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Correction

Correction to: Injective hulls for ordered algebras

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Abstract. We show that Lemma 4.4 and Theorem 4.5 in [Zhang, X., Laan, V., Injective hulls for ordered algebras, Algebra Universalis, 76 (2016), 339–349] are incorrect. These results can be corrected by replacing unary polynomial functions by linear functions.

Mathematics Subject Classification. 08B30, 06F99.

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Correction to: Algebra Univers. 76 (2016) 339–349

1. Introduction

In this note we will show that some results in the paper [5] are incorrect and explain how these results can be corrected. The key observation is that instead of arbitrary unary polynomial functions one has to use linear functions in order to define a closure operator. We will mostly use the same notation and terminology as in [5]. Let us just recall that

$$\mathsf{cl}(D) = \{u \in A \mid (\forall p \in P^1_{\mathcal{A}})(\forall a \in A)(p(D) \subseteq a{\downarrow} \implies p(u) \leqslant a)\},$$

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where $P^1_{\mathcal{A}}$ is the set of all unary polynomial functions on an ordered algebra $\mathcal{A}$ and $D \subseteq A$. We also point out that following [6] we prefer to use the term ‘lax morphism’ instead of ‘subhomomorphism’.

Throughout this text, a type Ω is fixed and all ordered algebras that we consider will be Ω-algebras even if Ω is not explicitly mentioned. If S is a subset of a poset P then we write $S{\downarrow} = \{a \in P \mid a \leqslant s \text{ for some } s \in S\}$ and denote the set of upper bounds of S by S^u. The next claim (where $a{\downarrow} := \{a\}{\downarrow}$) holds.

Lemma 1.1. *If $\mathcal{A}$ is an ordered algebra then $\mathsf{cl}(a{\downarrow}) = a{\downarrow}$ for all $a \in A$.*

Proof. Since cl is a closure operator on $\mathscr{P}(A)$ by [5, Lemma 4.1], we have $a{\downarrow} \subseteq \mathsf{cl}(a{\downarrow})$. Note that the identity mapping of A is a unary polynomial function. Thus, if $u \in \mathsf{cl}(a{\downarrow})$ then $\mathrm{id}_A(a{\downarrow}) \subseteq a{\downarrow}$ implies $u = \mathrm{id}_A(u) \leqslant a$, proving the inclusion $\mathsf{cl}(a{\downarrow}) \subseteq a{\downarrow}$. ☐

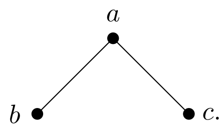
The following counter-example shows that Lemma 4.4 in [5] does not hold. More precisely, the equality

$$t_{\mathscr{P}(A)}(D_1, \dots, D_n) = t_{\mathcal{A}}(D_1, \dots, D_n){\downarrow}$$

need not hold for an arbitrary n-ary term t and $D_1, \dots, D_n \in \mathscr{P}(A)$.

Example 1.2. Let S be a posemigroup with the following multiplication and ordering:

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b



Then

$$\mathsf{cl}(\{b, c\}) = \{b, c\}\,.$$

Indeed, $a \notin \mathsf{cl}(\{b, c\})$, because, for the unary polynomial function $p(x) = x^2$ we have $p(\{b, c\}) = \{b^2, c^2\} = \{b\} \subseteq b{\downarrow}$, but $p(a) = a \nleqslant b$.

Using Lemma 1.1 we conclude that

$$\mathscr{Q}(S) = \mathscr{P}(S)_{\mathsf{cl}} = \{D \in \mathscr{P}(S) \mid \mathsf{cl}(D) = D\} = \{\emptyset, a{\downarrow}, b{\downarrow}, c{\downarrow}, \{b, c\}\} = \mathscr{P}(S)\,.$$

For the unary term $t = x^2$ we have

$$\begin{aligned} t_{\mathscr{P}(S)}(\{b, c\}) &= \{b, c\} \cdot_{\mathscr{P}(S)} \{b, c\} = \{u \cdot v \mid u, v \in \{b, c\}\}{\downarrow} = \{b, c\}{\downarrow} \\ &= \{b, c\} = \mathsf{cl}(\{b, c\})\,, \\ t_S(\{b, c\}){\downarrow} &= \{b^2, c^2\}{\downarrow} = \{b\}{\downarrow} = \{b\} = \mathsf{cl}(\{b\}). \end{aligned}$$

This contradicts Lemma 4.4 in [5].

