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Factor principal congruences and Boolean products in filtral varieties

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Dedicated to the memory of Tibor Katriňák and George McNulty.

Abstract. Motivated by Haviar and Ploščica's 2021 characterisation of Boolean products of simple De Morgan algebras, we investigate Boolean products of simple algebras in filtral varieties. We provide two main theorems. The first yields Werner's Boolean-product representation of algebras in a discriminator variety as an immediate application. The second, which applies to algebras in which the top congruence is compact, yields a generalisation of the Haviar–Ploščica result to semisimple varieties of Ockham algebras. The property of having factor principal congruences is fundamental to both theorems. While major parts of our general theorems can be derived from results in the literature, we offer new, self-contained and essentially elementary proofs.

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1. Introduction

In this section we give an overview of the genesis and the results of the paper.

Our results have their roots in a theorem by Haviar and Ploščica [16] on Boolean-product representations of De Morgan algebras. Let **A** be a De Morgan algebra and let **B**(**A**) be its Boolean skeleton. Haviar and Ploščica proved that the natural restriction map is an isomorphism between Con **A** and Con **B**(**A**) if and only if **A** is a Boolean product of the three simple De Morgan algebras—the two- and three-element chains and the four-element non-Boolean De Morgan algebra. (They also gave a characterisation of such algebras in

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terms of their natural-duality dual space.) The Boolean skeleton $\mathbf{B}(\mathbf{A})$ of $\mathbf{A}$ is isomorphic to the Boolean algebra $\mathrm{Con}_{\mathrm{f}}\,\mathbf{A}$ of factor congruences on $\mathbf{A}$ and consequently $\mathrm{Con}\,\mathbf{B}(\mathbf{A})$ is isomorphic to the lattice $\mathrm{Id}(\mathrm{Con}_{\mathrm{f}}\,\mathbf{A})$ of ideals of $\mathrm{Con}_{\mathrm{f}}\,\mathbf{A}$. Thus the Haviar–Ploščica result could be thought of as saying: *a De Morgan algebra* $\mathbf{A}$ *satisfies* $\mathrm{Con}\,\mathbf{A} \cong \mathrm{Id}(\mathrm{Con}_{\mathrm{f}}\,\mathbf{A})$ *if and only if* $\mathbf{A}$ *is isomorphic to a Boolean product of simple De Morgan algebras.*

Recall that a variety $\boldsymbol{\mathcal{V}}$ is *semisimple* if every subdirectly irreducible algebra in $\boldsymbol{\mathcal{V}}$ is simple, and is *sub-semisimple* if every non-trivial subalgebra of every subdirectly irreducible algebra in $\boldsymbol{\mathcal{V}}$ is simple. Since the variety $\boldsymbol{\mathcal{M}}$ of De Morgan algebras is finitely generated, sub-semisimple and congruence distributive, it is a filtral variety (meaning that every congruence on every subdirect product of subdirectly irreducible algebras from $\boldsymbol{\mathcal{M}}$ arises in a natural way from a filter on the index set of the product)—see Section 4, and Proposition 4.1(3) in particular. Note that 1_A is compact in $\mathrm{Con}\,\mathbf{A}$, for every De Morgan algebra $\mathbf{A}$; indeed, $1_A = \mathrm{Cg}^{\mathbf{A}}(0,1)$.

A setting in which *every* algebra is isomorphic to a Boolean product of simple algebras (with possibly one trivial stalk) is that of discriminator varieties: this is Werner's fundamental Boolean-product representation theorem for algebras in a discriminator variety (Werner [24, Thm 4.9]; see also Burris and Sankappanavar [5, Chapter IV, Thm 9.4]). Discriminator varieties are congruence distributive and sub-semisimple and their simple members form an elementary class (and are therefore closed under ultraproducts); see Subsection 2.2 below. Hence discriminator varieties are filtral; see the equivalence of (i) and (vi) in Proposition 4.1(2).

So a natural setting for a generalisation of both the Haviar–Ploščica theorem for De Morgan algebras and Werner's representation of algebras in a discriminator variety is to algebras in a filtral variety.

We present two theorems, one for algebras in which 1_A is not assumed to be compact in $\mathrm{Con}\,\mathbf{A}$, and a stronger one for algebras in which 1_A is compact. The first theorem yields Werner's Boolean-product representation theorem for discriminator varieties as an immediate application. The second theorem can be applied to extend the Haviar–Ploščica result from De Morgan algebras to every finitely generated semisimple variety of Ockham algebras.

We have attempted to make the paper as self-contained as possible. We give proofs, where they are short and illuminating, of some results that can be found elsewhere. The required universal-algebraic background is quite minimal and can be found, for example, in the texts by Bergman [1] and Burris and Sankappanavar [5].

The main theorems and the applications to discriminator varieties and to Ockham algebras are discussed in the next section. In Section 3 we investigate the relationship between compact, complemented and factor congruences, particularly in congruence-distributive algebras. Section 4 gives a brief summary of the results we need concerning filtral varieties. The natural duality for generalised Boolean algebras is essential to our results and is presented in Section 5. Boolean products are introduced in Section 6, where we also see how Boolean base spaces arise in filtral varieties. Section 7 presents four different representations for the congruence lattice of an algebra in a filtral variety and applies the representations to study complemented congruences. Finally, in Section 8 we present the proofs of our two main theorems.

It is our hope that this paper will serve as an introduction—indeed an invitation—to filtral varieties, to Boolean products and to Ockham algebras.

2. The theorems and their applications

Let $\mathbf{A}$ be an algebra. The congruence lattice of $\mathbf{A}$ is denoted by $\operatorname{Con}\mathbf{A}$, with top element 1_A and bottom 0_A. A congruence $\alpha \in \operatorname{Con}\mathbf{A}$ is a *factor congruence* if there exists $\beta \in \operatorname{Con}\mathbf{A}$ with $\alpha \cap \beta = 0_A$ and $\alpha \cdot \beta = 1_A$, and we then refer to (α, β) as a *pair of factor congruences*. This implies that $\alpha \vee \beta = 1_A$ in $\operatorname{Con}\mathbf{A}$, whence β is a complement of α in $\operatorname{Con}\mathbf{A}$. For a refresher on factor congruences, we refer forward to Section 3. We say that an algebra $\mathbf{A}$ has *factor principal congruences* if every principal congruence on $\mathbf{A}$ is a factor congruence. We denote the set of factor congruences on $\mathbf{A}$ by $\operatorname{Con}_{\mathrm{f}}\mathbf{A}$. For a congruence-distributive algebra $\mathbf{A}$, the set $\operatorname{Con}_{\mathrm{f}}\mathbf{A}$ is a sublattice of $\operatorname{Con}\mathbf{A}$ and forms a Boolean algebra (see Lemma 3.1). For a discussion of filtral varieties and Boolean products, we refer to Sections 4 and 6 respectively.

2.1. The main theorems

We now state abridged versions of our principal results. The difference between the versions stated here and the full versions in Section 8 is that the full versions include an item that gives a specific choice for the base space of the Boolean product.

Theorem 2.1 (See Theorem 8.1). *Let $\boldsymbol{\mathcal{V}}$ be a filtral variety and let $\mathbf{A}$ be a non-trivial algebra in $\boldsymbol{\mathcal{V}}$. The following are equivalent*:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) $\mathbf{A}$ *has permuting congruences*;
(3) $\mathbf{A}$ *is isomorphic to a Boolean product in which each non-trivial stalk is a simple algebra from $\boldsymbol{\mathcal{V}}$ and moreover at most one stalk is trivial.*

Theorem 2.2 (See Theorem 8.2). *Let $\boldsymbol{\mathcal{V}}$ be a filtral variety and let $\mathbf{A}$ be a non-trivial algebra in $\boldsymbol{\mathcal{V}}$ such that 1_A is compact in* $\operatorname{Con}\mathbf{A}$. *The following are equivalent*:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) $\mathbf{A}$ *has permuting congruences*;
(3) $\operatorname{Con}\mathbf{A}$ *is isomorphic to the lattice* $\operatorname{Id}(\operatorname{Con}_{\mathrm{f}}\mathbf{A})$ *of ideals of the Boolean algebra* $\operatorname{Con}_{\mathrm{f}}\mathbf{A}$ *of factor congruences on* $\mathbf{A}$ *via* $\psi\colon \alpha \mapsto {\downarrow}\alpha \cap \operatorname{Con}_{\mathrm{f}}\mathbf{A}$, *for all* $\alpha \in \operatorname{Con}\mathbf{A}$;
(4) $\mathbf{A}$ *is isomorphic to a Boolean product of simple algebras from $\boldsymbol{\mathcal{V}}$.*

Remark 2.3. The equivalence of the conditions in Theorem 2.1 (and of those in Theorem 8.1) follows from Theorem 4.3 of Campercholi and Vaggione [8]. The same is true of the equivalence of (1), (2) and (4) in Theorem 2.2 (and of (1), (2), (4) and (5) in Theorem 8.2). Campercholi and Vaggione's theorem applies to quasivarieties, but when applied to a variety $\boldsymbol{\mathcal{V}}$ their overriding assumptions become

(a) $\boldsymbol{\mathcal{V}}$ is semisimple,
(b) $\boldsymbol{\mathcal{V}}$ has restricted equationally definable principal congruences, and
(c) $\boldsymbol{\mathcal{V}}$ has equationally definable principal meets.

By Proposition 4.1(2) below, (a) and (b) hold if and only if $\boldsymbol{\mathcal{V}}$ is filtral. We will not define *equationally definable principal meets* but note that, by Blok and Pigozzi [3, Thm 1.5], $\boldsymbol{\mathcal{V}}$ has equationally definable principal meets if and only if $\boldsymbol{\mathcal{V}}$ is congruence distributive and the intersection of every pair of compact congruences on each algebra in $\boldsymbol{\mathcal{V}}$ is compact. It follows that every filtral variety has equationally definable principal meets; use (2) and (5)(i) of Proposition 4.1 below. Thus (c) follows from (a) and (b), and consequently their assumptions say exactly that $\boldsymbol{\mathcal{V}}$ is filtral.

Our Theorem 8.1 and the equivalence of (1), (2), (4) and (5) in Theorem 8.2 also follow from Vaggione [22, Thm 8.4], but it takes a little more work to see this. Our contribution is to provide new, elementary and self-contained proofs along with a new application to Ockham algebras.

2.2. An application to discriminator varieties

A variety $\boldsymbol{\mathcal{V}}$ is a *discriminator variety* if it has a *discriminator term*, that is, a ternary term t such that

$$t^{\mathbf{A}}(x, y, z) = \begin{cases} x, & \text{if } x \neq y, \\ z, & \text{if } x = y, \end{cases}$$

for every subdirectly irreducible algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{V}}$. We use Werner [24] as our standard reference on discriminator varieties; see also Burris and Sankappanavar [5, Chapter IV, §9]. (Unfortunately, Werner's excellent monograph [24] might be difficult for some readers to access.)

Let $\boldsymbol{\mathcal{V}}$ be a discriminator variety with discriminator term t. The following observations are very easily verified:

- The term t is a two-thirds minority term on every subdirectly irreducible algebra in $\boldsymbol{\mathcal{V}}$ and therefore $\boldsymbol{\mathcal{V}}$ satisfies
$$t(y, y, x) \approx t(x, y, x) \approx t(x, y, y) \approx x.$$
Hence, by a classic result of Pixley [20], $\boldsymbol{\mathcal{V}}$ is both congruence distributive and congruence permutable.
- $\boldsymbol{\mathcal{V}}$ is sub-semisimple and the simple algebras in $\boldsymbol{\mathcal{V}}$ form an elementary class and so are closed under ultraproducts.

