❙ Algebra Universalis

On the number of universal algebraic geometries

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Abstract. The *algebraic geometry* of a universal algebra $\mathbf{A}$ is defined as the collection of solution sets of systems of term equations. Two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ are called *algebraically equivalent* if they have the same algebraic geometry. We prove that on a finite set A with $|A|$ there are countably many algebraically inequivalent Mal'cev algebras and that on a finite set A with $|A|$ there are continuously many algebraically inequivalent algebras.

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1. Introduction

Universal algebraic geometry, introduced in [19,7,11], is based on the notion of *algebraic sets.* Given a universal algebra $\mathbf{A} = (A; (f_i)_{i \in I})$, a subset B of A^n is *algebraic* if it is the solution set of a system of (possibly infinitely many) term equations in the language of $\mathbf{A}$. Following [18], we denote the collection of the algebraic subsets of A^n by $\mathrm{Alg}_n \mathbf{A}$ and we define the *universal algebraic geometry* of $\mathbf{A}$ by $\mathrm{Alg}\,\mathbf{A} := \bigcup_{n \in \mathbb{N}} \mathrm{Alg}_n \mathbf{A}$. Clearly, for a universal algebra $\mathbf{A}$, $\mathrm{Alg}\,\mathbf{A}$ is completely determined by its clone of term functions (cf. [9,20]). We define the universal algebraic geometry of a clone $\mathcal{C}$ on a set A by $\mathrm{Alg}\,\mathcal{C} := \mathrm{Alg}(A; \mathcal{C})$. Following [18], we say that two clones $\mathcal{C}$ and $\mathcal{D}$ on the same set A are algebraically equivalent, $\mathcal{C} \sim_{\mathrm{alg}} \mathcal{D}$, if $\mathrm{Alg}\,\mathcal{C} = \mathrm{Alg}\,\mathcal{D}$. In [18,22] one finds examples of different clones that are algebraically equivalent; in fact Tóth and Waldhauser [22] proved that on the two element set there are only finitely many algebraically inequivalent clones. Pinus showed that on each finite set there are only finitely many universal algebraic geometries closed under taking

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unions [18, 6]. The aim of this note is to provide infinite families of algebraically inequivalent clones on a finite set.

Analysing the construction of $2^{\aleph_0}$ clones on $\{0, 1, 2\}$ in [14], we obtain:

Theorem 1.1. *Let A be a finite set with $|A| \geq 3$. Then there are $2^{\aleph_0}$ algebraically inequivalent clones on A.*

The proof will be given in Section 5.

A clone $\mathcal{C}$ is called a *Mal'cev clone* if it contains a ternary function such that for all $a, b \in A$,
$$d(a, b, b) = d(b, b, a) = a,$$
and it is called *constantive* if every unary constant function on A is in $\mathcal{C}$. Using Idziak's construction from [13], we prove:

Theorem 1.2. *Let A be a finite set with $|A| \geq 4$. Then there are exactly $\aleph_0$ algebraically inequivalent constantive Mal'cev clones on A.*

The proof will be given in Section 4. It relies on the following fact:

Theorem 1.3. *Let p be a prime, and let $n \in \mathbb{N}$. Then there are exactly $\aleph_0$ algebraically inequivalent clones that contain $\mathrm{Pol}(\mathbb{Z}_{np^2}, +)$.*

The proof will be given in Section 3.

In Table 1 we summarize our current knowledge on the number of algebraically inequivalent clones on a finite set.

2. Notation

We write $\mathbb{N}$ for the set of positive integers and for $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$. For a set A, the i-th component of $\boldsymbol{a} \in A^n$ is denoted by a_i and $\boldsymbol{a}(i)$, and for an algebra $\mathbf{A}$ with universe A, we will use the notions *clone*, $\mathrm{Pol}_n(\mathbf{A})$ and $\mathrm{Clo}_n(\mathbf{A})$ as they are commonly used in universal algebra [9, 17]. For $k \in \mathbb{N}$ with $k \leq n$, we define the k-th n-ary projection $\pi_k^{(n)} \colon A^n \to A$ by $\pi_k^{(n)}(a_1, \ldots, a_n) = a_k$ for all $\boldsymbol{a} \in A^n$, and we set $\mathcal{J}_A := \{\pi_k^{(n)} \mid n \in \mathbb{N} \text{ and } k \leq n\}$. For a clone $\mathcal{C}$ on A, $\mathcal{C}^{[n]}$ denotes the set of n-ary functions in $\mathcal{C}$. For $k \in \mathbb{N}$, for $\sigma \colon [k] \to [n]$ and for $f \in \mathcal{C}^{[k]}$, we define $f_\sigma \in \mathcal{C}^{[n]}$ by $f_\sigma(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for all $x_1, \ldots, x_n \in A$, and we call f_σ a *minor* of f. If t is a k-ary term in the language of a universal algebra $\mathbf{A}$, we write t_σ for the term $t(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. Observe that in this case $(t_\sigma)^{\mathbf{A}} = (t^{\mathbf{A}})_\sigma$.

Let A be a set and let $\mathcal{C}$ be a clone on A. For $n \in \mathbb{N}$ and for $X \subseteq A^n$ we say that X is *algebraic with respect to $\mathcal{C}$* if there exist an index set I and two families $(p_i)_{i \in I}$, $(q_i)_{i \in I}$ of functions in $\mathcal{C}^{[n]}$ such that $X = \{\boldsymbol{x} \in A^n \mid \forall i \in I \colon p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}$. We define $\mathrm{Alg}_n \mathcal{C}$ to be the collection of all the subsets of A^n that are algebraic with respect to $\mathcal{C}$ and we define the *algebraic geometry* of $\mathcal{C}$ by $\mathrm{Alg}\, \mathcal{C} := \bigcup_{n \in \mathbb{N}} \mathrm{Alg}_n \mathcal{C}$. For an algebra $\mathbf{A}$, we set $\mathrm{Alg}\, \mathbf{A} := \mathrm{Alg}\, \mathrm{Clo}(\mathbf{A})$.

For $n \in \mathbb{N}$ and for $X \subseteq A^n$, we define $\mathrm{V}_{\mathcal{C}}(X)$ to be the intersection of all the elements of $\mathrm{Alg}_n \mathcal{C}$ that contain X as a subset. Since the intersection

TABLE 1. The first column lists the properties of the clones we consider, the second column reports the number of such clones and the third column lists the number of such clones modulo $\sim_{\mathrm{alg}}$.

Property of the clone on an n-element set	Number of clones	Number of clones modulo $\sim_{\mathrm{alg}}$
$n = 2$	$\aleph_0$ [21]	Finite [22]
$n \geq 3$	$2^{\aleph_0}$ [14]	$2^{\aleph_0}$ (Theorem 1.1)
Constantive, $n \geq 3$	$2^{\aleph_0}$ [1]	?
Equationally additive	?	Finite [18,6]
Mal'cev, $n \leq 3$	Finite [8]	Finite
Constantive, Mal'cev, $n \geq 4$	$\aleph_0$ [2,13]	$\aleph_0$ (Theorem 1.2)
Contains $\mathrm{Clo}(\mathbb{Z}_n, +)$, n squarefree	Finite [12,4,16]	Finite
Contains $\mathrm{Pol}(\mathbb{Z}_{mp^2}, +)$, $m \geq 1$	$\aleph_0$ [2,13]	$\aleph_0$ (Theorem 1.3)

The number p is assumed to be prime. Question marks represent open problems.

of a collection of elements of $\mathrm{Alg}_n\,\mathcal{C}$ is an element of $\mathrm{Alg}_n\,\mathcal{C}$, we infer that $\mathrm{V}_{\mathcal{C}}(X) \in \mathrm{Alg}_n\,\mathcal{C}$. The following lemma will be used to assess whether a set X is algebraic with respect to a clone $\mathcal{C}$, which is equivalent to $X = \mathrm{V}_{\mathcal{C}}(X)$.