Lemma 4.4 is used in the proof of Theorem 4.5 in [5], more precisely, to justify the equality

$$\psi(\widetilde{t}_{\mathscr{Q}(A)}(a_1{\downarrow}, \dots, a_{m-1}{\downarrow}, D)) = \psi(\mathsf{cl}(\widetilde{t}_{\mathcal{A}}(a_1{\downarrow}, \dots, a_{m-1}{\downarrow}, D){\downarrow})).$$

Thus, this equality need not hold. We will show that this problem can be overcome if we replace $P^1_{\mathcal{A}}$ in the definition of $\mathsf{cl}(D)$ by the set of all *linear functions*.

2. Linear functions

We define *linear functions* on an ordered algebra $\mathcal{A}$ as follows.

L1. The identity mapping $A \to A, x \mapsto x$, is a linear function.

L2. If $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1,\ldots,n\}$, $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n \in A$ and $p : A \to A$ is a linear function, then the mapping

$$A \to A,\ \ x \mapsto \omega(a_1,\ldots,a_{i-1},p(x),a_{i+1},\ldots,a_n)$$

is a linear function.

Linear functions obtained by step L1 or by step L2 with $p = \mathrm{id}_A$ are called *elementary translations* of $\mathcal{A}$. We denote the set of all elementary translations on $\mathcal{A}$ by $E_{\mathcal{A}}$ and the set of all linear functions by $L_{\mathcal{A}}$. Linear functions are the composites of elementary translations and

$$E_{\mathcal{A}} \subseteq L_{\mathcal{A}} \subseteq P^1_{\mathcal{A}}.$$

Example 2.1. If R is a commutative semiring with identity, then unary polynomial functions have the form $p(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_0, a_1, \ldots, a_n \in R$, but linear functions have the form $p(x) = ax + b$, $a, b \in R$. Elementary translations are $x \mapsto x + b$ and $x \mapsto ax$, $a, b \in R$.

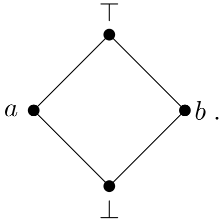
We call a term *linear* (cf. [2]) if it contains at least one variable and every variable occurring in it occurs precisely once. A function $p : A \to A$ on an ordered algebra $\mathcal{A}$ is linear if and only if there exists a linear term $t = t(x_1,\ldots,x_{n-1},x_n)$ and elements $a_1,\ldots,a_{n-1} \in A$ such that $p(x) = t_{\mathcal{A}}(a_1,\ldots,a_{n-1},x)$ for every $x \in A$.

An ordered Ω-algebra $\mathcal{Q} = (Q, \Omega_Q, \leqslant_Q)$ is called a *sup-algebra* (cf. [1, Definition 2.2.1]) if the poset $(Q, \leqslant_Q)$ is a complete lattice and all elementary translations preserve joins. Since linear functions are the composites of elementary translations, we have the following fact.

Lemma 2.2. *Let $\mathcal{A}$ be an ordered algebra such that $(A, \leqslant)$ is a complete lattice. Then $\mathcal{A}$ is a sup-algebra if and only if all linear functions on it preserve joins.*

Example 2.3. Arbitrary unary polynomial functions of sup-algebras need not preserve joins. Consider a commutative non-unital quantale $A = \{\top, \bot, a, b\}$ with the following multiplication table and ordering:

$\cdot$	$\bot$	$\top$	a	b
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$
a	$\bot$	$\top$	a	b
b	$\bot$	$\top$	b	a



For the unary polynomial function $p : A \to A, x \mapsto xx$, we have

$$p\left(\bigvee\{a,b\}\right) = p(\top) = \top\cdot\top = \top \neq a = \bigvee\{a\} = \bigvee\{a\cdot a, b\cdot b\} = \bigvee p(\{a,b\}).$$

A closure operator j on a sup-algebra $\mathcal{Q}$ is a *nucleus* if it is a lax endomorphism of $\mathcal{Q}$. The subset $Q_j = \{q \in Q \mid j(q) = q\}$ can be made into a sup-algebra called a *quantic quotient* of $\mathcal{Q}$ (see [1, Theorem 2.2.7] or [4, Proposition 16]).