Hence discriminator varieties are filtral. In addition, they are congruence permutable—indeed, by Fried and Kiss [15, Thm 4.16(a)], a variety is a discriminator variety if and only if it is filtral and congruence permutable.

An immediate application of Theorems 2.1 and 2.2 yields the representation theorem for discriminator varieties due to Werner [24, Thm 4.9] (see also Burris and Sankappanavar [5, Chapter IV, Thm 9.4]).

Theorem 2.4. *Let* $\mathbf{A}$ *be a non-trivial algebra in a discriminator variety* $\boldsymbol{\mathcal{V}}$*. Then* $\mathbf{A}$ *is isomorphic to a Boolean product in which each non-trivial stalk is a simple algebra from* $\boldsymbol{\mathcal{V}}$ *and moreover at most one stalk is trivial. If* 1_A *is compact in* $\operatorname{Con}\mathbf{A}$*, then* $\mathbf{A}$ *is isomorphic to a Boolean product of simple algebras from* $\boldsymbol{\mathcal{V}}$*.*

2.3. An application to Ockham algebras

Ockham algebras in general. An algebra $\mathbf{A} = \langle A; \vee, \wedge, f, 0, 1\rangle$ is an *Ockham algebra* if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $f\colon A \to A$ is a dual endomorphism of $\mathbf{A}^\flat$. We use the text by Blyth and Varlet [4] as our standard reference on Ockham algebras. It is straightforward to see that, given an Ockham algebra $\mathbf{A} = \langle A; \vee, \wedge, f, 0, 1\rangle$, the *Boolean skeleton*

$$B(\mathbf{A}) := \{\, a \in A \mid a \wedge f(a) = 0 \;\;\&\;\; a \vee f(a) = 1 \,\}$$

forms a Boolean subalgebra of $\mathbf{A}$.

Lemma 2.5. *Let* $\mathbf{A}$ *be a non-trivial Ockham algebra. For all* $a \in A$*, define an equivalence relation* θ_a *on* A *by*

$$(\forall x, y \in A) \quad (x, y) \in \theta_a \iff x \vee a = y \vee a.$$

(1) (a) θ_a *is a lattice congruence.*
  (b) θ_a *is an Ockham-algebra congruence if and only if* $a \vee f(a) = 1$*.*
  (c) *For all* $a \in B(\mathbf{A})$ *and all* $\alpha \in \operatorname{Con}\mathbf{A}$*, we have* $\theta_a \subseteq \alpha$ *if and only if* $(a, 0) \in \alpha$*.*
  (d) *A congruence on* $\mathbf{A}$ *is a factor congruence if and only if it is of the form* θ_a*, for some* $a \in B(\mathbf{A})$*.*

(2) *The map* $\mu\colon a \mapsto \theta_a$ *is an isomorphism between the Boolean algebras* $\mathbf{B}(\mathbf{A})$ *and* $\operatorname{Con_f}\mathbf{A}$*.*

Proof. (1) Parts (a)–(c) are left for the reader. We remark only that one can in fact show that a lattice $\mathbf{L}$ is distributive if and only if θ_a is a congruence on $\mathbf{L}$, for all $a \in L$.

(d) Let $a \in B(\mathbf{A})$; thus $a \wedge f(a) = 0$ and $a \vee f(a) = 1$. We claim that $(\theta_a, \theta_{f(a)})$ is a pair of factor congruences on $\mathbf{A}$, that is, $\theta_a \cap \theta_{f(a)} = 0_A$ and $\theta_a \cdot \theta_{f(a)} = 1_A$. Let $x, y \in A$. Then

$$\begin{aligned}(x, y) \in \theta_a \cap \theta_{f(a)} &\implies x \vee a = y \vee a \;\;\&\;\; x \vee f(a) = y \vee f(a)\\ &\implies x \vee (a \wedge f(a)) = y \vee (a \wedge f(a))\\ &\implies x = y \text{ since } a \wedge f(a) = 0.\end{aligned}$$

We now wish to prove that $(x, y) \in \theta_a \cdot \theta_{f(a)}$. Define $z := (x \vee a) \wedge (y \vee f(a))$. Then

$$z \vee a = ((x \vee a) \wedge (y \vee f(a))) \vee a = (x \vee a) \wedge (y \vee f(a) \vee a) = x \vee a,$$

as $f(a) \vee a = 1$. Thus $(x, z) \in \theta_a$. Similarly,

$$z \vee f(a) = ((x \vee a) \wedge (y \vee f(a))) \vee f(a) = (x \vee a \vee f(a)) \wedge (y \vee f(a)) = y \vee f(a),$$

as $a \vee f(a) = 1$. Thus $(z, y) \in \theta_{f(a)}$. Hence $\theta_a \cdot \theta_{f(a)} = 1_A$ and consequently θ_a is a factor congruence.

Now assume that α is a factor congruence. Then there is a congruence β such that $\alpha \cap \beta = 0_A$ and $\alpha \cdot \beta = 1_A$ and the map $\lambda\colon x \mapsto (x/\alpha, x/\beta)$ is an isomorphism from $\mathbf{A}$ to $\mathbf{A}/\alpha \times \mathbf{A}/\beta$. Let $a \in A$ satisfy $\lambda(a) = (0/\alpha, 1/\beta)$, that is, a is the unique element of A satisfying $(0, a) \in \alpha$ and $(a, 1) \in \beta$. Note that $\lambda(a)$ satisfies $\lambda(a) \wedge f(\lambda(a)) = 0$ and $\lambda(a) \vee f(\lambda(a)) = 1$ in $\mathbf{A}/\alpha \times \mathbf{A}/\beta$. As λ is an isomorphism, we conclude that $a \wedge f(a) = 0$ and $a \vee f(a) = 1$ in $\mathbf{A}$, that is, $a \in B(\mathbf{A})$. We claim that $\alpha = \theta_a$. Let $x, y \in A$. Since $(0, a) \in \alpha$, we have $\theta_a \subseteq \alpha$, by (c). To prove the reverse inclusion, let $(x, y) \in \alpha$. Then $x \vee a \equiv_\alpha y \vee a$, and, since $(a, 1) \in \beta$, we have

$$x \vee a \equiv_\beta x \vee 1 = 1 = y \vee 1 \equiv_\beta y \vee a,$$

giving $x \vee a \equiv_\beta y \vee a$. Since $\alpha \cap \beta = 0_A$, we have $x \vee a = y \vee a$, that is, $(x, y) \in \theta_a$. Hence $\alpha \subseteq \theta_a$. Thus $\alpha = \theta_a$, as claimed.

(2) By (1)(d), the map $\mu\colon \mathbf{B}(\mathbf{A}) \to \mathrm{Con_f}\, \mathbf{A}$ is well defined and surjective, and it remains to prove that μ is an order-embedding. Let $a, b \in B(\mathbf{A})$. Then, using (1)(c),

$$\theta_a \subseteq \theta_b \iff (a, 0) \in \theta_b \iff a \vee b = 0 \vee b = b \iff a \leqslant b,$$

as required. □

Since Ockham algebras have the congruence extension property, every non-trivial subalgebra of a simple Ockham algebra is simple. Hence every semisimple variety of Ockham algebras is sub-semisimple. Consequently, every finitely generated semisimple variety of Ockham algebras is filtral, by Proposition 4.1(3).

An algebra $\mathbf{A}$ is called a *perfect extension* of a subalgebra $\mathbf{B}$ if every congruence on $\mathbf{B}$ has a unique extension to $\mathbf{A}$, or equivalently, if the natural restriction map from $\mathrm{Con}\, \mathbf{A}$ to $\mathrm{Con}\, \mathbf{B}$ is an isomorphism.

Let $\mathcal{S}$ be a finite set of finite simple Ockham algebras and let $\mathsf{S}(\mathcal{S})$ denote the set of subalgebras of algebras in $\mathcal{S}$. It follows immediately from Jónsson's Lemma that every simple algebra in $\mathsf{Var}(\mathcal{S})$ is isomorphic to an algebra in $\mathsf{S}(\mathcal{S})$. Theorem 2.2 yields the following result.

Theorem 2.6. *Let $\mathcal{S}$ be a finite set of finite simple Ockham algebras and let $\mathbf{A}$ be a non-trivial algebra in $\mathcal{V} = \mathsf{Var}(\mathcal{S})$. Then the following are equivalent*:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) $\mathbf{A}$ *has permuting congruences*;
(3) $\mathbf{A}$ *is a perfect extension of its Boolean skeleton* $\mathbf{B}(\mathbf{A})$;
(4) $\mathbf{A}$ *is isomorphic to a Boolean product of algebras from* $\mathsf{S}(\mathcal{S})$.

Proof. Given Theorem 2.2, it remains to show that $\psi\colon\ \mathrm{Con}\, \mathbf{A} \to \mathrm{Id}(\mathrm{Con_f}\, \mathbf{A})$, defined by $\psi(\alpha) = {\downarrow}\alpha \cap \mathrm{Con_f}\, \mathbf{A}$, for all $\alpha \in \mathrm{Con}\, \mathbf{A}$, is an isomorphism if and only if $\mathbf{A}$ is a perfect extension of $\mathbf{B}(\mathbf{A})$.

Assume that ψ is an isomorphism. Since Ockham algebras have the congruence extension property, to prove that $\mathbf{A}$ is a perfect extension of $\mathbf{B}(\mathbf{A})$ we must show that $\alpha{\restriction}_{B(\mathbf{A})} = \beta{\restriction}_{B(\mathbf{A})}$ implies $\alpha = \beta$, for all $\alpha, \beta \in \mathrm{Con}\, \mathbf{A}$.

Let $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ with $\alpha{\restriction}_{B(\mathbf{A})} = \beta{\restriction}_{B(\mathbf{A})}$. To prove that $\alpha = \beta$, it suffices to show that $\psi(\alpha) = \psi(\beta)$. But

$$\begin{aligned}
\psi(\alpha) &= \{\, \gamma \in \operatorname{Con_f} \mathbf{A} \mid \gamma \subseteq \alpha \,\} && \text{definition of } \psi \\
&= \{\, \theta_a \mid a \in B(\mathbf{A}) \ \ \& \ \ \theta_a \subseteq \alpha \,\} && \text{by Lemma 2.5(1)(d)} \\
&= \{\, \theta_a \mid a \in B(\mathbf{A}) \ \ \& \ \ (a, 0) \in \alpha \,\} && \text{by Lemma 2.5(1)(c)} \\
&= \{\, \theta_a \mid a \in B(\mathbf{A}) \ \ \& \ \ (a, 0) \in \beta \,\} && \text{as } \alpha{\restriction}_{B(\mathbf{A})} = \beta{\restriction}_{B(\mathbf{A})} \\
&= \{\, \theta_a \mid a \in B(\mathbf{A}) \ \ \& \ \ \theta_a \subseteq \beta \,\} && \text{by Lemma 2.5(1)(c)} \\
&= \{\, \gamma \in \operatorname{Con_f} \mathbf{A} \mid \gamma \subseteq \beta \,\} && \text{by Lemma 2.5(1)(d)} \\
&= \psi(\beta),
\end{aligned}$$

as required.