Lemma 2.1. *Let A be a set, let $\mathcal{C}$ be a clone on A, let $n \in \mathbb{N}$, let $X \subseteq A^n$, and let $\boldsymbol{a} \in A^n$. Then we have*

$$\boldsymbol{a} \in \mathrm{V}_{\mathcal{C}}(X) \Longleftrightarrow \Big(\forall f, g \in \mathcal{C}^{[n]} : f|_X = g|_X \Rightarrow f(\boldsymbol{a}) = g(\boldsymbol{a})\Big).$$

Proof. "$\Leftarrow$": We assume that for all $f, g \in \mathcal{C}^{[n]}$ with $f|_X = g|_X$, we have $f(\boldsymbol{a}) = g(\boldsymbol{a})$, and we show that then $\boldsymbol{a} \in \mathrm{V}_{\mathcal{C}}(X)$. To this end, we show that for each $B \in \mathrm{Alg}_n\,\mathcal{C}$ with $X \subseteq B$, we have $\boldsymbol{a} \in B$. Since B is algebraic with respect to $\mathcal{C}$, there exist an index set I and two families $(p_i)_{i \in I}$, $(q_i)_{i \in I}$ of functions in $\mathcal{C}^{[n]}$ such that $B = \{\boldsymbol{x} \in A^n \mid \forall i \in I : p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}$. Since $X \subseteq B$, we have $p_i|_X = q_i|_X$ for all $i \in I$. Therefore, the assumption yields $p_i(\boldsymbol{a}) = q_i(\boldsymbol{a})$ for all $i \in I$, and thus $\boldsymbol{a} \in B$.

"$\Rightarrow$": We assume that there exist $f, g \in \mathcal{C}^{[n]}$ such that $f|_X = g|_X$ and $f(\boldsymbol{a}) \neq g(\boldsymbol{a})$. Then $B := \{\boldsymbol{x} \in A^n \mid f(\boldsymbol{x}) = g(\boldsymbol{x})\}$ satisfies $B \in \mathrm{Alg}_n\,\mathcal{C}$, $X \subseteq B$, and $\boldsymbol{a} \notin B$. Thus, $\boldsymbol{a} \notin \mathrm{V}_{\mathcal{C}}(X)$. $\qquad\square$

3. Countably many algebraically inequivalent constantive expansions of $\mathbb{Z}_{np^2}$

Let $\mathbf{A} = (A; +, -, 0, (f_i)_{i \in I})$ be an expanded group. Following [3], a function $f \colon A^n \to A$ is *absorbing* if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$ with $0 \in \{a_1, \ldots, a_n\}$.

Lemma 3.1. *Let p be a prime, let $n, d, k \in \mathbb{N}$ with $d \geq 2$ and $k \geq d+1$, let $\mathbf{A}_d$ be the algebra $(\mathbb{Z}_{np^2}; +, -, 0, f_d)$, where $f_d \colon \mathbb{Z}_{np^2}^d \to \mathbb{Z}_{np^2}$ is defined by*

$$f_d(x_1, \ldots, x_d) = np \prod_{j=1}^{d} x_j \ \text{for } x_1, \ldots, x_d \in \mathbb{Z}_{np^2},$$

and let $g \in \mathrm{Pol}_k(\mathbf{A}_d)$. If g is absorbing, then g is the constant zero function.

Lemma 3.1 states that the algebra $\mathbf{A}_d$ is d-supernilpotent (cf. [5]). We provide a proof that makes no use of the notion of supernilpotency.

Proof. Let $\mathcal{C}_d$ be the set of all operations on $\mathbb{Z}_{np^2}$ that are induced by a sum of monomials of the form $a x_{i_1}^{n_1} \ldots x_{i_r}^{n_r}$, where $n_1, \ldots, n_r \in \mathbb{N}$, $x_{i_1}, \ldots, x_{i_r}$ are pairwise distinct variables, $a \in \mathbb{Z}_{np^2}$, and the following property is satisfied:

$$(r = 0) \text{ or } (r = n_1 = 1) \text{ or } (2 \leq n_1 + \cdots + n_r \leq d \text{ and } np \text{ divides } a). \ (3.1)$$

It is clear that $\{+, -, 0, f_d\} \subseteq \mathcal{C}_d$, that the projection clone satisfies $\mathcal{J}_{\mathbb{Z}_{np^2}} \subseteq \mathcal{C}_d$, and that $\mathcal{C}_d$ contains the constant functions. Furthermore, it is straightforward to verify that for $t_1, t_2, \ldots, t_d \in \mathcal{C}_d$ we have that $t_1 + t_2 \in \mathcal{C}_d$, $t_1 - t_2 \in \mathcal{C}_d$ and $f_d(t_1, \ldots, t_d) \in \mathcal{C}_d$. Hence $\mathrm{Pol}(\mathbf{A}_d) \subseteq \mathcal{C}_d$. We assume that $g \in \mathrm{Pol}_k(\mathbf{A}_d)$ is absorbing, and consider the representation of g as a sum of monomials, each one satisfying (3.1). Since g is absorbing, $g(0, x_2, \ldots, x_k)$ is the constant

0-function. Thus, the sum of those monomials of g that do not involve x_1 induces the constant zero function. Therefore, there exists a representation of g as a sum of monomials of the above form, each of them involving the variable x_1. A similar argument applies to all the other variables $x_2, \ldots, x_k$. Therefore, there exists a (possibly empty) sum of monomials that induces the same polynomial function as g such that each monomial involves all the variables $x_1, \ldots, x_k$ and satisfies (3.1). If $x_{i_1}^{n_1} \ldots x_{i_r}^{n_r}$ is such a monomial, then $r \geq k$, and therefore $n_1 + \cdots + n_r \geq k$. Since $k \geq 3$, (3.1) implies $d \geq n_1 + \cdots + n_r \geq k$, contradicting the assumption $k \geq d + 1$. Hence g is induced by the empty sum of monomials, and therefore it is the constant 0-function. $\quad\square$

Proof of Theorem 1.3. For each $d \in \mathbb{N} \setminus \{1\}$, let $\mathbf{A}_d$ and f_d be as in the statement of Lemma 3.1. For $d \in \mathbb{N} \setminus \{1\}$, for $m \in \mathbb{N}$ and for $X \subseteq \mathbb{Z}_{np^2}^m$, we use $\mathrm{V}_d(X)$ as a shorthand for $\mathrm{V}_{\mathrm{Pol}(\mathbf{A}_d)}(X)$. Let $l, i \in \mathbb{N}$ with $l > i \geq 2$. We claim that $\mathrm{Pol}(\mathbf{A}_i) \not\sim_{\mathrm{alg}} \mathrm{Pol}(\mathbf{A}_l)$. For proving this, we let

$$Q := \{\boldsymbol{a} \in \mathbb{Z}_{np^2}^l \mid \exists s \in [l] : \boldsymbol{a}(s) = 0\}$$

and show that $\mathrm{V}_i(Q) \neq \mathrm{V}_l(Q)$; as a consequence $\mathrm{Pol}(\mathbf{A}_i) \not\sim_{\mathrm{alg}} \mathrm{Pol}(\mathbf{A}_l)$.