If $\mathcal{A}$ is an ordered algebra then $\mathscr{P}(A)$ can be considered as a sup-algebra in a certain canonical way (see [5]). A nucleus j on $\mathscr{P}(A)$ is called *topological* (cf. [3, Definition 2.2]), if $j(a{\downarrow}) = a{\downarrow}$ for all $a \in A$.

For an ordered algebra $\mathcal{A}$ and a subset $D \subseteq A$ we will use the notation

$$\overline{D} = \{u \in A \mid (\forall p \in L_{\mathcal{A}})(\forall a \in A)(p(D) \subseteq a{\downarrow} \implies p(u) \leqslant a)\}.$$

Since $L_{\mathcal{A}} \subseteq P^1_{\mathcal{A}}$, we conclude that, for each subset D of A,

$$\mathsf{cl}(D) \subseteq \overline{D}\,.$$

Lemma 2.4. *For every subset D of an ordered algebra $\mathcal{A}$, $\overline{D{\downarrow}} = \overline{D}$.*

Proof. The inclusion $\overline{D} \subseteq \overline{D{\downarrow}}$ is obvious. To prove the opposite inclusion, take $a \in \overline{D{\downarrow}}$ and let $p \in L_{\mathcal{A}}$, $b \in A$ and $p(D) \subseteq b{\downarrow}$. Since p is monotone, also $p(D{\downarrow}) \subseteq b{\downarrow}$. By assumption, $p(a) \leqslant b$, and hence $a \in \overline{D}$. □

Lemma 2.5. *If $\mathcal{A}$ is an ordered algebra then the mapping $\mathscr{P}(A) \to \mathscr{P}(A)$, $D \mapsto \overline{D}$, is a topological nucleus on the sup-algebra $\mathscr{P}(A)$.*

Proof. Very similarly to Lemma 4.1 in [5] one can prove that $D \mapsto \overline{D}$ is a nucleus. Precisely as in Lemma 1.1 one can see that $\overline{a{\downarrow}} = a{\downarrow}$ for every $a \in A$, so the nucleus is topological. □

Since the mapping $\overline{D} : \mathscr{P}(A) \to \mathscr{P}(A), D \mapsto \overline{D}$, is a nucleus on the sup-algebra $\mathscr{P}(A)$, we may consider the quotient sup-algebra

$$\mathscr{L}(A) := \mathscr{P}(A)_{-} = \{D \in \mathscr{P}(A) \mid \overline{D} = D\}.$$

Operations in $\mathscr{L}(A)$ are defined by

$$\omega_{\mathscr{L}(A)}(D_1, \ldots, D_n) = \overline{\omega_{\mathscr{P}(A)}(D_1, \ldots, D_n)} = \overline{\omega_A(D_1, \ldots, D_n){\downarrow}} \tag{2.1}$$

if $n \in \mathbb{N}$, $\omega \in \Omega_n$, $D_1, \ldots, D_n \in \mathscr{L}(A)$, and $\omega_{\mathscr{L}(A)} = \overline{\omega_{\mathscr{P}(A)}} = \overline{\omega_A{\downarrow}} = \omega_A{\downarrow}$ if $\omega \in \Omega_0$.

By [5, Lemma 4.1], cl is also a nucleus on the sup-algebra $\mathscr{P}(A)$, and we have the quotient sup-algebra

$$\mathscr{Q}(A) := \mathscr{P}(A)_{\mathsf{cl}} = \{D \in \mathscr{P}(A) \mid \mathsf{cl}(D) = D\}.$$

Lemma 2.6. *For every ordered algebra $\mathcal{A}$,*

$$\mathscr{L}(A) \subseteq \mathscr{Q}(A).$$

Proof. If $\overline{D} = D$ then $\mathsf{cl}(D) \subseteq \overline{D} = D \subseteq \mathsf{cl}(D)$ implying $\mathsf{cl}(D) = D$. □

Example 2.7. It can happen that $\mathsf{cl}(D) \subset \overline{D}$ and $\mathscr{L}(A) \subset \mathscr{Q}(A)$. Consider the po-semigroup S of Example 1.2. Then

$$\overline{\{b, c\}} = \{a, b, c\}.$$

To see this we only need to prove that $a \in \overline{\{b,c\}}$. Because of commutativity, the linear functions are id_S, $p_a(x) = ax$, $p_b(x) = bx$ and $p_c(x) = cx$. We need to prove that

$$(\forall p \in L_S)(\forall s \in S)\,(s \in p(\{b,c\})^u \implies p(a) \leqslant s)\,.$$

It is straightforward to calculate that $p(\{b,c\})^u = \{a\}$ for every $p \in L_S$, and the inequality $p(a) \leqslant a$ clearly holds.