Conversely, assume that $\mathbf{A}$ is a perfect extension of $\mathbf{B}(\mathbf{A})$, that is, $\rho\colon \operatorname{Con} \mathbf{A} \to \operatorname{Con} \mathbf{B}(\mathbf{A})$, given by $\rho(\alpha) = \alpha{\restriction}_{B(\mathbf{A})}$, for all $\alpha \in \operatorname{Con} \mathbf{A}$, is an isomorphism. Let $\sigma\colon \operatorname{Con} \mathbf{B}(\mathbf{A}) \to \operatorname{Id}(\mathbf{B}(\mathbf{A}))$ be the isomorphism given by $\sigma(\gamma) = 0/\gamma$, for all $\gamma \in \operatorname{Con} \mathbf{B}(\mathbf{A})$, and let $\overline{\mu}\colon \operatorname{Id}(\mathbf{B}(\mathbf{A})) \to \operatorname{Id}(\operatorname{Con_f} \mathbf{A})$ be the natural isomorphism induced by the isomorphism $\mu\colon \mathbf{B}(\mathbf{A}) \to \operatorname{Con_f} \mathbf{A}$ of Lemma 2.5(2). We claim that $\psi = \overline{\mu} \circ \sigma \circ \rho$. Since ρ, σ and $\overline{\mu}$ are isomorphisms, it will then follow that ψ is an isomorphism, as required.

Let $\alpha \in \operatorname{Con} \mathbf{A}$. Then

$$\begin{aligned}
\overline{\mu}(\sigma(\rho(\alpha))) &= \overline{\mu}(\sigma(\alpha{\restriction}_{B(\mathbf{A})})) \\
&= \overline{\mu}(0/\alpha{\restriction}_{B(\mathbf{A})}) \\
&= \{\, \theta_a \mid a \in 0/\alpha{\restriction}_{B(\mathbf{A})} \,\} && \text{definition of } \overline{\mu} \\
&= \{\, \theta_a \mid a \in B(\mathbf{A}) \ \ \& \ \ (a, 0) \in \alpha \,\} \\
&= \{\, \gamma \in \operatorname{Con_f} \mathbf{A} \mid \gamma \subseteq \alpha \,\} && \text{by Lemma 2.5(1)(c)(d)} \\
&= \psi(\alpha),
\end{aligned}$$

as claimed. □

De Morgan algebras in particular. A *De Morgan algebra* is an Ockham algebra that satisfies the double negation law: $f(f(x)) \approx x$. The result of Haviar and Ploščica [16] that motivated this work states that a De Morgan algebra $\mathbf{A}$ is a perfect extension of its Boolean skeleton $\mathbf{B}(\mathbf{A})$ if and only if $\mathbf{A}$ is a Boolean product of simple De Morgan algebras. This follows immediately from Theorem 2.6 by choosing $\mathfrak{S}$ to contain just the four-element non-Boolean De Morgan algebra.

3. Compact, complemented and factor congruences

We begin by recording, without proof, some well-known and straightforward facts about factor congruences on an algebra $\mathbf{A}$; see the start of Section 2 for the definition.

Pairs of factor congruences correspond to factorisations of $\mathbf{A}$: if (α, β) is a pair of factor congruences, then $\mathbf{A} \cong \mathbf{A}/\alpha \times \mathbf{A}/\beta$, and if $\mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2$, then $(\ker\pi_1, \ker\pi_2)$ is a pair of factor congruences, where $\pi_i\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{A}_i$ is the projection for $i \in \{1,2\}$. Recall that $\mathrm{Con_f}\,\mathbf{A}$ denotes the set of factor congruences on $\mathbf{A}$. We denote the set of complemented congruences on $\mathbf{A}$ by $\mathrm{Con}'\mathbf{A}$. Thus $\mathrm{Con_f}\,\mathbf{A} \subseteq \mathrm{Con}'\mathbf{A}$ always holds.

The following is a straightforward and well-known exercise.

Lemma 3.1. *Let* $\mathbf{A}$ *be a congruence-distributive algebra.*

(1) $\alpha \in \mathrm{Con}\,\mathbf{A}$ *is a factor congruence if and only if* α *has a complement in* $\mathrm{Con}\,\mathbf{A}$ *and* $\alpha \cdot \theta = \theta \cdot \alpha$, *for all* $\theta \in \mathrm{Con}\,\mathbf{A}$.
(2) *The set* $\mathrm{Con_f}\,\mathbf{A}$ *is a sublattice of* $\mathrm{Con}\,\mathbf{A}$ *and forms a Boolean algebra.*

As an almost immediate consequence we have the following.

Lemma 3.2. *Let* $\mathbf{A}$ *be a congruence-distributive algebra with factor principal congruences. Then* $\mathbf{A}$ *has permuting congruences.*

Proof. By Lemma 3.1, every principal congruence on $\mathbf{A}$ permutes with every congruence on $\mathbf{A}$. In particular, every pair of principal congruences permute. Since every congruence on $\mathbf{A}$ is a join of principal congruences, it follows easily from the description via relational products of joins in $\mathrm{Con}\,\mathbf{A}$ that each pair of congruences on $\mathbf{A}$ permute. □

To generalise the Haviar–Ploščica result, we will work with algebras $\mathbf{A}$ that have the property that 1_A is compact in $\mathrm{Con}\,\mathbf{A}$. At the variety level, the following result is a useful characterisation.

Lemma 3.3. *Let* $\mathcal{V}$ *be a variety. The following are equivalent*:

(1) 1_A *is compact in* $\mathrm{Con}\,\mathbf{A}$, *for every algebra* $\mathbf{A}$ *in* $\mathcal{V}$;
(2) *no non-trivial algebra in* $\mathcal{V}$ *has a trivial subalgebra*;
(3) *no subdirectly irreducible algebra in* $\mathcal{V}$ *has a trivial subalgebra.*

Proof. The equivalence of (1) and (2) is the main result of Kollár [18], and (2) is equivalent to (3) since every algebra in $\mathcal{V}$ is a subdirect product of its subdirectly irreducible homomorphic images. See also Csákány [9]. □

The following remark recalls some well-known facts about congruence lattices and establishes some notation.

Remark 3.4. Let $\mathbf{A}$ be an algebra. The set $\mathrm{Con_c}\,\mathbf{A}$ of compact congruences on $\mathbf{A}$ forms a join-subsemilattice of $\mathrm{Con}\,\mathbf{A}$ and contains 0_A. For all $\alpha \in \mathrm{Con}\,\mathbf{A}$, the set $I := {\downarrow}\alpha \cap \mathrm{Con_c}\,\mathbf{A}$ is an ideal of $\mathrm{Con_c}\,\mathbf{A}$ with $\alpha = \bigvee_{\mathrm{Con}\,\mathbf{A}} I = \bigcup I$. The map

$$\varphi\colon \alpha \mapsto {\downarrow}\alpha \cap \mathrm{Con_c}\,\mathbf{A}$$

is an isomorphism between $\mathrm{Con}\,\mathbf{A}$ and the lattice $\mathrm{Id}(\mathrm{Con_c}\,\mathbf{A})$ of ideals of $\mathrm{Con_c}\,\mathbf{A}$, with inverse given by $\varphi^{-1}\colon I \mapsto \bigvee_{\mathrm{Con}\,\mathbf{A}} I = \bigcup I$. Where needed, we shall use φ rather than $\varphi{\restriction}_X$ to denote the restriction of φ to a subset X of $\mathrm{Con}\,\mathbf{A}$.

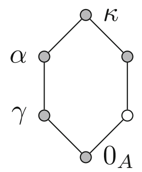

FIGURE 1. The pentagon sublattice of $\operatorname{Con}\mathbf{A}$

An algebra $\mathbf{A}$ has *complemented principal congruences* if, for all $a, b \in A$, the principal congruence $\operatorname{Cg}^{\mathbf{A}}(a, b)$ has a complement in $\operatorname{Con}\mathbf{A}$, and a variety $\boldsymbol{\mathcal{V}}$ has complemented principal congruences if every algebra in $\boldsymbol{\mathcal{V}}$ does. This is a weakening of factor principal congruences. It is very easy to see that if a variety $\boldsymbol{\mathcal{V}}$ has complemented principal congruences, then it is semisimple—just use the fact that the monolith of a subdirectly irreducible algebra is principal.

Lemma 3.5. *Let* $\mathbf{A}$ *be a congruence-modular algebra.*

(1) *Assume that* κ *is compact in* $\operatorname{Con}\mathbf{A}$ *and let* $\alpha \in \operatorname{Con}\mathbf{A}$ *with* $\alpha \leqslant \kappa$. *If* α *has a complement* β *in the interval* ${\downarrow_{\operatorname{Con}\mathbf{A}}}\kappa$, *then both* α *and* β *are compact in* $\operatorname{Con}\mathbf{A}$.
(2) *Assume that* 1_A *is compact in* $\operatorname{Con}\mathbf{A}$. *Then every complemented congruence is compact, i.e.,* $\operatorname{Con}'\mathbf{A} \subseteq \operatorname{Con_c}\mathbf{A}$.

Proof. Clearly (2) follows from (1). To prove (1), let $\alpha \in {\downarrow_{\operatorname{Con}\mathbf{A}}}\kappa$ and assume that β is a complement of α in ${\downarrow_{\operatorname{Con}\mathbf{A}}}\kappa$. Define $I := \varphi(\alpha)$, $J := \varphi(\beta)$ and $K := \varphi(\kappa)$. Then $I \cap J = \{0_A\}$ and $I \vee J = K$ in the lattice of ideals of $\operatorname{Con_c}\mathbf{A}$. Since, by assumption, $\kappa \in \operatorname{Con_c}\mathbf{A}$, we have $\kappa \in K$, and hence there exist $\gamma \in I$ and $\delta \in J$ with $\gamma \vee \delta = \kappa$. Since $I \cap J = \{0_A\}$, we also have $\gamma \wedge \delta = 0_A$. As $\gamma \in I = \varphi(\alpha) = {\downarrow}\alpha \cap \operatorname{Con_c}\mathbf{A}$, we have $\gamma \leqslant \alpha$, and similarly, $\delta \leqslant \beta$. If $\gamma < \alpha$, then $\{0_A, \gamma, \alpha, \beta, \kappa\}$ forms a pentagon sublattice of $\operatorname{Con}\mathbf{A}$, contradicting the fact that $\operatorname{Con}\mathbf{A}$ is modular; see Figure 1. Hence $\alpha = \gamma \in \operatorname{Con_c}\mathbf{A}$, and by symmetry, $\beta \in \operatorname{Con_c}\mathbf{A}$. ☐

Recall that a *generalised Boolean algebra* is a distributive lattice $\mathbf{L}$ with zero in which ${\downarrow}a$ is a complemented lattice, for all $a \in L$. We will see in Proposition 4.1(2) that, at the variety level, condition (1) of the next lemma is equivalent to filtrality.