First we prove that $(1, \ldots, 1) \notin \mathrm{V}_l(Q)$. To this end, we consider the term equation $f_l(x_1, \ldots, x_l) \approx 0$. We have that $f_l(1, \ldots, 1) = np$ and $f_l|_Q = 0$. Hence Lemma 2.1 yields $(1, \ldots, 1) \notin \mathrm{V}_l(Q)$.

We now prove that $\mathrm{V}_i(Q) = \mathbb{Z}_{np^2}^l$. To this end, let us consider an equation of the form $t_1(x_1, \ldots, x_l) \approx t_2(x_1, \ldots, x_l)$ with $t_1, t_2 \in \mathrm{Pol}_l(\mathbf{A}_i)$. This equation is equivalent to $(t_1 - t_2)(x_1, \ldots, x_l) \approx 0$. Hence it suffices to consider equations of the form $t(x_1, \ldots, x_l) \approx 0$. If $t(x_1, \ldots, x_l) \approx 0$ is satisfied by all elements of Q, then the function t is absorbing and has arity greater than i. Lemma 3.1 implies that t is the constant zero function. Therefore, the equation $t(x_1, \ldots, x_l) \approx 0$ is satisfied by all elements in $\mathbb{Z}_{np^2}^l$. Hence Lemma 2.1 yields $(1, \ldots 1) \in \mathrm{V}_i(Q)$ and therefore $\mathrm{V}_i(Q) \neq \mathrm{V}_l(Q)$.

We conclude that $(\mathrm{Pol}(\mathbf{A}_i))_{i \in \mathbb{N} \setminus \{1\}}$ is an infinite family of algebraically inequivalent clones. $\quad\square$

4. Countably many algebraically inequivalent constantive Mal'cev clones on finite sets with at least 4 elements

We report a construction from [13, Section 3]: Let A be a finite set such that $|A| \geq 4$ and let $\mathcal{C}$ be a constantive clone on A with a Mal'cev term p. The construction picks an element $1 \in A$ and an element $0 \notin A$ and constructs a new clone $\mathcal{C}^\oplus$ on the set $A^\oplus := A \cup \{0\}$.

For all $k \in \mathbb{N}$ and for all $f \in \mathcal{C}$, the operation $f^\oplus \colon (A^\oplus)^k \to A^\oplus$ is defined by

$$f^\oplus(x_1, \ldots, x_k) := \begin{cases} f(x_1, \ldots, x_k) & \text{if } (x_1, \ldots, x_k) \in A^k, \\ 0 & \text{otherwise.} \end{cases}$$

A binary operation $\cdot$ on $A^{\oplus}$ is defined by

$$x \cdot y := \begin{cases} 0 & \text{if } (x, y) \in A^2, \\ 1 & \text{if } x = y = 0, \\ x & \text{if } x \in A \text{ and } y = 0, \\ y & \text{if } x = 0 \text{ and } y \in A. \end{cases}$$

Finally, $\mathcal{C}^{\oplus}$ is defined as the clone on $A^{\oplus}$ generated by $\{f^{\oplus} \mid f \in \mathcal{C}\} \cup \{\cdot\}$ and all unary constant operations.

Lemma 4.1. *Let A be a finite set with $|A| \geq 4$, let $\mathcal{C}$ be a constantive clone on A and let $f \in \mathcal{C}^{\oplus}$. Then*

$$f|_{A^k} = 0 \text{ or there exists } \hat{f} \in \mathcal{C} \text{ such that } f|_{A^k} = \hat{f}. \tag{4.1}$$

Proof. Let K be the set of all unary constant functions on $A^{\oplus}$. We will show that all projections on $A^{\oplus}$ satisfy (4.1), and that for all $k, n \in \mathbb{N}$, for all n-ary

$$s \in \{f^{\oplus} \mid f \in \mathcal{C}\} \cup \{\cdot\} \cup K$$

and for all k-ary $t_1, \ldots, t_n$ that satisfy (4.1), also $s(t_1, \ldots, t_n)$ satisfies (4.1). From this, we conclude that every function in $\mathcal{C}^{\oplus}$ satisfies (4.1).

Clearly, all the projections satisfy (4.1).

Let $k, n \in \mathbb{N}$, let $t_1, \ldots, t_n$ be k-ary functions on $A^{\oplus}$ that satisfy (4.1), and let s be an n-ary function on $A^{\oplus}$ from $\{f^{\oplus} \mid f \in \mathcal{C}\} \cup \{\cdot\} \cup K$. We will show that $s(t_1, \ldots, t_n)$ satisfies (4.1).

We first consider the case that $s = f^{\oplus}$ with $f \in \mathcal{C}^{[n]}$. If there exists $i \in [n]$ such that $t_i|_{A^k} = 0$, then by the definition of $f^{\oplus}$, $f^{\oplus}(t_1, \ldots, t_n)|_{A^k} = 0$, and hence $s(t_1, \ldots, t_n)$ satisfies (4.1). If for all $i \in [n]$, there exists $\hat{t}_i \in \mathcal{C}^{[k]}$ such that $t_i|_{A^k} = \hat{t}_i$, then by the definition of $f^{\oplus}$ we have

$$s(t_1, \ldots, t_n)|_{A^k} = f^{\oplus}(t_1, \ldots, t_n)|_{A^k} = f(\hat{g}_1, \ldots, \hat{g}_n) \in \mathcal{C}.$$

Therefore $s(t_1, \ldots, t_n)$ satisfies (4.1).

Now we consider the case that s is the function $\cdot$ (and $n = 2$). Let t_1, t_2 be k-ary functions on $A^{\oplus}$ satisfying (4.1). We show that $t_1 \cdot t_2$ satisfies (4.1). Since t_1, t_2 satisfy (4.1), we have to consider the following cases.

Case 1: $t_1|_{A^k} = t_2|_{A^k} = 0$: In this case, the definition of $\cdot$ yields $t_1 \cdot t_2|_{A^k} = 1$. Since $\mathcal{C}$ is constantive, $t_1 \cdot t_2|_{A^k} \in \mathcal{C}$, and thus $t_1 \cdot t_2$ satisfies (4.1).

Case 2: $t_1|_{A^k} = 0$ and there exists $\hat{t}_2 \in \mathcal{C}^{[k]}$ such that $t_2|_{A^k} = \hat{t}_2$: In this case, the definition of $\cdot$ yields $t_1 \cdot t_2|_{A^k} = \hat{t}_2$. This implies that $t_1 \cdot t_2|_{A^k} \in \mathcal{C}$, and thus $t_1 \cdot t_2$ satisfies (4.1).

Case 3: $t_2|_{A^k} = 0$ *and there exists* $\hat{t}_1 \in \mathcal{C}^{[k]}$ *such that* $t_1|_{A^k} = \hat{t}_1$: In this case, the definition of $\cdot$ yields $t_1 \cdot t_2|_{A^k} = \hat{t}_1$. This implies that $t_1 \cdot t_2|_{A^k} \in \mathcal{C}$, and thus $t_1 \cdot t_2$ satisfies (4.1).