For this posemigroup S we have

$$\mathscr{L}(S) = \{a{\downarrow},\ b{\downarrow},\ c{\downarrow},\ \emptyset\} \subset \{a{\downarrow},\ b{\downarrow},\ c{\downarrow},\ \emptyset,\ \{b,c\}\} = \mathscr{Q}(S).$$

3. Injective hulls

In this section we will prove that $\mathscr{L}(A)$ is the $\mathcal{M}^{\leqslant}$-injective hull of an ordered algebra $\mathcal{A}$ in the category $\mathsf{OAlg}^{\leqslant}$, thereby correcting Theorem 4.5 in [5].

Recall that $\mathcal{M}^{\leqslant}$ is the class of mappings $h\colon \mathcal{A} \to \mathcal{B}$ between ordered Ω-algebras that satisfy the following conditions:

M1. h is monotone,
M2. $\omega_B(h(a_1), \dots, h(a_n)) \leqslant h(\omega_A(a_1, \dots, a_n))$ for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \dots, a_n \in A$,
M3. $\omega_B = h(\omega_A)$ for all $\omega \in \Omega_0$,
M4. for all $n \in \mathbb{N},\ t \in T_\Omega^n$, $a_1, \dots, a_n, a \in A$,

$$t_B(h(a_1), \dots, h(a_n)) \leqslant h(a) \Longrightarrow t_A(a_1, \dots, a_n) \leqslant a.$$

It turns out that arbitrary terms in condition M4 can be replaced by linear terms.

Lemma 3.1. *A mapping $h\colon \mathcal{A} \to \mathcal{B}$ between ordered Ω-algebras satisfies condition M4 if and only if it satisfies the following condition:*

M4′. *for all $n \in \mathbb{N}$, $a_1, \dots, a_n, a \in A$, and a linear term $t = t(x_1, \dots, x_n)$,*

$$t_B(h(a_1), \dots, h(a_n)) \leqslant h(a) \Longrightarrow t_A(a_1, \dots, a_n) \leqslant a.$$

Proof. Obviously M4 implies M4′. Conversely, assume that M4′ holds and consider an arbitrary term $t = t(x_1, \dots, x_n)$, where a variable x_i appears r_i times, $i = 1, \dots, n$. We replace each occurrence of x_i by a distinct new variable x_i^j, $j \in \{1, \dots, r_i\}$. In the resulting term

$$\hat{t} = \hat{t}(x_1^1, \dots, x_1^{r_1}, \dots, x_n^1, \dots, x_n^{r_n})$$

each variable occurs precisely once, thus it is a linear term. We say that $\hat{t}$ is obtained from t by *linearizing*. Now if $t_B(h(a_1), \dots, h(a_n)) \leqslant h(a)$ then

$$\hat{t}_B(h(a_1), \dots, h(a_1), \dots, h(a_n), \dots, h(a_n)) \leqslant a.$$

Applying condition M4′, we obtain $\hat{t}_A(a_1, \dots, a_1, \dots, a_n, \dots, a_n) \leqslant a$, but then also $t_A(a_1, \dots, a_n) \leqslant a$. □

We will use the following lemma.

Lemma 3.2. *For an ordered algebra $\mathcal{A}$, $D_1, \ldots, D_n \in \mathscr{P}(A)$ and a linear term $t = t(x_1, \ldots, x_n)$ we have*

$$t_{\mathscr{P}(A)}(D_1, \ldots, D_n) = t_A(D_1, \ldots, D_n){\downarrow}$$

and

$$t_{\mathscr{L}(A)}(D_1, \ldots, D_n) = \overline{t_A(D_1, \ldots, D_n){\downarrow}}.$$

Proof. If $t = x$ then $t_{\mathscr{P}(A)}(D_1) = D_1 = D_1{\downarrow} = t_A(D_1){\downarrow}$.