Lemma 3.6. *The following are related by* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *for every algebra* $\mathbf{A}$:

(1) $\mathbf{A}$ *is congruence distributive and has complemented principal congruences*;
(2) $\mathbf{A}$ *is congruence distributive and* $\operatorname{Con_c}\mathbf{A} \subseteq \operatorname{Con}'\mathbf{A}$;
(3) (a) $\operatorname{Con_c}\mathbf{A}$ *is a sublattice of* $\operatorname{Con}\mathbf{A}$, *and*
    (b) $\operatorname{Con_c}\mathbf{A}$ *is a generalised Boolean algebra.*

Moreover, (3)(b) *implies that* $\mathbf{A}$ *is congruence distributive.*

If 1_A *is compact in* $\operatorname{Con}\mathbf{A}$, *then the three conditions are mutually equivalent, and we can replace 'generalised Boolean algebra' by 'Boolean algebra' in* (3)(b) *and we can write* $\operatorname{Con_c}\mathbf{A} = \operatorname{Con}'\mathbf{A}$ *in* (2).

Proof. (1) $\Leftrightarrow$ (2): Assume (1). It is easy to see that every finite join of complemented elements in a bounded distributive lattice is complemented. Since every compact congruence is a join of principal congruences, we conclude that every compact congruence on $\mathbf{A}$ has a complement. Since principal congruences are compact, (2) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (3): Assume (2). Let $\alpha, \beta \in \mathrm{Con_c}\,\mathbf{A}$. By (2), α and β have complements, say α' and β', in $\mathrm{Con}\,\mathbf{A}$. The distributivity of $\mathrm{Con}\,\mathbf{A}$ guarantees that $\alpha' \wedge \beta$ is a complement of $\alpha \wedge \beta$ in the interval $\downarrow_{\mathrm{Con}\,\mathbf{A}}\beta$. By Lemma 3.5(1), both $\alpha \wedge \beta$ and $\alpha' \wedge \beta$ are compact in $\mathrm{Con}\,\mathbf{A}$. Hence (a) holds. When $\alpha \leqslant \beta$, we have shown that $\alpha' \wedge \beta$ is a complement of α in the interval $\downarrow_{\mathrm{Con_c}\,\mathbf{A}}\beta$. Since $\mathrm{Con}\,\mathbf{A}$ is distributive, it now follows that $\downarrow_{\mathrm{Con_c}\,\mathbf{A}}\beta$ is a Boolean lattice, for all $\beta \in \mathrm{Con_c}\,\mathbf{A}$, whence $\mathrm{Con_c}\,\mathbf{A}$ is a generalised Boolean algebra, proving (b).

Now assume (3)(b). Since, by Remark 3.4, $\mathrm{Con}\,\mathbf{A}$ is isomorphic to $\mathrm{Id}(\mathrm{Con_c}\,\mathbf{A})$ and since the lattice of ideals of a distributive lattice is itself distributive, we conclude that $\mathrm{Con}\,\mathbf{A}$ is distributive.

Finally, assume (3) and assume that $1_A \in \mathrm{Con_c}\,\mathbf{A}$. It follows that the generalised Boolean algebra $\mathrm{Con_c}\,\mathbf{A}$ is in fact a Boolean algebra and the complement in $\mathrm{Con_c}\,\mathbf{A}$ of a principal congruence is also a complement in $\mathrm{Con}\,\mathbf{A}$. Hence (3) implies (1); moreover, since $\mathrm{Con}\,\mathbf{A}$ is distributive and therefore modular, $\mathrm{Con}'\mathbf{A} \subseteq \mathrm{Con_c}\,\mathbf{A}$ by Lemma 3.5(2). ☐

Since $\mathrm{Con_f}\,\mathbf{A} \subseteq \mathrm{Con}'\mathbf{A}$, the previous lemma applies whenever $\mathbf{A}$ is congruence distributive and has factor principal congruences. When 1_A is compact in $\mathrm{Con}\,\mathbf{A}$, we can say a little more.

Lemma 3.7. *Let* $\mathbf{A}$ *be a congruence-distributive algebra. The following conditions are related by* (1) $\Leftrightarrow$ (2) *and* (3) $\Rightarrow$ (4) $\Rightarrow$ (1), *and are equivalent if* 1_A *is compact in* $\mathrm{Con}\,\mathbf{A}$:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) *every compact congruence on* $\mathbf{A}$ *is a factor congruence*;
(3) $\mathrm{Con_c}\,\mathbf{A} = \mathrm{Con_f}\,\mathbf{A} = \mathrm{Con}'\mathbf{A}$;
(4) $\mathrm{Con}\,\mathbf{A}$ *is isomorphic to the lattice of ideals of the Boolean algebra* $\mathrm{Con_f}\,\mathbf{A}$ *of factor congruences on* $\mathbf{A}$ *via* $\psi\colon \alpha \mapsto {\downarrow}\alpha \cap \mathrm{Con_f}\,\mathbf{A}$, *for all* $\alpha \in \mathrm{Con}\,\mathbf{A}$.

Proof. Since the compact congruences are the finite joins of principal congruences, (1) $\Rightarrow$ (2) follows from Lemma 3.1(2), and (2) $\Rightarrow$ (1) is trivial.

(3) $\Rightarrow$ (4): Assume (3); then $\mathrm{Con_f}\,\mathbf{A} = \mathrm{Con_c}\,\mathbf{A}$, and (4) follows at once from Remark 3.4 since $\psi = \varphi$.

(4) $\Rightarrow$ (1): Assume (4) and let α be a principal congruence on $\mathbf{A}$. Then α is compact in $\mathrm{Con}\,\mathbf{A}$ and hence, by (4), $\psi(\alpha)$ is compact in $\mathrm{Id}(\mathrm{Con_f}\,\mathbf{A})$. Since an ideal of $\mathrm{Con_f}\,\mathbf{A}$ is compact if and only if it is principal, there exists $\gamma \in \mathrm{Con_f}\,\mathbf{A}$ with $\psi(\alpha) = \downarrow_{\mathrm{Con_f}\,\mathbf{A}}\gamma$. Hence

$$\begin{aligned}\psi(\alpha) &= \downarrow_{\mathrm{Con_f}\,\mathbf{A}}\gamma\\ &= {\downarrow}\gamma \cap \mathrm{Con_f}\,\mathbf{A}\\ &= \psi(\gamma),\end{aligned}$$

whence $\alpha = \gamma \in \mathrm{Con_f}\,\mathbf{A}$ since ψ is one-to-one. Thus α is a factor congruence, as required.

Finally assume $1_A \in \mathrm{Con_c}\,\mathbf{A}$ and assume (2); so $\mathrm{Con_c}\,\mathbf{A} \subseteq \mathrm{Con_f}\,\mathbf{A}$. Since $\mathrm{Con_f}\,\mathbf{A} \subseteq \mathrm{Con}'\mathbf{A}$ always holds and $\mathrm{Con}'\mathbf{A} \subseteq \mathrm{Con_c}\,\mathbf{A}$ by Lemma 3.5(2), (3) follows. □

We note that the converse property, namely 'every factor congruence is compact', has been studied by Vaggione and Sánchez Terraf [23].

4. Filtral varieties

Let $\mathbf{A}$ be a subalgebra of a product $\prod_{x\in X}\mathbf{A}_x$, with $X \neq \varnothing$. For all $a, b \in A$, define

$$[\![a = b]\!] := \{\, x \in X \mid a(x) = b(x) \,\}.$$

Let $\mathcal{F}$ be a filter on X. The congruence $\theta_{\mathcal{F}}$ on $\mathbf{A}$ induced by $\mathcal{F}$ is defined by: for all $a, b \in A$,

$$(a, b) \in \theta_{\mathcal{F}} \iff [\![a = b]\!] \in \mathcal{F}.$$

For all $N \subseteq X$, the set $\uparrow\! N := \{\, U \subseteq X \mid N \subseteq U \,\}$ is the principal filter generated by N and we abbreviate $\theta_{\uparrow N}$ to θ_N; thus,

$$(a, b) \in \theta_N \iff N \subseteq [\![a = b]\!].$$

When necessary for clarity, we shall use $\theta_{\mathcal{F}}$ to denote the congruence induced by $\mathcal{F}$ on $\prod_{x\in X}\mathbf{A}_x$ and use $\theta_{\mathcal{F}}{\restriction}_A$ to denote its restriction to $\mathbf{A}$.

Congruences of the form $\theta_{\mathcal{F}}$, for some filter $\mathcal{F}$ on X, are called *filtral*. A variety $\boldsymbol{\mathcal{V}}$ is *filtral* if every congruence on every subdirect product of subdirectly irreducible algebras from $\boldsymbol{\mathcal{V}}$ is filtral. Filtral varieties were introduced by Magari [19] in the late 1960s and were intensively studied, particularly during the 1970s and 1980s.

We now collect together from the literature some important characterisations and properties of filtral varieties.

A class $\boldsymbol{\mathcal{K}}$ of algebras has *restricted equationally definable principal congruences* if there exist 4-ary terms p_i, q_i, $i = 1, \ldots, n$, such that, for every algebra $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ and all $a, b, c, d \in A$,

$$(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b) \iff \mathop{\&}_{i=1}^{n} p_i(a, b, c, d) = q_i(a, b, c, d).$$

While it is obvious that filtral varieties are semisimple, it is far from clear that they are sub-semisimple. We include a short proof based on an argument given by Franci [12, Thm 1.1].

Proposition 4.1. *Let $\boldsymbol{\mathcal{V}}$ be a variety.*

(1) *Every filtral variety is sub-semisimple.*
(2) *The following are equivalent*:
  (i) *$\boldsymbol{\mathcal{V}}$ is a filtral variety*;
  (ii) *$\boldsymbol{\mathcal{V}}$ is semisimple and every congruence on every subalgebra of a product of simple algebras from $\boldsymbol{\mathcal{V}}$ is filtral*;

(iii) $\mathcal{V}$ *is semisimple and for every subalgebra* $\mathbf{A}$ *of a product* $\prod_{x\in X}\mathbf{A}_x$ *of simple algebras from* $\mathcal{V}$, *we have* $\mathrm{Cg}^{\mathbf{A}}(a,b) = \theta_N$, *where* $N = [\![a=b]\!]$, *for all* $a, b \in A$;
(iv) $\mathcal{V}$ *is congruence distributive and has complemented principal congruences*;
(v) $\mathcal{V}$ *is semisimple and has restricted equationally definable principal congruences*;
(vi) $\mathcal{V}$ *is sub-semisimple and congruence distributive, and the class consisting of the simple algebras from* $\mathcal{V}$ *is closed under ultraproducts.*

(3) *A finitely generated variety is filtral if and only if it is sub-semisimple and congruence distributive.*
(4) *If* $\mathcal{V}$ *is filtral, then the class consisting of the simple algebras from* $\mathcal{V}$ *along with the one-element algebras is a universal class.*
(5) *Assume that* $\mathcal{V}$ *is filtral and let* $\mathbf{A}$ *be a non-trivial algebra in* $\mathcal{V}$.
(i) $\mathrm{Con_c}\,\mathbf{A}$ *is a sublattice of* $\mathrm{Con}\,\mathbf{A}$, *and*
(ii) $\mathrm{Con_c}\,\mathbf{A}$ *is a generalised Boolean algebra.*
If 1_A *is compact in* $\mathrm{Con}\,\mathbf{A}$, *then*
(i) $\mathrm{Con_c}\,\mathbf{A}$ *is a* $\{0,1\}$*-sublattice of* $\mathrm{Con}\,\mathbf{A}$, *and*
(ii) $\mathrm{Con_c}\,\mathbf{A}$ *is a Boolean algebra.*
(6) *An algebra in a filtral variety has factor principal congruences if and only if it has permuting congruences.*

Proof. (1) Assume that $\mathcal{V}$ is a non-trivial filtral variety. Let $\mathbf{A} \in \mathcal{V}$ be subdirectly irreducible, let $\mathbf{B}$ be a non-trivial subalgebra of $\mathbf{A}$ and let $\beta \in \mathrm{Con}\,\mathbf{B}$. We shall prove that $\beta \in \{0_B, 1_B\}$. Define

$$C := \{\, c \in A^{\mathbb{N}} \mid (\exists b \in B)\,\{\, n \in \mathbb{N} \mid c(n) \neq b \,\} \text{ is finite} \,\}.$$

Then C is a subuniverse of $\mathbf{A}^{\mathbb{N}}$ and $\mathbf{C}$ is a subdirect subalgebra of $\mathbf{A}^{\mathbb{N}}$. Define $\mu\colon \mathbf{C} \to \mathbf{B}$ by $\mu(c) := \lim_{n\to\infty} c(n)$ and define $\widehat{\beta} \in \mathrm{Con}\,\mathbf{C}$ by

$$(\forall c_1, c_2 \in C) \quad (c_1, c_2) \in \widehat{\beta} \iff (\mu(c_1), \mu(c_2)) \in \beta. \qquad (*)$$

Since $\mathbf{A}$ is subdirectly irreducible and $\mathcal{V}$ is filtral, there exists a filter $\mathcal{F}$ on $\mathbb{N}$ with $\theta_{\mathcal{F}}{\restriction}_C = \widehat{\beta}$. For all $b \in B$, let $\underline{b} \in C$ be the constant map onto $\{b\}$.