Case 4: *There exist* $\hat{g}_1, \hat{g}_2 \in \mathcal{C}^{[k]}$ *such that* $g_1|_{A^k} = \hat{g}_1$ *and* $g_2|_{A^k} = \hat{g}_2$: In this case, the definition of $\cdot$ yields $t_1 \cdot t_2|_{A^k} = 0$, thus $t_1 \cdot t_2$ satisfies (4.1).

Finally, let us consider the case $s \in K$ (and $n = 1$). Let $a \in A^{\oplus}$ be the unique function value of s, and let t be a k-ary function on $A^{\oplus}$ that satisfies (4.1). Clearly, $s(t)(x_1, \ldots, x_k) = a$ for all $x_1, \ldots, x_k \in A^{\oplus}$. If $a \in A$, since $\mathcal{C}$ is

constantive, we have that $s(t)$ satisfies (4.1). If $a = 0$, then $s(t)(x_1, \ldots, x_k) = 0$ for all $(x_1, \ldots, x_k) \in A^k$. Thus, $s(t)$ satisfies (4.1). $\qquad \square$

Lemma 4.2. *Let A be a finite set with $|A| \geq 4$, let $\mathcal{C}$ be a constantive clone on A, let $f \in (\mathcal{C}^{\oplus})^{[k]}$, and let $\boldsymbol{a}, \boldsymbol{b} \in A^k$. Then $f(\boldsymbol{a}) = 0$ if and only if $f(\boldsymbol{b}) = 0$.*

Proof. If $f(\boldsymbol{a}) \neq 0$, then $f|_{A^k} \neq 0$. Thus by Lemma 4.1, there exists $\hat{f} \in \mathcal{C}$ such that $f|_{A^k} = \hat{f}$, and therefore $f(\boldsymbol{b}) = \hat{f}(\boldsymbol{b}) \in A$, which implies $f(\boldsymbol{b}) \neq 0$. $\qquad \square$

Lemma 4.3. *[13, Section 3] Let A be a finite set with $|A| \geq 4$ and let $\mathcal{C}$ be a constantive Mal'cev clone on A. Then $\mathcal{C}^{\oplus}$ is a constantive Mal'cev clone on a set of cardinality $|A| + 1$.*

Proposition 4.4. *Let A be a finite set with $|A| \geq 4$, and let $\mathcal{C}_1, \mathcal{C}_2$ be two constantive clones on A. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are not algebraically equivalent, then $\mathcal{C}_1^{\oplus}$ and $\mathcal{C}_2^{\oplus}$ are not algebraically equivalent.*

Proof. Without loss of generality, we assume that there exists $B \subseteq A^k$ that is algebraic with respect to $\mathcal{C}_1$ but not with respect to $\mathcal{C}_2$. Note that then $B \neq A^k$. Since $\mathcal{C}_2$ is constantive, B is not empty. Then there exists $\boldsymbol{a} \in A^k \setminus B$ such that $\boldsymbol{a} \in \mathrm{V}_{\mathcal{C}_2}(B)$.

Let $B_0 := B \cup ((A^{\oplus})^k \setminus A^k)$. We claim that

$$B_0 \in \mathrm{Alg}\,\mathcal{C}_1^{\oplus}. \tag{4.2}$$

To prove (4.2), let $\boldsymbol{c} \notin B_0$. By Lemma 2.1 it suffices to show that there are two functions $p_1, p_2 \in \mathcal{C}_1^{\oplus}$ such that $p_1|_{B_0} = p_2|_{B_0}$ and $p_1(\boldsymbol{c}) \neq p_2(\boldsymbol{c})$. Since $\boldsymbol{c} \in A^k \setminus B$ and B is algebraic with respect to $\mathcal{C}_1$, Lemma 2.1 implies that there are $f_1, f_2 \in \mathcal{C}_1^{[k]}$ such that $f_1|_B = f_2|_B$ and $f_1(\boldsymbol{c}) \neq f_2(\boldsymbol{c})$. We set $p_1 = f_1^{\oplus}$ and $p_2 = f_2^{\oplus}$. Then $p_1|_{B_0} = p_2|_{B_0}$ and $p_1(\boldsymbol{c}) \neq p_2(\boldsymbol{c})$, which concludes the proof of (4.2).

We now claim that

$$\boldsymbol{a} \in \mathrm{V}_{\mathcal{C}_2^{\oplus}}(B_0). \tag{4.3}$$

Suppose that $\boldsymbol{a} \notin \mathrm{V}_{\mathcal{C}_2^{\oplus}}(B_0)$. Then Lemma 2.1 implies that there are $f_1, f_2 \in (\mathcal{C}_2^{\oplus})^{[k]}$ such that $f_1|_{B_0} = f_2|_{B_0}$ and $f_1(\boldsymbol{a}) \neq f_2(\boldsymbol{a})$. Since $B \subseteq B_0$ we infer that $f_1|_B = f_2|_B$. From Lemma 4.2 and $B \neq \emptyset$, we deduce that $f_1|_B = f_2|_B \neq 0$. Moreover, by Lemma 4.1, there are $\hat{f}_1, \hat{f}_2 \in \mathcal{C}_2^{[k]}$ such that $f_1|_{A^k} = \hat{f}_1$ and $f_2|_{A^k} = \hat{f}_2$. Therefore, $\hat{f}_1|_B = \hat{f}_2|_B$ and $\hat{f}_1(\boldsymbol{a}) \neq \hat{f}_2(\boldsymbol{a})$. Hence by Lemma 2.1, $\boldsymbol{a} \notin \mathrm{V}_{\mathcal{C}_2}(B)$, contradicting the choice of $\boldsymbol{a}$.

From (4.2) and (4.3) we obtain $\mathrm{Alg}\,\mathcal{C}_1^{\oplus} \neq \mathrm{Alg}\,\mathcal{C}_2^{\oplus}$. $\qquad \square$

Proof of Theorem 1.2. We proceed by induction on $n = |A|$. For the base case $n = 4$, we use Theorem 1.3 with $n = 1$ and $p = 2$. For the induction step we let $n \geq 4$ and $\{\mathcal{C}_i \mid i \in \mathbb{N}\}$ be a collection of pairwise algebraically inequivalent constantive Mal'cev clones on a set of cardinality n. Then Proposition 4.4 and Lemma 4.3 imply that $\{\mathcal{C}_i^{\oplus} \mid i \in \mathbb{N}\}$ is a collection of pairwise algebraically inequivalent constantive Mal'cev clones on a set of cardinality $n + 1$. $\qquad \square$

5. Continuously many algebraically inequivalent clones on sets with at least 3 elements

Let $Z = \{0,1,2\}$. For $n \in \mathbb{N}$ and for $i \leq n$, a function $f\colon Z^n \to Z$ *depends on its i-th argument* if there exist $\boldsymbol{a} \in Z^n$ and $b \in Z$ such that $f(\boldsymbol{a}) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$. If f depends on exactly one of its arguments, then f is called *essentially unary*. The *essential arity* of f is defined by $\mathrm{essArity}(f) := |\{i \in [n] : f$ depends on its i-th argument$\}|$. Let $\mathcal{F}$ be a set of finitary functions on Z. We say that f *belongs to $\mathcal{F}$ up to inessential arguments* if there exist $l \in \mathbb{N}$, $1 \leq i_1 < \cdots < i_l \leq n$ and a function $g \in \mathcal{F}^{[l]}$ such that $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_l})$ for all $\boldsymbol{x} \in Z^n$. We note that then $\mathrm{essArity}(f) = \mathrm{essArity}(g)$.

