

Theorem 7 ([7, 25, 22]). *There exist simply connected complex surfaces of general type, with $b^+=1$, $c_1^2=1,2,3$ or 4 and ample canonical bundle.*

Proof of Theorem B. If we want to construct a smooth structure on the manifold X which has positive Yamabe invariant and admits non-singular solutions for the normalized Ricci flow, then we can just consider the canonical smooth and complex structures of the complex projective plane blown-up at l points, where $3\leq l\leq 8$. The existence of an Einstein metric is given by a famous result of Tian [30]. Hence, on these Del Pezzo surfaces, there are non-singular solutions (fixed points) of the normalized Ricci flow by taking the Kähler-Einstein metrics with positive scalar curvature as initial metrics. Since the scalar curvature of these metrics is positive, Lemma 1.5 in [15] tells us that the Yamabe invariants of these manifolds must be also positive.

In the second case of the theorem, we are going to consider the smooth structures associated to the complex structures of general type found in Theorem 7. On these manifolds, a nice result of Cao [5, 6] tells us that non-singular solutions to the normalized Ricci flow exist if we start with a Kähler metric whose Kähler form is in the cohomology class of the canonical line bundle. Moreover, for surfaces of general type the Yamabe invariant is strictly negative [18].

For the proof of the third part of the theorem, we use the exotic structures on manifolds with small topology constructed by Akhmedov and Park [1]. In Section 6 of [1], they exhibit infinitely many smooth structures, X_i, on the topological space $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. The smooth structures are distinguished by their Seiberg-Witten invariants. On $M:=\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}, 5\leq\ell\leq 8$, we consider the smooth structures $M_i:=X_i\#(\ell-2)\overline{\mathbb{C}P^2}, i\in\mathbb{N}$. Each M_i is homeomorphic to $\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$ as each

X_i is homeomorphic to $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. Moreover, we know that the Seiberg-Witten invariants of the manifolds X_i are non-trivial and distinct. We can use the gluing formula for the connected sum with copies of $\overline{\mathbb{C}P^2}$ to conclude that infinitely many Seiberg-Witten invariants of M_i remain distinct. Hence on M, we have constructed infinitely many distinct smooth structures. As $c_1^2(M_i) = 9 - l > 0$, the Yamabe invariant of each smooth structure is negative [18]. The manifolds X_i have $c_1^2(X_i) = 7$, non-trivial Seiberg-Witten invariant by construction and of course $3(\ell - 2) > c_1^2(X_i) = 7$, as $5 \leq \ell \leq 8$. Hence, Theorem A tells us that there are no non-singular solutions to the normalized Ricci flow on any M_i for any initial metric. □

Acknowledgements. We would like to express our deep gratitude to Claude LeBrun for his warm encouragement. The first author is partially supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 20540090. The second author would like to thank CNRS and IRMA Strasbourg for support and the excellent conditions provided while this work was completed.

References

[1] A. Akhmedov and B. Park, Exotic smooth structures on small 4-manifolds with-odd signatures, arXiv:math/0701829

[2] K. Akutagawa, M. Ishida, and C. LeBrun, Perelman's invariant, Ricci flow, and the Yamabe invariants of smooth manifolds, Arch. Math. **88**, 71–76 (2007).

[3] T. Aubin, Equations du type Monge-Ampère sur les variétés Kählériennes compactes, C. R. Acad. Sci. Paris. **283**, 119–121 (1976).

[4] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. (9) **55**, 269–296 (1976).

[5] H.-D. Cao, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. math. **81**, 359–372 (1985).

[6] H.-D. Cao and B. Chow, Recent developments on the Ricci flow, Bull. Amer. Math. Soc. (N.S) (1) **36**, 59–74 (1995).

[7] F. Catanese and C. LeBrun, On the scalar curvature of Einstein manifolds, Math. Res. Lett. **4**, 843–854 (1997).

[8] S. K. Donalsdon, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Differential. Geom. **26**, 397–428 (1987).

[9] F. Fang, Y. Zhang, and Z. Zhang, Non-singular solutions to the normalized Ricci flow equation, Math. Ann. **340**, 647–674 (2008).