Case (a): $\varnothing \in \mathcal{F}$ *or equivalently* $\mathcal{F} = \wp(\mathbb{N})$. In this case $\theta_{\mathcal{F}}{\restriction}_C = 1_C$. Hence, for all $b_1, b_2 \in B$, we have $(\underline{b_1}, \underline{b_2}) \in \theta_{\mathcal{F}}{\restriction}_C = \widehat{\beta}$ and therefore $(b_1, b_2) = (\mu(\underline{b_1}), \mu(\underline{b_2})) \in \beta$ by $(*)$. Thus, in this case, we have $\beta = 1_B$.

Case (b): $\varnothing \notin \mathcal{F}$. Let $b_1, b_2 \in B$. Then, using $(*)$ and the fact that $\theta_{\mathcal{F}}{\restriction}_C = \widehat{\beta}$,

$$(b_1, b_2) \in \beta \iff (\mu(\underline{b_1}), \mu(\underline{b_2})) \in \beta \iff (\underline{b_1}, \underline{b_2}) \in \widehat{\beta} \iff [\![\underline{b_1} = \underline{b_2}]\!] \in \mathcal{F}.$$

Since $[\![\underline{b_1} = \underline{b_2}]\!] \in \{\varnothing, \mathbb{N}\}$, for all $b_1, b_2 \in B$, and $\varnothing \notin \mathcal{F}$, we conclude that

$$(b_1, b_2) \in \beta \iff [\![\underline{b_1} = \underline{b_2}]\!] = \mathbb{N} \iff b_1 = b_2.$$

Thus, in this case, we have $\beta = 0_B$.

We have proved $\beta \in \{0_B, 1_B\}$, as required.

(2) That (i) implies (ii) is an easy consequence of (1): use the fact that a subalgebra of a product is a subdirect product of subalgebras of the factors.

Of course, it is trivial that (ii) implies (i). (The equivalence of (i) and (ii) is due to Magari [19]; see also Franci [12, Thm 1.1].)

That a filtral variety is congruence distributive was proved by Köhler and Pigozzi [17, Cor. 6], and is also proved in Fried and Kiss [15, Thm 4.9].

The equivalence of (i), (iii) and (iv) is due to Fried and Kiss [15, Thms 4.11 and 4.13]. The equivalence of (i) and (v) is due to Fried, Grätzer and Quackenbush [14, Thms 4.5, 5.4 and Cor. 5.6].

We now prove that a filtral variety $\boldsymbol{\mathcal{V}}$ satisfies the conditions given in (vi). By (1), $\boldsymbol{\mathcal{V}}$ is sub-semisimple, and $\boldsymbol{\mathcal{V}}$ is congruence distributive by (iv). Finally, we show that the class of simple algebras in $\boldsymbol{\mathcal{V}}$ is closed under ultraproducts. It is very easy to see that if $\prod_{x\in X} \mathbf{A}_x$ is a product of simple algebras from $\boldsymbol{\mathcal{V}}$ with $X \neq \varnothing$, then the map $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is an isomorphism between the lattice of filters on X and the congruence lattice of $\prod_{x\in X} \mathbf{A}_x$. Hence, if $\mathcal{F}$ is an ultrafilter on X, then $\theta_{\mathcal{F}}$ is a maximal congruence on the product and consequently the ultraproduct $\big(\prod_{x\in X} \mathbf{A}_x\big)/\theta_{\mathcal{F}}$ is simple.

For the converse, assume that $\boldsymbol{\mathcal{V}}$ satisfies the conditions in (vi). Let $\mathbf{A}$ be a subalgebra of a product $\prod_{x\in X} \mathbf{A}_x$, where $X \neq \varnothing$, with each $\mathbf{A}_x$ a subdirectly irreducible algebra in $\boldsymbol{\mathcal{V}}$. As $\boldsymbol{\mathcal{V}}$ is semisimple, each $\mathbf{A}_x$ is simple. Again since $\boldsymbol{\mathcal{V}}$ is semisimple, every congruence on $\mathbf{A}$ is an intersection of maximal congruences. Therefore to prove that the congruences on $\mathbf{A}$ are filtral, it suffices to prove that every maximal congruence on $\mathbf{A}$ is filtral. Let β be a maximal congruence on $\mathbf{A}$. By Jónsson's Lemma, there is an ultrafilter $\mathcal{F}$ on X such that $\theta_{\mathcal{F}}{\restriction}_A \leqslant \beta$. By assumption, the ultraproduct $\big(\prod_{x\in X} \mathbf{A}_x\big)/\theta_{\mathcal{F}}$ is simple. Since $\boldsymbol{\mathcal{V}}$ is sub-semisimple, it follows that the subalgebra $\mathbf{A}/\theta_{\mathcal{F}}{\restriction}_A$ of $\big(\prod_{x\in X} \mathbf{A}_x\big)/\theta_{\mathcal{F}}$ is simple, whence $\theta_{\mathcal{F}}{\restriction}_A$ is maximal in $\operatorname{Con}\mathbf{A}$. Consequently $\theta_{\mathcal{F}}{\restriction}_A \leqslant \beta$ implies that $\theta_{\mathcal{F}}{\restriction}_A = \beta$; whence β is filtral.

(3) This is an immediate consequence of the equivalence of (i) and (vi) in (2), as a finitely generated congruence-distributive variety has only finitely many subdirectly irreducible algebras each of which is finite—another consequence of Jónsson's Lemma.

(4) Assume that $\boldsymbol{\mathcal{V}}$ is a filtral variety. By (2)(vi), the class $\boldsymbol{\mathcal{S}}$ consisting of the simple algebras from $\boldsymbol{\mathcal{V}}$ along with the one-element algebras is closed under ultraproducts. By sub-semisimplicity, $\boldsymbol{\mathcal{S}}$ is closed under forming subalgebras. Hence $\boldsymbol{\mathcal{S}}$ is a universal class by the Łos–Tarski Theorem. (That $\boldsymbol{\mathcal{S}}$ is a universal class was first observed by Bergman [2].)

(5) This follows at once from Lemma 3.6 since a filtral variety is congruence distributive and has complemented principal congruences by (2). (This result was first proved by Fried and Grätzer [13]; see their Claims 1 and 2.)

(6) Since a filtral variety is congruence distributive, the forward direction follows from Lemma 3.2. The reverse direction follows at once from the equivalence of (i) and (iv) in (2). $\square$

A wealth of information about filtral varieties may be found in Fried and Kiss [15, Thm 4.9].

5. Duality for generalised Boolean algebras

Denote the class of Boolean algebras by $\boldsymbol{\mathcal{B}}$. A topological space $\langle X; \mathcal{T}\rangle$ is a *Boolean space* if it is compact and each pair of distinct points in X can be separated by a clopen set. Denote the category of Boolean spaces and continuous maps between them by $\boldsymbol{\mathcal{X}}$. For a discussion of Boolean spaces and their role in Stone's duality between $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{X}}$ that is sufficient for our needs we refer to Davey and Priestley [11, pp. 247–250].

To encompass algebras $\mathbf{A}$ in which 1_A is not compact, we shall require the duality for the class $\boldsymbol{\mathcal{B}}_0$ of generalised Boolean algebras. There are various ways to describe $\boldsymbol{\mathcal{B}}_0$ algebraically. Perhaps the simplest, due to M.H. Stone [21], is to identify $\boldsymbol{\mathcal{B}}_0$ with the class of $\boldsymbol{\mathcal{R}}_0$ of *Boolean rings*, that is, rings satisfying $x^2 \approx x$. We now sketch the details.

- Every Boolean ring satisfies $x + x \approx 0$ and $xy \approx yx$.
- Let $\mathbf{R} = \langle R; +, \cdot, 0\rangle$ be a Boolean ring and define $a \vee b := a + b + a \cdot b$ and $a \wedge b := a \cdot b$. Then $F(\mathbf{R}) := \langle R; \vee, \wedge, 0\rangle$ is a generalised Boolean algebra; if $b \leqslant a$, then the complement of b in ${\downarrow}a$ is $a + a \cdot b$.
- Conversely, let $\mathbf{B} = \langle B; \vee, \wedge, 0\rangle$ be a generalised Boolean algebra and define $a \cdot b := a \wedge b$ and define $a + b$ to be the complement of $a \wedge b$ in ${\downarrow}(a \vee b)$. Then $G(\mathbf{B}) := \langle B; +, \cdot, 0\rangle$ is a Boolean ring.
- Moreover, $GF(\mathbf{R}) = \mathbf{R}$ and $FG(\mathbf{B}) = \mathbf{B}$, for all $\mathbf{R} \in \boldsymbol{\mathcal{R}}_0$ and all $\mathbf{B} \in \boldsymbol{\mathcal{B}}_0$. In fact, $\boldsymbol{\mathcal{B}}_0$ (with 0-preserving lattice homomorphisms) and $\boldsymbol{\mathcal{R}}_0$ (with ring homomorphisms) are isomorphic as categories.
- For all $\mathbf{R} \in \boldsymbol{\mathcal{R}}_0$, a subset J of R is an ideal of the ring $\mathbf{R}$ if and only if it is an ideal of the generalised Boolean algebra $F(\mathbf{R})$.
- Every prime ideal of a generalised Boolean algebra is maximal.
- Define $\mathbf{R}_0 = \langle \{0, 1\}; +, \cdot, 0\rangle$, where $+$ and $\cdot$ are the usual operations on $\{0, 1\}$. Let $\mathbf{R}$ be a Boolean ring and let $a \neq b$ in R. There exists a prime ideal P of $\mathbf{R}$ that contains exactly one of a and b. The characteristic function of $R \backslash P$ is a ring homomorphism from $\mathbf{R}$ to $\mathbf{R}_0$ that separates a and b. It follows that $\mathbf{R}$ embeds into a power of $\mathbf{R}_0$ and hence $\boldsymbol{\mathcal{R}}_0 = \mathsf{ISP}(\mathbf{R}_0)$.

We leave the missing details to the reader.