Let us consider $t = \omega(t^1(x_{11}, \ldots, x_{1n_1}), \ldots, t^k(x_{k1}, \ldots, x_{kn_k}))$ where t^1, $\ldots, t^k$ are linear terms and $x_{ij} \in \{x_1, \ldots, x_n\}$. Assume that for the terms $t^1, \ldots, t^k$ the claim holds. Then

$$\begin{aligned}
&t_{\mathscr{P}(A)}(D_1, \ldots, D_n)\\
&\quad= \omega_{\mathscr{P}(A)}(t^1_{\mathscr{P}(A)}(D_{11}, \ldots, D_{1n_1}), \ldots, t^k_{\mathscr{P}(A)}(D_{k1}, \ldots, D_{kn_k}))\\
&\quad= \{\omega_A(u_1, \ldots, u_k) \mid u_i \in t^i_{\mathscr{P}(A)}(D_{i1}, \ldots, D_{in_i})\}{\downarrow}\\
&\quad= \{\omega_A(u_1, \ldots, u_k) \mid u_i \in t^i_A(D_{i1}, \ldots, D_{in_i}){\downarrow}\}{\downarrow}\\
&\quad= \{\omega_A(u_1, \ldots, u_k) \mid u_i \in t^i_A(D_{i1}, \ldots, D_{in_i})\}{\downarrow}\\
&\quad= \{\omega_A(t^1_A(d_{11}, \ldots, d_{1n_1}), \ldots, t^k_A(d_{k1}, \ldots, d_{kn_k})) \mid d_{ij} \in D_{ij}\}{\downarrow}\\
&\quad= \{t_A(d_1, \ldots, d_n) \mid d_i \in D_i\}{\downarrow}\\
&\quad= t_A(D_1, \ldots, D_n){\downarrow}
\end{aligned}$$

and

$$t_{\mathscr{L}(A)}(D_1, \ldots, D_n) = \overline{t_{\mathscr{P}(A)}(D_1, \ldots, D_n)} = \overline{t_A(D_1, \ldots, D_n){\downarrow}}. \qquad \square$$

Theorem 3.3. *If $\mathcal{A}$ is an ordered algebra then $\mathscr{L}(A)$ is the $\mathcal{M}^{\leqslant}$-injective hull of $\mathcal{A}$ in the category $\mathsf{OAlg}^{\leqslant}$.*

Proof. By [5, Theorem 2.6] we know that the sup-algebra $\mathscr{L}(A)$ is an injective object in the category $\mathsf{OAlg}^{\leqslant}$. The mapping $\eta\colon A \to \mathscr{L}(A), a \mapsto a{\downarrow}$ is clearly an order-embedding. It is also an ordered algebra homomorphism, because, by (2.1)

$$\begin{aligned}
\omega_{\mathscr{L}(A)}(\eta(a_1), \ldots, \eta(a_n)) &= \overline{\omega_A(a_1{\downarrow}, \ldots, a_n{\downarrow}){\downarrow}} = \overline{\omega_A(a_1, \ldots, a_n){\downarrow}}\\
&= \omega_A(a_1, \ldots, a_n){\downarrow} = \eta(\omega_A(a_1, \ldots, a_n))
\end{aligned}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$, and $\omega_{\mathscr{L}(A)} = \omega_A{\downarrow} = \eta(\omega_A)$ for every $\omega \in \Omega_0$. Thus η belongs to $\mathcal{M}^{\leqslant}$ by [5, Lemma 1.1].

We will prove that η is an $\mathcal{M}^{\leqslant}$-essential morphism in the category $\mathsf{OAlg}^{\leqslant}$. Assume that $\psi\colon \mathscr{L}(A) \to \mathcal{B}$ is a morphism in $\mathsf{OAlg}^{\leqslant}$ such that the composite

$$\mathcal{A} \xrightarrow{\eta} \mathscr{L}(A) \xrightarrow{\psi} \mathcal{B}$$

belongs to $\mathcal{M}^{\leqslant}$. Denote $\phi := \psi\eta$. We have to show that $\psi \in \mathcal{M}^{\leqslant}$. Clearly conditions M1 and M2 are satisfied. If $\omega \in \Omega_0$ then $\omega_B = (\psi\eta)(\omega_A) = \psi(\omega_A{\downarrow}) = \psi(\omega_{\mathscr{L}(A)})$, so condition M3 also holds.