Let $D_1 := \emptyset$ and for each $m \in \mathbb{N} \setminus \{1\}$, let
$$D_m := \{(1, 2, \ldots, 2), \ldots, (2, \ldots, 2, 1)\}.$$

For each $n \in \mathbb{N}$, let F_n be the set consisting of all the functions $f\colon Z^n \to Z$ such that

$$f(Z^n) \subseteq \{0,1\}, \text{ and} \tag{5.1}$$

$$f^{-1}(\{1\}) \subseteq D_n. \tag{5.2}$$

We set $F := \bigcup_{n \in \mathbb{N}} F_n$ and

$$F' := \bigcup_{n \in \mathbb{N}} \{f\colon Z^n \to Z \mid f \text{ belongs to } F \text{ up to inessential arguments}\} \cup \mathcal{J}_Z.$$

Lemma 5.1. *The sets F and F' satisfy the following properties:*

(1) *For all $f \in F$ with $\mathrm{essArity}(f) \leq 1$, we have $f = 0$.*
(2) *For all $n \in \mathbb{N}$ and for all $f \in F_n$ with $\mathrm{essArity}(f) > 0$, we have that $\mathrm{essArity}(f) = n$ and $n \geq 2$.*
(3) *For all $f \in F'$ with $\mathrm{essArity}(f) = 1$, we have $f \in \mathcal{J}_Z$.*
(4) *For all $l, n \in \mathbb{N}$, for all $g \in F_l$ and for all $\rho\colon [l] \to [n]$, we have $g_\rho \in F'_n$.*
(5) *For all $k, n \in \mathbb{N}$, for all $f \in F'_k$ and for all $\sigma\colon [k] \to [n]$, we have $f_\sigma \in F'_n$.*

Proof. (1) Let $n \in \mathbb{N}$, let $f \in F_n$ with $\mathrm{essArity}(f) \leq 1$ and let $\boldsymbol{x} \in Z^n$. We prove that $f(\boldsymbol{x}) = 0$. If $\mathrm{essArity}(f) \leq 1$, then there exists $i \in [n]$ such that for all $j \in [n] \setminus \{i\}$ the function f does not depend on its j-th argument. Let us define $f' \in Z^Z$ by $f'(y) = f(y, \ldots, y)$ for all $y \in Z$. We prove that $f' = 0$. To this end, let $a \in Z$. Since $(a, \ldots, a) \notin D_n$, we have $f'(a) = f(a, \ldots, a) = 0$. Moreover, since for all $j \neq i$, f does not depend on its j-th argument, $f(\boldsymbol{x}) = f(x_i, \ldots, x_i) = f'(x_i) = 0$. This concludes the proof of (1). (2) Let $n \in \mathbb{N}$ and let $f \in F_n$. If f is not constantly zero, then there exists $\boldsymbol{a} \in D_n$ such that $f(\boldsymbol{a}) = 1$. Changing any one of the coordinates of $\boldsymbol{a}$ to 0, the value of f changes from 1 to 0. Hence f depends on all of its arguments and (1) implies that it cannot be essentially unary. This concludes the proof of (2). (3) Let $f \in F'_n$ with $\mathrm{essArity}(f) = 1$. Seeking a contradiction, we suppose that f is not a projection. Then f belongs to F up to inessential arguments. Thus, there exist $1 \leq j_1 < \cdots < j_k \leq n$ and $\hat{f} \in F$ such that $f(x_1, \ldots, x_n) = \hat{f}(x_{j_1}, \ldots, x_{j_k})$ for all $\boldsymbol{x} \in Z^n$; then $\mathrm{essArity}(\hat{f}) = \mathrm{essArity}(f) = 1$. Thus, (1)

yields $\hat{f} = 0$, contradicting $\mathrm{essArity}(f) = 1$. (4) Let us assume that the image of ρ is $\{j_1, \ldots, j_k\} \subseteq [n]$, with $j_1 < \cdots < j_k$, and let us define

$$h := \{ \big((x_{j_1}, \ldots, x_{j_k}), g(x_{\rho(1)}, \ldots, x_{\rho(l)})\big) \mid (x_1, \ldots, x_n) \in Z^n \}.$$

We first prove that h is a functional relation. To this end, let $\boldsymbol{x}, \boldsymbol{y} \in Z^n$ be such that $(x_{j_1}, \ldots, x_{j_k}) = (y_{j_1}, \ldots, y_{j_k})$. Since $\{j_1, \ldots, j_k\}$ is the image of ρ, if $(x_{j_1}, \ldots, x_{j_k}) = (y_{j_1}, \ldots, y_{j_k})$, we have $(x_{\rho(1)}, \ldots, x_{\rho(l)}) = (y_{\rho(1)}, \ldots, y_{\rho(l)})$, and therefore $g(x_{\rho(1)}, \ldots, x_{\rho(l)}) = g(y_{\rho(1)}, \ldots, y_{\rho(l)})$. This concludes the proof that h is a functional relation. Now we prove that $h \in F_k$. Condition (5.1) is clearly satisfied since $g \in F_l$. Next we prove that h satisfies (5.2). To this end, we let $\boldsymbol{x} \in A^n$ with $g(x_{\rho(1)}, \ldots, x_{\rho(l)}) = 1$ and show that $(x_{j_1}, \ldots, x_{j_k}) \in D_k$. Since $g \in F_l$ and $g(x_{\rho(1)}, \ldots, x_{\rho(l)}) = 1$, exactly one of $x_{\rho(1)}, \ldots, x_{\rho(l)}$ is equal to 1, and all the others are equal to 2, which can be expressed formally by saying that there exists $a \in [l]$ such that $x_{\rho(a)} = 1$ and $x_{\rho(b)} = 2$ for all $b \in [l] \setminus \{a\}$. We have to show that exactly one of $x_{j_1}, \ldots, x_{j_k}$ is equal to 1. Since $j_1 < \cdots < j_k$ and $\{j_1, \ldots, j_k\}$ is the image of ρ, there exists a unique $i \in [k]$ with $j_i = \rho(a)$, and this is also the unique $i \in [k]$ with $x_{j_i} = 1$. For each $r \in [k] \setminus \{i\}$ there exists $b \in [l] \setminus \{a\}$ such that $j_r = \rho(b)$, and therefore $x_{j_r} = x_{\rho(b)} = 2$. Thus, $(x_{j_1}, \ldots, x_{j_k}) \in D_k$, and so h satisfies (5.2). This proves that $h \in F_k$. Since $j_1 < \cdots < j_k$ and $g_\rho(\boldsymbol{a}) = h(a_{j_1}, \ldots, a_{j_k})$ for all $\boldsymbol{a} \in A^n$, g_ρ belongs to F up to inessential arguments. Hence $g_\rho \in F'_n$. This concludes the proof of (4). (5) Let $k, n \in \mathbb{N}$, let $f \in F'_k$ and let $\sigma \colon [k] \to [n]$. We prove that $f_\sigma \in F'_n$. By the definition of F', there exist $l \in \mathbb{N}$, $g \in F_l$ and $\tau \colon [l] \to [k]$ injective and increasing such that $f(y_1, \ldots, y_k) = g(y_{\tau(1)}, \ldots, y_{\tau(l)})$ for all $\boldsymbol{y} \in Z^k$. Therefore, for all $\boldsymbol{x} \in Z^n$ we have

$$f_\sigma(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = g(x_{\sigma(\tau(1))}, \ldots, x_{\sigma(\tau(l))}).$$