By an easy application of the theory of natural dualities, a strong duality for $\boldsymbol{\mathcal{R}}_0$ is obtained by using the alter ego $\mathbb{R}_0 := \langle \{0, 1\}; 0, \mathcal{T}\rangle$, where $\mathcal{T}$ is the discrete topology; use [10, Cor. 3.3.9]. The dual category $\boldsymbol{\mathcal{X}}_0 := \mathsf{IS_cP}(\mathbb{R}_0)$ consists of isomorphic copies of closed substructures of powers of $\mathbb{R}_0$. The next result contains the salient points, which we present without proof.

Proposition 5.1. *The duality between $\boldsymbol{\mathcal{R}}_0$ and $\boldsymbol{\mathcal{X}}_0$ has the following properties.*

(1) *The dual category $\boldsymbol{\mathcal{X}}_0 := \mathsf{IS_cP}(\mathbb{R}_0)$ is the category of pointed Boolean spaces.*

(2) *The dual of $\mathbf{R} \in \boldsymbol{\mathcal{R}}_0$ is the pointed Boolean space*

$$D(\mathbf{R}) := \langle \hom(\mathbf{R}, \mathbf{R}_0); \underline{0}, \mathcal{T}\rangle,$$

which inherits its structure from $\mathbb{R}_0^R$; the distinguished point is the constant homomorphism $\underline{0}\colon \mathbf{R} \to \mathbf{R}_0$ onto $\{0\}$.

(3) *Equivalently, we can take* $D(\mathbf{R})$ *to be the set*

$$Y_{\mathbf{R}} := \{\, I \in \mathrm{Id}(\mathbf{R}) \mid I \text{ is maximal in } \mathrm{Id}(\mathbf{R}) \,\} \cup \{R\}$$

of maximal or improper ideals of $\mathbf{R}$ *appropriately topologised, with* R *as the distinguished point. A subbasis for the topology on* $Y_{\mathbf{R}}$ *consists of all sets of the form*

$$U_a := \{\, y \in Y_{\mathbf{R}} \mid a \in y \,\} \quad \text{and} \quad V_a := \{\, y \in Y_{\mathbf{R}} \mid a \notin y \,\},$$

for some $a \in R$. *A homeomorphism between* $\hom(\mathbf{R}, \mathbf{R}_0)$ *and* $Y_{\mathbf{R}}$ *is given by mapping a homomorphism* $u\colon \mathbf{R} \to \mathbf{R}_0$ *to the set* $u^{-1}(0)$.

(4) *The dual of a pointed Boolean space* $\mathbb{X} = \langle X; 0, \mathfrak{T}\rangle$ *is the subalgebra*

$$E(\mathbb{X}) := \langle \hom(\mathbb{X}, \mathbb{R}_0); +, \cdot, \underline{0}\rangle$$

of $\mathbf{R}_0^X$.

(5) *Equivalently, we can take* $E(\mathbb{X})$ *to be the (ring corresponding to the) generalised Boolean algebra consisting of all clopen subsets of* $\mathbb{X}$ *not containing* 0.

(6) *For all* $\mathbf{R} \in \mathfrak{R}_0$, *a subset* U *of* $Y_{\mathbf{R}}$ *is clopen and contains* R *if and only if it is of the form* U_a, *for some* $a \in R$.

(7) *For all* $\mathbf{R} \in \mathfrak{R}_0$, *the map* $a \mapsto V_a$ *is an isomorphism from* $\mathbf{R}$ *to the generalised Boolean algebra of clopen subsets of* $Y_{\mathbf{R}}$ *that do not contain* R.

Let $\mathbf{B}$ be a generalised Boolean algebra. The duality gives us a convenient route to the ideal lattice $\mathrm{Id}(\mathbf{B})$ of $\mathbf{B}$ or equivalently to the lattice of congruences on the Boolean ring $G(\mathbf{B})$. Since $\mathbf{B}$ and $G(\mathbf{B})$ have the same ideals, to lighten the notation, we shall denote the set of maximal or improper ideals of $G(\mathbf{B})$ by $Y_{\mathbf{B}}$ (rather than $Y_{G(\mathbf{B})}$).

Given a pointed Boolean space $\langle X; 0, \mathfrak{T}\rangle$, we shall denote the lattice of closed subsets of X that contain 0 by $\mathsf{Cl}_0(X)$. We denote the order-theoretic dual of a lattice $\mathbf{L}$ by $\mathbf{L}^\partial$.

Lemma 5.2. *Let* $\mathbf{B}$ *be a generalised Boolean algebra. Then*

$$\mathrm{Id}(\mathbf{B}) \cong \mathrm{Con}\, G(\mathbf{B}) \cong \mathsf{Cl}_0(Y_{\mathbf{B}})^\partial.$$

A lattice isomorphism $\nu\colon \mathsf{Cl}_0(Y_{\mathbf{B}})^\partial \to \mathrm{Con}\, G(\mathbf{B})$ *is given by defining*

$$(a,b) \in \nu(Z) \iff \big[(\forall z \in Z)\ a \in z \iff b \in z\big],$$

for all $a, b \in B$ *and all* $Z \in \mathsf{Cl}_0(Y_{\mathbf{B}})$.

Proof. The lattice $\mathrm{Id}(\mathbf{B})$ of ideals of $\mathbf{B}$ *equals* the lattice of ideals of the ring $G(\mathbf{B})$. As the lattice of ideals of every ring is isomorphic to its lattice of congruences, we have $\mathrm{Id}(\mathbf{B}) \cong \mathrm{Con}\, G(\mathbf{B})$. We have a strong duality between $\mathfrak{R}_0$ and $\mathfrak{X}_0$, and $\mathsf{ISP}(\mathbf{R}_0)$ is the variety of Boolean rings and is therefore closed under homomorphic images. Hence we may apply Theorem 3.2.1 of Clark and Davey [10] to conclude that the lattice of congruences on $G(\mathbf{B})$ is isomorphic to the order-theoretic dual of the lattice $\mathsf{Cl}_0(D(G(\mathbf{B})))$ of closed substructures of $D(G(\mathbf{B}))$. Of course, here closed substructure means simply a closed subset containing the distinguished point. Finally, $\mathsf{Cl}_0(D(G(\mathbf{B}))) \cong \mathsf{Cl}_0(Y_{\mathbf{B}})$ by Proposition 5.1(3).

The description of the isomorphism $\nu\colon \mathsf{Cl}_0(Y_{\mathbf{B}})^{\partial} \to \operatorname{Con} G(\mathbf{B})$ is obtained by interpreting the proof of Clark and Davey [10, Thm 3.2.1] in the context of generalised Boolean algebras/Boolean rings. □

6. Boolean products

Boolean products, as an alternative to sheaves over Boolean spaces, were introduced by Burris and Werner in [6,7] and were popularised in the text by Burris and Sankappanavar [5].

Let $\mathbf{A} \leqslant \prod_{x\in X} \mathbf{A}_x$ be a subdirect product with $X \neq \varnothing$. Then $\mathbf{A}$ (or more correctly $\langle \mathbf{A}, \mathcal{T}_{\mathbf{A}}\rangle$) is a *Boolean product of* $\{\, \mathbf{A}_x \mid x \in X \,\}$ (relative to $\mathcal{T}_{\mathbf{A}}$) if $\mathcal{T}_{\mathbf{A}}$ is a Boolean topology on X such that

(BP$_1$) $[\![a = b]\!]$ is clopen, for all $a, b \in A$,

(BP$_2$) if $a, b \in A$ and N is a clopen subset of X, then $a{\restriction}_N \cup b{\restriction}_{X\setminus N} \in A$.

For (BP$_1$) we say that $\mathbf{A}$ *has clopen equalisers*, and for (BP$_2$) we say that $\mathbf{A}$ satisfies the *patchwork property*. Because of the connection between Boolean products and representations as algebras of global sections over Boolean spaces, the algebra $\mathbf{A}_x$ is referred to as the *stalk* at x. (Much has been written about Boolean products, but our presentation will be self-contained and will require little more than the definition.)

The patchwork property is intimately connected to factor congruences, as the following observation shows.

Lemma 6.1. *Let $\mathbf{A} \leqslant \prod_{x\in X} \mathbf{A}_x$ be a subdirect product, with $X \neq \varnothing$, and let $N \subseteq X$. Then $(\theta_N, \theta_{X\setminus N})$ is a pair of factor congruences on $\mathbf{A}$ if and only if, for all $a, b \in A$, we have $a{\restriction}_N \cup b{\restriction}_{X\setminus N} \in A$.*

Proof. Assume that $(\theta_N, \theta_{X\setminus N})$ is a pair of factor congruences on $\mathbf{A}$ and let $a, b \in A$. Then there exists $c \in A$ with $(a, c) \in \theta_N$ and $(c, b) \in \theta_{X\setminus N}$. Thus, $N \subseteq [\![a = c]\!]$ and $X\setminus N \subseteq [\![c = b]\!]$. Hence $c{\restriction}_N = a{\restriction}_N$ and $c{\restriction}_{X\setminus N} = b{\restriction}_{X\setminus N}$, that is, $a{\restriction}_N \cup b{\restriction}_{X\setminus N} = c$. Thus $a{\restriction}_N \cup b{\restriction}_{X\setminus N} \in A$.

Conversely, assume that $a{\restriction}_N \cup b{\restriction}_{X\setminus N} \in A$, for all $a, b \in A$. Let $a, b \in A$ and define $c := a{\restriction}_N \cup b{\restriction}_{X\setminus N}$. Reversing the argument just given, we see that $(a, c) \in \theta_N$ and $(c, b) \in \theta_{X\setminus N}$. It follows that $\theta_N \cdot \theta_{X\setminus N} = 1_A$. Since it is trivial that $\theta_N \cap \theta_{X\setminus N} = 0_A$, we conclude that $(\theta_N, \theta_{X\setminus N})$ is a pair of factor congruences on $\mathbf{A}$. □

Let $\mathbf{A}$ be a non-trivial algebra in a filtral variety. By Proposition 4.1(5), $\operatorname{Con_c} \mathbf{A}$ is a generalised Boolean algebra. We will abbreviate $Y_{\operatorname{Con_c}\mathbf{A}}$ to $Y_{\mathbf{A}}$. Hence

$$Y_{\mathbf{A}} := \{\, I \in \operatorname{Id}(\operatorname{Con_c} \mathbf{A}) \mid I \text{ is maximal in } \operatorname{Id}(\operatorname{Con_c} \mathbf{A}) \,\} \cup \{\operatorname{Con_c} \mathbf{A}\}$$

is the set of maximal or improper ideals of $\operatorname{Con_c} \mathbf{A}$; see Proposition 5.1(3). Let

$$X_{\mathbf{A}} := \{\, \alpha \in \operatorname{Con} \mathbf{A} \mid \alpha \text{ is maximal in } \operatorname{Con} \mathbf{A} \,\} \cup \{1_A\}$$

be the set of maximal or improper congruences on $\mathbf{A}$. By Remark 3.4, the map $\varphi\colon \alpha \mapsto {\downarrow}\alpha \cap \operatorname{Con_c} \mathbf{A}$ is a bijection between $X_{\mathbf{A}}$ and $Y_{\mathbf{A}}$ that maps the distinguished point 1_A of $X_{\mathbf{A}}$ to the distinguished point $\operatorname{Con_c} \mathbf{A}$ of $Y_{\mathbf{A}}$.