[10] M. Freedman, On the topology of 4-manifolds, J. Diff. Geom. **17**, 357–454 (1982).

[11] R. Hamilton, Non-singular solutions of the Ricci flow on three-manifolds, Comm. Anal. Geom. **7**, 695–729 (1999).

[12] M. Ishida, The normalized Ricci flow on four-manifolds and exotic smooth structures, arXiv:0807.2169 [math.DG] (2008).

[13] M. Ishida and I. Şuvaina, Smooth structures and normalized Ricci flows on non-simply connected four-manifolds, to appear in Ann. Anal. Geom., arXiv:0808.0316 [math.DG] (2008).

[14] B. Kleiner and J. Lott, Notes on Perelman's papers. arXiv: math.DG/0605667 (2006).

[15] O. Kobayashi, Scalar curvature of a metric of unit volume, Math. Ann. **279**, 253–265 (1987).

[16] D. Kotschick, J. W. Morgan, and C. H. Taubes, Four-manifolds without symplectic structures but with non-trivial Seiberg-Witten invariants, Math. Res. Lett. **2**, 119–124 (1995).

[17] C. LeBrun, Polarized 4-manifolds, extremal Kähler metrics, and Seiberg-Witten theory, Math. Res. Lett. **2**, 653–662 (1995).

[18] C. LeBrun, Four-manifolds without Einstein metrics, Math. Res. Lett. **3**, 137–147 (1996).

[19] C. LeBrun, Weyl curvature, Einstein metrics, and Seiberg-Witten theory, Math. Res. Lett. **5**, 423–438 (1998).

[20] C. LeBrun, Ricci curvature, minimal volumes, and Seiberg-Witten theory, Invent. Math. **145**, 279–316 (2001).

[21] J. W. Morgan, The Seiberg-Witten equations and applications to the topology of smooth four-manifolds, Mathematical Notes, 44. Princeton University Press, Princeton, NJ, 1996.

[22] H. Park, J. Park, and D. Shin, A simply connected surface of general type with $p_g = 0$ and $K^2 = 4$, arXiv:0803.3667v3 [math.AG] (2008).

[23] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, math.DG/0211159 (2002).

[24] G. Perelman, Ricci flow with surgery on three-manifolds, math.DG/0303109 (2003).

[25] R. Răsdeaconu and I. Şuvaina, Smooth structures and Einstein metrics on $\mathbb{C}P^2\#5, 6, 7\overline{\mathbb{C}P^2}$, to appear in Math. Proc. Cambridge Philos. Soc., arXiv:0806.1424 [math.DG].

[26] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. **20**, 478–495 (1984).

[27] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Lec. Notes Math. **1365**, 120–154 (1987).

[28] C. Sung, Surgery, Yamabe invariant, and Seiberg-Witten theory, arXiv:0710.2375v1 [math.DG].

[29] C. H. Taubes, The Seiberg-Witten invariants and symplectic forms. Math. Res. Lett. **1**, 809–822 (1994).

[30] G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, Invent. Math. **101**, 101–172 (1990).

[31] N. Trudinger, Remarks concerning the conformal deformation of metrics to constant scalar curvature, Ann. Scuola Norm. Sup. Pisa, **22**, 265–274 (1968).

[32] E. Witten, Monopoles and four-manifolds. Math. Res. Lett. **1**, 809–822 (1994).

[33] H. Yamabe, On the deformation of Riemannian structures on compact manifolds, Osaka Math. J. **12**, 21–37 (1960).

[34] S.-T. Yau, Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. USA. **74**, 1789–1799 (1977).

Masashi Ishida, Department of Mathematics, Sophia University, 7-1 Kioi-Cho, Chiyoda-Ku, Tokyo 102-8554, Japan
e-mail: ishida@mm.sophia.ac.jp

Rareş Răsdeaconu, Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, Jerusalem, 91904, Israel
e-mail: rares@math.huji.ac.il

Ioana Şuvaina, Courant Institute of Mathematical Sciences, 251 Mercer St. New York 10012, USA
e-mail: ioana@cims.nyu.edu

Received: 13 November 2008