Since $g \in F_l$, (4) implies that $g_{\sigma \circ \tau} \in F'_n$, and thus $f_\sigma \in F'_n$. This concludes the proof of (5). $\qquad\square$

Lemma 5.2. *The set F' is a clone.*

Proof. We prove that F' is closed under ζ, τ, Δ, ∇ and $*$ as they are defined in [15] (cf. [10,20]). To this end, let $f \in F'_n$. Lemma 5.1(5) yields that $\zeta f \in F'_n$, $\tau f \in F'_n$, $\Delta f \in F'_{n-1}$, and $\nabla f \in F'_{n+1}$. Let $g \in F'_m$ and let h be the function defined by $h(\boldsymbol{x}) = (f * g)(\boldsymbol{x}) = f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$ for all $\boldsymbol{x} \in Z^{m+n-1}$. We prove that h belongs to F'. If f is constant, then h is a constant mapping with the same function value as f. Hence h is a minor of f, and thus by Lemma 5.1(5), $h \in F'$. If f is a projection, then either h is a projection or $h = g$. In both cases h belongs to F'. If g is a projection, then h is a minor of f. Hence Lemma 5.1(5) yields $h \in F'$. Finally, let us assume that both f and g belong to F up to inessential arguments and f is not constant. Then there exist $1 \leq j_1 < \cdots < j_{n'} \leq n$, $1 \leq i_1 < \cdots < i_{m'} \leq m$, $\hat{f} \in F_{n'}$ and $\hat{g} \in F_{m'}$ such that for all $\boldsymbol{x} \in Z^n$ and for all $\boldsymbol{y} \in Z^m$ we have $f(\boldsymbol{x}) = \hat{f}(x_{j_1}, \ldots, x_{j_{n'}})$ and $g(\boldsymbol{y}) = \hat{g}(y_{i_1}, \ldots, y_{i_{m'}})$. If $j_1 \geq 2$, then for all $\boldsymbol{x} \in Z^{m+n-1}$ we have $h(\boldsymbol{x}) = \hat{f}(x_{m+j_1-1}, \ldots, x_{m+j_{n'}-1})$. Thus, h is a minor of $\hat{f}$ and Lemma 5.1(4) yields $h \in F'$. Let us now assume that $j_1 = 1$ and let

$\hat{h}$ be defined by $\hat{h}(x_1, \ldots, x_{m'+n'-1}) = \hat{f}(\hat{g}(x_1, \ldots, x_{m'}), x_{m'+1}, \ldots, x_{m'+n'-1})$ for all $\boldsymbol{x} \in Z^{m'+n'-1}$. We now prove that $\hat{h} \in F$. Since $\hat{f}$ belongs to F, $\hat{h}$ satisfies (5.1). We now prove that $\hat{h}$ satisfies (5.2). To this end, let $\boldsymbol{a} \in Z^{m'+n'-1}$ be such that $\hat{h}(\boldsymbol{a}) = 1$. Then, since $\hat{f}$ satisfies (5.2), we infer that $(g(a_1, \ldots, a_{m'}), a_{m'+1}, \ldots, a_{m'+n'-1}) \in D_{n'}$. Moreover, since g satisfies (5.1) and (5.2), we have that $(a_1, \ldots, a_{m'}) \in D_{m'}$ and $(a_{m'+1}, \ldots, a_{m'+n'-1}) = (2, \ldots, 2)$. Thus, there exists $r \in [m']$ such that $a_r = 1$ and for all $k \in [m' + n' - 1] \setminus \{r\}$ we have $a_k = 2$. This proves that $\boldsymbol{a} \in D_{m'+n'-1}$. Thus, $\hat{h}$ satisfies (5.2), and so $\hat{h} \in F$. Moreover, we have that for all $\boldsymbol{x} \in Z^{m+n-1}$

$$h(x_1, \ldots, x_{m+n-1}) = \hat{f}(\hat{g}(x_{i_1}, \ldots, x_{i_{m'}}), x_{m+j_2-1}, \ldots, x_{m+j_{n'}-1}).$$

Therefore h belongs to F up to inessential arguments, and so $h \in F'$. Therefore, F' is closed under $*$. $\square$

We make use of the following construction from [14] (cf. [20, Chapter 3]). For each $i \in \mathbb{N} \setminus \{1\}$, the function $g_i \colon Z^i \to Z$ is defined by

$$g_i(x_1, \ldots, x_i) = \begin{cases} 1 & \text{if } (x_1, \ldots, x_i) \in D_i, \\ 0 & \text{otherwise.} \end{cases}$$

For each $I \subseteq \mathbb{N} \setminus \{1\}$, $\mathbf{Z}_I$ is defined as $(Z; (g_i)_{i \in I})$.

Lemma 5.3. *Let $I \subseteq \mathbb{N} \setminus \{1\}$, let $k \in \mathbb{N}$ and let $g \in \mathrm{Clo}_k(\mathbf{Z}_I)$. Then we have:*
 (1) *If g is constant, then $g = 0$.*
 (2) *If g is essentially unary, then g is a projection.*
 (3) *If g depends exactly on the arguments $i_1 < \cdots < i_l$ with $l \geq 2$, then the following two conditions are satisfied:*
    (a) *For all $\boldsymbol{b} \in Z^k$ we have $g(\boldsymbol{b}) \neq 2$.*
    (b) *For all $\boldsymbol{a} \in Z^k$ with $(a_{i_1}, \ldots, a_{i_l}) \notin D_l$, we have $g(\boldsymbol{a}) = 0$.*

Proof. Equations (5.1) and (5.2) imply that for all $i \in \mathbb{N} \setminus \{1\}$, $g_i \in F$. Thus, Lemma 5.2 yields that for all $I \subseteq \mathbb{N} \setminus \{1\}$, $\mathrm{Clo}(\mathbf{Z}_I) \subseteq F'$. Since $g \in \mathrm{Clo}_k(\mathbf{Z}_I) \subseteq F'$, then either $g \in \mathcal{J}_Z$ or g belongs to F up to inessential arguments. If g is a constant, then it cannot be a projection. Hence it belongs to F up to inessential arguments. Thus, by Lemma 5.1(1), it is the constant zero function. This proves (1). Statement (2) follows directly from Lemma 5.1(3). We now prove (3). Let us assume that g depends on its arguments $i_1 < \cdots < i_l$, let $\boldsymbol{b} \in Z^k$ and let $\boldsymbol{a} \in Z^k$ be such that $(a_{i_1}, \ldots, a_{i_l}) \notin D_l$. Since $l \geq 2$, g is not a projection. Since $g \in F'$, g belongs to F up to inessential arguments. This, together with Lemma 5.1(2), implies that there exists $\hat{g} \in F_l$ such that $g(x_1, \ldots, x_k) = \hat{g}(x_{i_1}, \ldots, x_{i_l})$ for all $\boldsymbol{x} \in Z^k$, and $\hat{g}$ depends on all of its arguments. Since $\hat{g} \in F$, (5.1) implies that $g(\boldsymbol{b}) = \hat{g}(b_{i_1}, \ldots, b_{i_l}) \in \{0, 1\}$. Moreover, since $(a_{i_1}, \ldots, a_{i_l}) \notin D_l$, (5.2) and (5.1) yield $g(\boldsymbol{a}) = \hat{g}(a_{i_1}, \ldots, a_{i_l}) = 0$. This proves (3). $\square$

Proposition 5.4. *Let $I \subseteq \mathbb{N} \setminus \{1\}$, let $i \in \mathbb{N} \setminus \{1\}$ and let*

$$g_i^\bullet = \{(x_1, \ldots, x_i, x_{i+1}) \in Z^{i+1} \mid x_{i+1} = g(x_1, \ldots, x_i)\}.$$

Then $g_i^\bullet \in \mathrm{Alg}\,\mathbf{Z}_I$ if and only if $i \in I$.