Lemma 6.2. *Let* $\mathbf{A}$ *be a non-trivial algebra in a filtral variety* $\boldsymbol{\mathcal{V}}$ *and define* $\eta\colon \mathbf{A} \to \prod_{\alpha\in X_{\mathbf{A}}} \mathbf{A}/\alpha$ *to be the natural map.*

(1) *A subbasis for the topology on the set* $Y_{\mathbf{A}}$ *of maximal or improper ideals of the generalised Boolean algebra* $\operatorname{Con_c}\mathbf{A}$ *consists of all sets of the form*

$$U_\kappa := \{\, y \in Y_{\mathbf{A}} \mid \kappa \in y \,\} \quad \text{and} \quad V_\kappa := \{\, y \in Y_{\mathbf{A}} \mid \kappa \notin y \,\},$$

for some $\kappa \in \operatorname{Con_c}\mathbf{A}$.

(2) *The map* η *is a subdirect embedding.*

(3) *The bijection* $\varphi\colon X_{\mathbf{A}} \to Y_{\mathbf{A}}$ *satisfies* $\varphi\big([\![\eta(a) = \eta(b)]\!]\big) = U_{\mathrm{Cg}^{\mathbf{A}}(a,b)}$, *for all* $a, b \in A$.

Proof. (1) follows directly from Proposition 5.1(3).

(2) Since $\boldsymbol{\mathcal{V}}$ is semisimple, we have $\bigcap X_{\mathbf{A}} = 0_A$ as $X_{\mathbf{A}}$ includes all of the maximal congruences on $\mathbf{A}$. Hence η is a subdirect embedding.

(3) Let $a, b \in A$. We have

$$\begin{aligned}
\varphi\big([\![\eta(a) = \eta(b)]\!]\big) &= \{\, \varphi(\gamma) \mid \gamma \in X_{\mathbf{A}} \ \ \&\ \ a/\gamma = b/\gamma \,\}\\
&= \{\, {\downarrow}\gamma \cap \operatorname{Con_c}\mathbf{A} \mid \gamma \in X_{\mathbf{A}} \ \ \&\ \ a/\gamma = b/\gamma \,\}\\
&= \{\, {\downarrow}\gamma \cap \operatorname{Con_c}\mathbf{A} \mid \gamma \in X_{\mathbf{A}} \ \ \&\ \ \mathrm{Cg}^{\mathbf{A}}(a,b) \subseteq \gamma \,\}\\
&= \{\, y \in Y_{\mathbf{A}} \mid \mathrm{Cg}^{\mathbf{A}}(a,b) \in y \,\}\\
&= U_{\mathrm{Cg}^{\mathbf{A}}(a,b)}.
\end{aligned}$$

□

Lemma 6.3. *Let* $\mathbf{A}$ *be a non-trivial algebra in a filtral variety and define* $\eta\colon \mathbf{A} \to \prod_{\alpha\in X_{\mathbf{A}}} \mathbf{A}/\alpha$ *to be the natural embedding. Define* $\mathcal{T}_{\mathbf{A}}$ *to be the topology on* $X_{\mathbf{A}}$ *with subbasis*

$$\mathcal{S}_{\mathbf{A}} := \big\{\, [\![\eta(a) = \eta(b)]\!] \;\big|\; a, b \in A \,\big\} \cup \big\{\, X_{\mathbf{A}} \backslash [\![\eta(a) = \eta(b)]\!] \;\big|\; a, b \in A \,\big\}.$$

(1) *The map* $\varphi\colon X_{\mathbf{A}} \to Y_{\mathbf{A}} : \alpha \mapsto {\downarrow}\alpha \cap \operatorname{Con_c}\mathbf{A}$ *is a homeomorphism of pointed spaces.*

(2) $\langle X_{\mathbf{A}}; 1_A, \mathcal{T}_{\mathbf{A}}\rangle$ *is a pointed Boolean space.*

Proof. (1) We will instead prove that $\varphi^{-1}\colon Y_{\mathbf{A}} \to X_{\mathbf{A}}$ is a homeomorphism. By Proposition 5.1, $Y_{\mathbf{A}}$ is a Boolean space and therefore is compact. Let $\alpha, \beta \in X_{\mathbf{A}}$ with $\alpha \neq \beta$. Without loss of generality, we may assume that $\beta \neq 1_A$. Thus there exist $a, b \in A$ with $a \neq b$ and $(a, b) \in \alpha \backslash \beta$, whence

$$[\![\eta(a) = \eta(b)]\!] := \{\, \gamma \in X_{\mathbf{A}} \mid a/\gamma = b/\gamma \,\}$$

is a clopen set containing α but not β. Hence the topology on $X_{\mathbf{A}}$ is Hausdorff. Thus, to prove that $\varphi^{-1}\colon Y_{\mathbf{A}} \to X_{\mathbf{A}}$ is a homeomorphism, it suffices to prove that φ^{-1} is continuous, for which it suffices to prove that φ maps subbasic open sets in $\mathcal{S}_{\mathbf{A}}$ to open subsets of $Y_{\mathbf{A}}$. (We are using the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.)

Let $a, b \in A$ and consider the subbasic open set $[\![\eta(a) = \eta(b)]\!]$ in $\mathcal{S}_{\mathbf{A}}$. By Lemma 6.2(3), we have $\varphi\big([\![\eta(a) = \eta(b)]\!]\big) = U_{\mathrm{Cg}^{\mathbf{A}}(a,b)}$, which is open in $Y_{\mathbf{A}}$ by Lemma 6.2(1). Now consider the subbasic open set $X_{\mathbf{A}} \backslash [\![\eta(a) = \eta(b)]\!]$ in $\mathcal{S}_{\mathbf{A}}$. By Lemma 6.2(3), we have

$$\varphi\big(X_{\mathbf{A}} \backslash [\![\eta(a) = \eta(b)]\!]\big) = Y_{\mathbf{A}} \backslash \varphi\big([\![\eta(a) = \eta(b)]\!]\big) = Y_{\mathbf{A}} \backslash U_{\mathrm{Cg}^{\mathbf{A}}(a,b)},$$

which is open in $Y_{\mathbf{A}}$ by Lemma 6.2(1). Hence $\varphi^{-1}\colon Y_{\mathbf{A}} \to X_{\mathbf{A}}$ is continuous, as required.

(2) follows immediately from (1) since $Y_{\mathbf{A}}$ is a Boolean space. □

7. The many lives of Con A

Let $\boldsymbol{\mathcal{V}}$ be a filtral variety and let $\mathbf{A}$ be a non-trivial algebra in $\boldsymbol{\mathcal{V}}$. Since $\operatorname{Con_c}\mathbf{A}$ is a generalised Boolean algebra, we can apply Lemma 5.2 to yield information about $\operatorname{Con}\mathbf{A}$.

Lemma 7.1. *Let $\mathbf{A}$ be a non-trivial algebra in a filtral variety and define $\eta\colon \mathbf{A} \to \prod_{\alpha\in X_{\mathbf{A}}} \mathbf{A}/\alpha$ to be the natural embedding.*

(1) $\operatorname{Con}\mathbf{A} \cong \operatorname{Id}(\operatorname{Con_c}\mathbf{A}) \cong \operatorname{Con} G(\operatorname{Con_c}\mathbf{A}) \cong \mathsf{Cl}_0(Y_{\mathbf{A}})^\partial \cong \mathsf{Cl}_0(X_{\mathbf{A}})^\partial$.

(2) *An isomorphism $\rho\colon \mathsf{Cl}_0(X_{\mathbf{A}})^\partial \to \operatorname{Con}\mathbf{A}$ is given by*

$$\rho(N) := \bigcap N, \quad \text{for all } N \in \mathsf{Cl}_0(X_{\mathbf{A}}).$$

(3) *An isomorphism $\mu\colon \mathsf{Cl}_0(X_{\mathbf{A}})^\partial \to \operatorname{Con}\eta(\mathbf{A})$ is given by*

$$\mu(N) := \theta_N, \quad \text{for all } N \in \mathsf{Cl}_0(X_{\mathbf{A}}).$$

(4) *Let N be clopen in $X_{\mathbf{A}}$ with $1_A \in N$. Then the unique complement of θ_N in $\operatorname{Con}\eta(\mathbf{A})$ is $\theta_{X_{\mathbf{A}}\setminus N} = \theta_{(X_{\mathbf{A}}\setminus N)\cup\{1_A\}}$.*

Proof. The lattice $\operatorname{Con}\mathbf{A}$ is isomorphic to $\operatorname{Id}(\operatorname{Con_c}\mathbf{A})$, by Remark 3.4, and the pointed space $Y_{\mathbf{A}}$ is isomorphic in $\boldsymbol{\mathfrak{X}}_0$ to $X_{\mathbf{A}}$, by Lemma 6.3. Hence (1) follows at once from Lemma 5.2 applied to the generalised Boolean algebra $\mathbf{B} = \operatorname{Con_c}\mathbf{A}$.

(2) To prove that ρ is an isomorphism, we work backwards along the chain of isomorphisms in (1). Let N be a closed subset of $X_{\mathbf{A}}$ containing 1_A. The corresponding subset of $Y_{\mathbf{A}}$ is $\varphi(N) = \{\, {\downarrow}\alpha \cap \operatorname{Con_c}\mathbf{A} \mid \alpha \in N \,\}$. By Lemma 5.2, the corresponding congruence $\nu(\varphi(N))$ on $G(\operatorname{Con_c}\mathbf{A})$ is given by

$$\begin{aligned}(\theta_1,\theta_2) \in \nu(\varphi(N)) &\iff \big[\big(\forall z \in \varphi(N)\big)\ \theta_1 \in z \iff \theta_2 \in z\big]\\ &\iff \big[\big(\forall \alpha \in N\big)\ \theta_1 \in {\downarrow}\alpha \iff \theta_2 \in {\downarrow}\alpha\big],\end{aligned}$$

for all $\theta_1, \theta_2 \in \operatorname{Con_c}\mathbf{A}$. The corresponding ideal of $\operatorname{Con_c}\mathbf{A}$ is

$$\begin{aligned}0_A/\nu(\varphi(N)) &= \{\, \theta \in \operatorname{Con_c}\mathbf{A} \mid (\theta, 0_A) \in \nu(\varphi(N)) \,\}\\ &= \{\, \theta \in \operatorname{Con_c}\mathbf{A} \mid \big(\forall \alpha \in N\big)\ \theta \in {\downarrow}\alpha \iff 0_A \in {\downarrow}\alpha \,\}\\ &= \{\, \theta \in \operatorname{Con_c}\mathbf{A} \mid \big(\forall \alpha \in N\big)\ \theta \in {\downarrow}\alpha \,\}\\ &= \bigcap \{\, {\downarrow}\alpha \cap \operatorname{Con_c}\mathbf{A} \mid \alpha \in N \,\}\\ &= \bigcap_{\alpha\in N} \varphi(\alpha).\end{aligned}$$

Finally, the congruence on $\mathbf{A}$ corresponding to the ideal $\bigcap_{\alpha\in N}\varphi(\alpha)$ of $\operatorname{Con_c}\mathbf{A}$ is

$$\varphi^{-1}\Bigg(\bigcap_{\alpha\in N}\varphi(\alpha)\Bigg) = \varphi^{-1}\Bigg(\bigwedge_{\alpha\in N}\varphi(\alpha)\Bigg) = \bigwedge_{\alpha\in N}\varphi^{-1}(\varphi(\alpha)) = \bigwedge_{\alpha\in N}\alpha = \bigcap N,$$

as required.

(3) will follow from (2) once we show that θ_N is the congruence on $\eta(\mathbf{A})$ corresponding to the congruence $\bigcap N$ on $\mathbf{A}$, for every subset N of $X_{\mathbf{A}}$. Let $N \subseteq X_{\mathbf{A}}$. Then, for all $a, b \in A$,

$$\begin{aligned}(a,b) \in \bigcap N &\iff (a,b) \in \alpha, \text{ for all } \alpha \in N\\ &\iff a/\alpha = b/\alpha, \text{ for all } \alpha \in N\\ &\iff \eta(a)(\alpha) = \eta(b)(\alpha), \text{ for all } \alpha \in N\\ &\iff N \subseteq [\![\eta(a) = \eta(b)]\!]\\ &\iff (\eta(a), \eta(b)) \in \theta_N,\end{aligned}$$

as required.

(4) Let N be clopen in $X_{\mathbf{A}}$ with $1_A \in N$. Then $(X_{\mathbf{A}} \backslash N) \cup \{1_A\}$ is a complement of N in $\mathsf{Cl}_0(X_{\mathbf{A}})$, and consequently $\theta_{(X_{\mathbf{A}} \backslash N) \cup \{1_A\}}$ is a complement of θ_N in $\mathrm{Con}\,\eta(\mathbf{A})$, by (3). Since $\eta(a)(1_A) = a/1_A = b/1_A = \eta(b)(1_A)$, for all $a, b \in A$, we have $\theta_{X_{\mathbf{A}} \backslash N} = \theta_{(X_{\mathbf{A}} \backslash N) \cup \{1_A\}}$. The uniqueness of the complement follows from the fact that $\mathrm{Con}\,\eta(\mathbf{A})$ is distributive. □

8. Proofs of the main theorems

Let $\mathbf{A}$ be a non-trivial algebra in a filtral variety. Following Werner [24], we refer to the topological space $X_{\mathbf{A}}$ of Lemma 6.3 as the *spectrum* of $\mathbf{A}$ and the subspace $X^{\flat}_{\mathbf{A}} := X_{\mathbf{A}} \backslash \{1_A\}$, consisting of the maximal congruences on $\mathbf{A}$, as the *proper spectrum* of $\mathbf{A}$. We refer to the embedding $\eta\colon \mathbf{A} \to \prod_{\alpha \in X_{\mathbf{A}}} \mathbf{A}/\alpha$ as the *spectral representation of* $\mathbf{A}$ and the embedding $\eta\colon \mathbf{A} \to \prod_{\alpha \in X^{\flat}_{\mathbf{A}}} \mathbf{A}/\alpha$ as the *proper spectral representation of* $\mathbf{A}$.

We now prove our main result.

Theorem 8.1. *Let* $\boldsymbol{\mathcal{V}}$ *be a filtral variety and let* $\mathbf{A}$ *be a non-trivial algebra in* $\boldsymbol{\mathcal{V}}$*. The following are equivalent*:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) $\mathbf{A}$ *has permuting congruences*;
(3) *the spectral representation* η *of* $\mathbf{A}$ *is an isomorphism onto a Boolean product of* $\{\, \mathbf{A}/\alpha \mid \alpha \in X_{\mathbf{A}} \,\}$;
(4) $\mathbf{A}$ *is isomorphic to a Boolean product in which each non-trivial stalk is a simple algebra from* $\boldsymbol{\mathcal{V}}$ *and moreover at most one stalk is trivial.*

Proof. (1) is equivalent to (2) by Proposition 4.1(6). Since (3) implies (4) is trivial, it remains to prove that (1) implies (3) and that (4) implies (1).

(1) ⇒ (3): Assume $\mathbf{A}$ has factor principal congruences. Since $\eta(\mathbf{A})$ satisfies (BP$_1$) by construction (see Lemma 6.3), it remains to prove that $\eta(\mathbf{A})$ satisfies (BP$_2$). Let N be a clopen subset of $X_{\mathbf{A}}$. By Lemma 6.1, it suffices to prove that $(\theta_N, \theta_{X_{\mathbf{A}} \backslash N})$ is a pair of factor congruences on $\eta(\mathbf{A})$. By symmetry we may assume that $1_A \in N$. By Lemma 6.3(1), $\varphi(N)$ is a clopen subset of $Y_{\mathbf{A}}$

containing $\mathrm{Con_c}\,\mathbf{A}$ and, by Proposition 5.1(6), there exists $\kappa \in \mathrm{Con_c}\,\mathbf{A}$ with $\varphi(N) = U_\kappa$. For all $a, b \in A$,

$$\begin{aligned}
(\eta(a), \eta(b)) \in \theta_N &\iff N \subseteq [\![\eta(a) = \eta(b)]\!] \\
&\iff \varphi(N) \subseteq \varphi([\![\eta(a) = \eta(b)]\!]) \\
&\iff U_\kappa \subseteq \varphi([\![\eta(a) = \eta(b)]\!]) \\
&\iff U_\kappa \subseteq U_{\mathrm{Cg}^{\mathbf{A}}(a,b)} \qquad \text{by Lemma 6.2(3)} \\
&\iff (\forall y \in Y_{\mathbf{A}})\ \big[\kappa \in y \implies \mathrm{Cg}^{\mathbf{A}}(a,b) \in y\big] \\
&\iff \mathrm{Cg}^{\mathbf{A}}(a,b) \subseteq \kappa \\
&\iff (a,b) \in \kappa.
\end{aligned}$$

The second-last equivalence follows from the fact that if $u \nleqslant v$ in a generalised Boolean algebra $\mathbf{B}$, then there exists a prime and therefore maximal ideal y of $\mathbf{B}$ with $v \in y$ but $u \notin y$. We have proved that $\theta_N = \eta(\kappa)$. Since $\kappa \in \mathrm{Con_c}\,\mathbf{A}$ and $\mathbf{A}$ has factor principal congruences, κ is a factor congruence, by Lemma 3.7, and hence θ_N is a factor congruence on $\eta(\mathbf{A})$. Thus there is a congruence β on $\eta(\mathbf{A})$ such that (θ_N, β) is a pair of factor congruences on $\eta(\mathbf{A})$. Since $\mathrm{Con}\,\eta(\mathbf{A})$ is distributive, β is the unique complement of θ_N in $\mathrm{Con}\,\eta(\mathbf{A})$. Hence, by Lemma 7.1(4), $\beta = \theta_{X_{\mathbf{A}} \setminus N}$. Thus $(\theta_N, \theta_{X_{\mathbf{A}} \setminus N})$ is a pair of factor congruences on $\eta(\mathbf{A})$ and, by Lemma 6.1, $\eta(\mathbf{A})$ satisfies (BP$_2$).

(4) ⇒ (1): Without loss of generality, we may assume that, for some non-empty set X, the algebra $\mathbf{A}$ is a Boolean product of a family $\{\, \mathbf{A}_x \mid x \in X \,\}$ of algebras from $\boldsymbol{\mathcal{V}}$ each of which is either simple or trivial with at most one trivial. Let $a, b \in A$. We must prove that $\mathrm{Cg}^{\mathbf{A}}(a,b)$ is a factor congruence. We claim that $\mathrm{Cg}^{\mathbf{A}}(a,b) = \theta_N$, where $N = [\![a = b]\!]$. In the case where each $\mathbf{A}_x$ is simple, this follows at once from Proposition 4.1(2)(iii). Now assume that $\mathbf{A}_{x_0}$ is the unique trivial factor. Let $\pi\colon \prod_{x \in X} \mathbf{A}_x \to \prod_{x \in X\setminus\{x_0\}} \mathbf{A}_x$ be the natural projection and note that π is an isomorphism. Again by Proposition 4.1(2)(iii), we have

$$\mathrm{Cg}^{\pi(\mathbf{A})}(\pi(a), \pi(b)) = \theta_{N'}, \quad \text{where } N' = \{\, x \in X\setminus\{x_0\} \mid \pi(a)(x) = \pi(b)(x) \,\}.$$

Hence $\mathrm{Cg}^{\mathbf{A}}(a,b) = \theta_N$, where $N = N' \cup \{x_0\} = [\![a = b]\!]$, as claimed. Since, by (BP$_1$), N is clopen, (BP$_2$) along with Lemma 6.1 guarantees that θ_N, and therefore $\mathrm{Cg}^{\mathbf{A}}(a,b)$, is a factor congruence. □

When 1_A is compact, we can move the focus from $\mathrm{Con_c}\,\mathbf{A}$ to $\mathrm{Con_f}\,\mathbf{A}$ and obtain a stronger result.

Theorem 8.2. *Let* $\boldsymbol{\mathcal{V}}$ *be a filtral variety and let* $\mathbf{A}$ *be a non-trivial algebra in* $\boldsymbol{\mathcal{V}}$ *such that* 1_A *is compact in* $\mathrm{Con}\,\mathbf{A}$. *The following are equivalent*:

(1) $\mathbf{A}$ *has factor principal congruences*;
(2) $\mathbf{A}$ *has permuting congruences*;
(3) $\mathrm{Con}\,\mathbf{A}$ *is isomorphic to the lattice* $\mathrm{Id}(\mathrm{Con_f}\,\mathbf{A})$ *of ideals of the Boolean algebra* $\mathrm{Con_f}\,\mathbf{A}$ *of factor congruences on* $\mathbf{A}$ *via* $\psi\colon \alpha \mapsto {\downarrow}\alpha \cap \mathrm{Con_f}\,\mathbf{A}$, *for all* $\alpha \in \mathrm{Con}\,\mathbf{A}$;

(4) *the proper spectral representation of* $\mathbf{A}$ *is an isomorphism onto a Boolean product of* $\{\, \mathbf{A}/\alpha \mid \alpha \in X^{\flat}_{\mathbf{A}} \,\}$;

(5) $\mathbf{A}$ *is isomorphic to a Boolean product of simple algebras from* $\boldsymbol{\mathcal{V}}$.

Proof. Again, (1) is equivalent to (2) by Proposition 4.1(6), and (1) is equivalent to (3) by Lemma 3.7. Now assume (1). The previous theorem implies that the spectral representation η of $\mathbf{A}$ is an isomorphism onto a Boolean product of $\{\, \mathbf{A}/\alpha \mid \alpha \in X_{\mathbf{A}} \,\}$. Since 1_A is compact in $\operatorname{Con} \mathbf{A}$, it is easy to see that $\operatorname{Con_c} \mathbf{A}$ is an isolated point of $Y_{\mathbf{A}}$. Hence, by Lemma 6.3(1), 1_A is an isolated point of $X_{\mathbf{A}}$ and consequently $X^{\flat}_{\mathbf{A}}$ is a closed subspace of $X_{\mathbf{A}}$. Using the fact that (BP$_1$) and (BP$_2$) hold with respect to $X_{\mathbf{A}}$, it follows easily that (BP$_1$) and (BP$_2$) hold with respect to $X^{\flat}_{\mathbf{A}}$, and consequently the proper spectral representation of $\mathbf{A}$ is an isomorphism onto a Boolean product of $\{\, \mathbf{A}/\alpha \mid \alpha \in X^{\flat}_{\mathbf{A}} \,\}$. Hence (4) holds. Finally, (4) implies (5) is trivial, and, by the previous theorem, (5) implies (1). □

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