Proof. If $i \in I$, then $g_i^\bullet \in \text{Alg}\,\mathbf{Z}_I$ by definition. We now assume that $g_i^\bullet \in \text{Alg}\,\mathbf{Z}_I$ and prove that $i \in I$. If $g_i^\bullet \in \text{Alg}\,\mathbf{Z}_I$, since $(1,\dots,1) \notin g_i^\bullet$, Lemma 2.1 implies that there exist $f_1, f_2 \in \text{Clo}_{i+1}(\mathbf{Z}_I)$ such that $f_1|_{g_i^\bullet} = f_2|_{g_i^\bullet}$ and $f_1(1,\dots,1) \neq f_2(1,\dots,1)$. We now prove that one of the two is a projection and the other one depends on at least two of its arguments. Seeking contradictions, let us suppose that this is not the case. We distinguish cases according to the essential arities of f_1 and f_2.

Case 1: f_1 *and* f_2 *are constant*: In this case $f_1(1,\dots,1) = f_2(1,\dots,1)$.

Case 2: f_1 *is constant and* f_2 *is essentially unary*: In this case, Lemma 5.3 yields that f_1 is the constant 0 function and f_2 is a projection. Let $\boldsymbol{a} = (1,2,\dots,2,1) \in Z^{i+1}$. Since $g(a_1,\dots,a_i) = 1$, $\boldsymbol{a} \in g_i^\bullet$. Lemma 5.3 yields $f_1(\boldsymbol{a}) = 0$ and $f_2(\boldsymbol{a}) \in \{1,2\}$. Hence $f_1(\boldsymbol{a}) \neq f_2(\boldsymbol{a})$. Thus, $f_1|_{g_i^\bullet} \neq f_2|_{g_i^\bullet}$.

Case 3: f_1 *is essentially unary and* f_2 *is constant*: This case is symmetric to Case 2.

Case 4: f_1 *is constant and* f_2 *depends on at least two of its arguments*: In this case Lemma 5.3 yields $f_1(1,\dots,1) = 0 = f_2(1,\dots,1)$.

Case 5: f_1 *depends on at least two of its arguments and* f_2 *is constant*: This case is symmetric to Case 4.

Case 6: f_1 *and* f_2 *are both essentially unary*: In this case Lemma 5.3(2) implies that both f_1 and f_2 are projections. Thus, we have $f_1(1,\dots,1) = 1 = f_2(1,\dots,1)$.

Case 7: f_1 *and* f_2 *both depend on at least two of their arguments*: In this case Lemma 5.3(3) yields $f_1(1,\dots,1) = 0 = f_2(1,\dots,1)$.

Thus, we have proved that one among f_1 and f_2 is a projection, while the other one depends on at least two of its arguments. Without loss of generality, let us assume that f_2 is a projection and that f_1 depends on at least two of its arguments. Let

$$T := \{\boldsymbol{a} \in Z^{i+1} \mid a_{i+1} = 1 \text{ and } (a_1,\dots,a_i) \in D_i\}.$$

Note that $T \subseteq g_i^\bullet$. Let $j \in \{1,\dots,i,i+1\}$ be such that $f_2(x_1,\dots,x_{i+1}) = x_j$ for all $\boldsymbol{x} \in Z^{i+1}$. We claim $j = i+1$. Seeking a contradiction, let us suppose that $j \leq i$. Let $\boldsymbol{x} \in T$ be such that $\boldsymbol{x}(j) = 2$. Then, since $f_1|_T = f_2|_T$, we have $f_1(\boldsymbol{x}) = 2$. On the other hand, Lemma 5.3(3) implies that $f_1(\boldsymbol{x}) \neq 2$. Thus we get the desired contradiction and deduce that f_2 is the $(i+1)$-th projection. Next, we prove that f_1 does not depend on its $(i+1)$-th argument. Seeking a contradiction, let us suppose that f_1 depends on its $(i+1)$-th argument. Then, since f_1 depends on at least two of its arguments, there exists $j \in [i]$ such that f_1 depends also on its j-th argument. Let $\boldsymbol{a} \in T$ be such that $\boldsymbol{a}(j) = 1$. Since $f_1|_T = f_2|_T$, we have that $f_1(\boldsymbol{a}) = f_2(\boldsymbol{a}) = a_{i+1} = 1$. On the other hand, Lemma 5.3 implies that $f_1(\boldsymbol{a}) = 0$ because $a_j = a_{i+1} = 1$. From this contradiction, we conclude that f_1 does not depend on its $(i+1)$-th argument.

Let $f\colon Z^i \to Z$ be defined by $f(x_1,\dots x_i) := f_1(x_1,\dots,x_i,x_1)$. Since f_1 does not depend on its $(i+1)$-th argument, for each $\boldsymbol{b} \in g_i^\bullet$, we have

$$
\begin{aligned}
f(b_1,\dots,b_i) &= f_1(b_1,\dots,b_i,b_1) = \\
f_1(b_1,\dots,b_{i+1}) &= f_2(b_1,\dots,b_{i+1}) = b_{i+1}.
\end{aligned}
\tag{5.3}
$$

We show that $g_i = f$. To this end, let $\boldsymbol{c} \in Z^i$. Then $(c_1, \ldots, c_i, g_i(\boldsymbol{c})) \in g_i^\bullet$, and thus (5.3) yields $f(\boldsymbol{c}) = g_i(\boldsymbol{c})$. Since $f \in \mathrm{Clo}_i(\mathbf{Z}_I)$, we deduce that $g_i \in \mathrm{Clo}_i(\mathbf{Z}_I)$. We now apply [20, Theorem 3.1.4] and deduce that $i \in I$.   □

Corollary 5.5. *On the set $Z = \{0, 1, 2\}$ there are exactly $2^{\aleph_0}$ algebraically inequivalent clones.*

Proof. Let $\mathcal{L}_Z$ be the set of all clones on Z and let $\psi \colon \mathcal{P}(\mathbb{N} \setminus \{1\}) \to \mathcal{L}_Z/\sim_{\mathrm{alg}}$ be defined by $I \mapsto \mathrm{Clo}(\mathbf{Z}_I)/\sim_{\mathrm{alg}}$. Proposition 5.4 yields that for all $I \neq J \in \mathcal{P}(\mathbb{N} \setminus \{1\})$ we have $\mathrm{Clo}(\mathbf{Z}_I) \sim_{\mathrm{alg}} \mathrm{Clo}(\mathbf{Z}_J)$ if and only if $I = J$. Therefore, ψ is injective. Hence there are at least $2^{\aleph_0}$ algebraically inequivalent clones on Z. Since by [20, Theorem 3.1.4] there are at most $2^{\aleph_0}$ clones on Z, we can conclude that there are exactly $2^{\aleph_0}$ algebraically inequivalent clones on Z.   □

Proof of Theorem 1.1. We first show that for a set A and an element $u \notin A$, $A \cup \{u\}$ has at least as many algebraically inequivalent clones as A. To this end, let A be a finite set with $|A| \geq 3$, let $\mathcal{L}_A$ be the set of all clones on A, let $u \notin A$, let $A^\oplus = A \cup \{u\}$, let $\mathcal{L}_{A^\oplus}$ be the set of all clones on $A^\oplus$, and let $\Phi \colon \mathcal{L}_A \to \mathcal{L}_{A^\oplus}$ be defined by

$$\Phi(\mathcal{C}) = \bigcup_{n \in \mathbb{N}} \{f : (A^\oplus)^n \to A^\oplus \mid f|_{A^n} \in \mathcal{C}\}.$$

We first prove that for all $\mathcal{C} \in \mathcal{L}_A$, $\Phi(\mathcal{C})$ is a clone on $A^\oplus$. Clearly, all the projections belong to $\Phi(\mathcal{C})$. Let $f \in \Phi(\mathcal{C})^{[n]}$, let $t_1, \ldots, t_n \in \Phi(\mathcal{C})^{[k]}$. Then

$$f(t_1, \ldots, t_n)|_{A^k} = f(t_1|_{A^k}, \ldots, t_n|_{A^k}) = f|_{A^n}(t_1|_{A^k}, \ldots, t_n|_{A^k}).$$

The definition of Φ yields $f|_{A^n}, t_1|_{A^k}, \ldots, t_n|_{A^k} \in \mathcal{C}$. Thus, the function $f|_{A^n}(t_1|_{A^k}, \ldots, t_n|_{A^k})$ belongs to $\mathcal{C}$ and $f(t_1, \ldots, t_n) \in \Phi(\mathcal{C})$. Therefore, $\Phi(\mathcal{C})$ is a clone on $A^\oplus$.

Next, we prove that Φ satisfies

$$\forall \mathcal{C} \in \mathcal{L}_A, \forall n \in \mathbb{N}, \forall B \subseteq (A^\oplus)^n : B \in \mathrm{Alg}\, \Phi(\mathcal{C}) \Leftrightarrow B \cap A^n \in \mathrm{Alg}\, \mathcal{C}. \quad (5.4)$$

For proving (5.4), let $\mathcal{C} \in \mathcal{L}_A$, let $n \in \mathbb{N}$, let $B \subseteq (A^\oplus)^n$ and let us assume that $B \in \mathrm{Alg}_n \Phi(\mathcal{C})$. We prove that $B \cap A^n \in \mathrm{Alg}_n \mathcal{C}$. To this end, let $\boldsymbol{b} \in A^n \setminus B$. By Lemma 2.1 it suffices to prove that there are $\hat{f}_1, \hat{f}_2 \in \mathcal{C}$ such that $\hat{f}_1|_{A^n \cap B} = \hat{f}_2|_{A^n \cap B}$ and $\hat{f}_1(\boldsymbol{b}) \neq \hat{f}_2(\boldsymbol{b})$. Since $B \in \mathrm{Alg}_n \Phi(\mathcal{C})$ and $\boldsymbol{b} \notin B$, Lemma 2.1 yields that there exist $f_1, f_2 \in \Phi(\mathcal{C})$ such that $f_1|_B = f_2|_B$ and $f_1(\boldsymbol{b}) \neq f_2(\boldsymbol{b})$. Let $\hat{f}_1 = f_1|_{A^n}$ and let $\hat{f}_2 = f_2|_{A^n}$. Then clearly $\hat{f}_1, \hat{f}_2 \in \mathcal{C}$, $\hat{f}_1|_{A^n \cap B} = \hat{f}_2|_{A^n \cap B}$ and $\hat{f}_1(\boldsymbol{b}) \neq \hat{f}_2(\boldsymbol{b})$. Thus, $B \cap A^n \in \mathrm{Alg}\, \mathcal{C}$.

Let us now assume that $B \cap A^n \in \mathrm{Alg}\, \mathcal{C}$. We prove that $B \in \mathrm{Alg}_n \Phi(\mathcal{C})$. Let $\boldsymbol{a} \in (A^\oplus)^n \setminus B$. By Lemma 2.1 it suffices to show that there are $f_1, f_2 \in \Phi(\mathcal{C})$ such that $f_1|_B = f_2|_B$ and $f_1(\boldsymbol{a}) \neq f_2(\boldsymbol{a})$. We split the proof into two cases.

Case 1: $\boldsymbol{a} \in A^n$: In this case Lemma 2.1 implies that there are $g_1, g_2 \in \mathcal{C}$ such that $g_1|_{B \cap A^n} = g_2|_{B \cap A^n}$ and $g_1(\boldsymbol{a}) \neq g_2(\boldsymbol{a})$. Then for $i \in \{1, 2\}$, we set

$$f_i(\boldsymbol{x}) = \begin{cases} g_i(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in A^n, \\ u & \text{otherwise.} \end{cases}$$

By the definition of $\Phi(\mathcal{C})$, both f_1 and f_2 belong to $\Phi(\mathcal{C})$, $f_1|_B = f_2|_B$ and, since $\boldsymbol{a} \in A^n$, $f_1(\boldsymbol{a}) \neq f_2(\boldsymbol{a})$.

Case 2: $\boldsymbol{a} \notin A^n$: Let $v_1, v_2 \in A$ with $v_1 \neq v_2$. For $i \in \{1, 2\}$, we define

$$
f_i(\boldsymbol{x}) = \begin{cases} u & \text{if } \boldsymbol{x} \in (A^{\oplus})^n \setminus (A^n \cup \{\boldsymbol{a}\}), \\ x_1 & \text{if } \boldsymbol{x} \in A^n, \\ v_i & \text{if } \boldsymbol{x} \in \{\boldsymbol{a}\}. \end{cases}
$$

By construction, we have that $f_1|_{A^n}, f_2|_{A^n} \in \mathcal{C}$. Hence $f_1, f_2 \in \Phi(\mathcal{C})$. Moreover, $f_1|_B = f_2|_B$ and $f_1(\boldsymbol{a}) \neq f_2(\boldsymbol{a})$. Thus, we can conclude that $B \in \operatorname{Alg} \Phi(\mathcal{C})$. Equation (5.4) implies that for all $\mathcal{C}, \mathcal{D} \in \mathcal{L}_A$, if $\mathcal{D} \not\sim_{\text{alg}} \mathcal{C}$ then $\Phi(\mathcal{C}) \not\sim_{\text{alg}} \Phi(\mathcal{D})$.

Since by Corollary 5.5 there are $2^{\aleph_0}$ algebraically inequivalent clones on the set $\{0, 1, 2\}$ and since there are at most $2^{\aleph_0}$ clones on a finite set, we deduce that on each finite set A with $|A| \geq 3$ there are exactly $2^{\aleph_0}$ algebraically inequivalent clones. $\qquad\square$

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