It remains to verify condition M4. Suppose that

$$t_B(\psi(D_1), \ldots, \psi(D_n)) \leqslant \psi(D) \tag{3.1}$$

in B for an n-ary linear term $t = t(x_1, \ldots, x_n)$, where $D_1, \ldots, D_n, D \in \mathscr{L}(A)$. Our aim is to establish the inclusion $t_{\mathscr{L}(A)}(D_1, \ldots, D_n) \subseteq D$. Take

$$u \in t_{\mathscr{L}(A)}(D_1, \ldots, D_n) = \overline{t_{\mathscr{P}(A)}(D_1, \ldots, D_n)}.$$

We need to prove that $u \in \overline{D} = D$. Suppose that $p(D) \subseteq a{\downarrow}$ where $a \in A$, $p \in L_{\mathcal{A}}$. We are done if $p(u) \leqslant a$.

Since p is a linear function, there exists a linear term $\widetilde{t}(y_1, \ldots, y_m, y)$ (we may assume that $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_m, y\} = \emptyset$) and $a_1, \ldots, a_m \in A$ such that $p = \widetilde{t}_A(a_1, \ldots, a_m, \_)$. Then $p' = \widetilde{t}_{\mathscr{L}(A)}(\eta(a_1), \ldots, \eta(a_m), \_) \in L_{\mathscr{L}(A)}$. Using Lemma 3.2 and Lemma 2.4 we obtain

$$\begin{aligned} p'(D) &= \widetilde{t}_{\mathscr{L}(A)}(a_1{\downarrow}, \ldots, a_m{\downarrow}, D) = \overline{\widetilde{t}_A(a_1{\downarrow}, \ldots, a_m{\downarrow}, D){\downarrow}} \\ &= \overline{\widetilde{t}_A(a_1, \ldots, a_m, D){\downarrow}} = \overline{p(D)}. \end{aligned}$$

We wish to prove the inclusion

$$p(t_{\mathscr{P}(\mathcal{A})}(D_1, \ldots, D_n)) \subseteq a{\downarrow}. \tag{3.2}$$

To this end, we take an arbitrary $v \in t_{\mathscr{P}(\mathcal{A})}(D_1, \ldots, D_n)$. By Lemma 3.2,

$$v \leqslant t_A(d_1, \ldots, d_n),$$

where $d_i \in D_i$ for every $i \in \{1, \ldots, n\}$. Then $\widetilde{t}(y_1, \ldots, y_m, t(x_1, \ldots, x_n))$ is a linear term and

$$\begin{aligned} &\widetilde{t}_B(\phi(a_1), \ldots, \phi(a_m), t_B(\phi(d_1), \ldots, \phi(d_n))) \\ &\quad = \widetilde{t}_B(\psi(a_1{\downarrow}), \ldots, \psi(a_m{\downarrow}), t_B(\psi(d_1{\downarrow}), \ldots, \psi(d_n{\downarrow}))) && (\phi = \psi\eta) \\ &\quad \leqslant \widetilde{t}_B(\psi(a_1{\downarrow}), \ldots, \psi(a_m{\downarrow}), t_B(\psi(D_1), \ldots, \psi(D_n))) && (d_i{\downarrow} \subseteq D_i) \\ &\quad \leqslant \widetilde{t}_B(\psi(a_1{\downarrow}), \ldots, \psi(a_m{\downarrow}), \psi(D)) && (\text{by } (3.1)) \\ &\quad \leqslant \psi(\widetilde{t}_{\mathscr{L}(A)}(a_1{\downarrow}, \ldots, a_m{\downarrow}, D)) && (\psi \text{ is a lax morphism}) \\ &\quad = \psi(p'(D)) && (\text{def. of } p') \\ &\quad = \psi\left(\overline{p(D)}\right) && (p'(D) = \overline{p(D)}) \\ &\quad \leqslant \psi\left(\overline{a{\downarrow}}\right) && (p(D) \subseteq a{\downarrow}) \\ &\quad = \psi(a{\downarrow}) && (\text{Lemma } 2.5) \\ &\quad = \phi(a). && (\psi\eta = \phi) \end{aligned}$$

Since ϕ satisfies condition M4′, we conclude that

$$p(v) \leqslant \widetilde{t}(a_1, \ldots, a_m, t_A(d_1, \ldots, d_n)) \leqslant a,$$

which proves (3.2). From $p(t_{\mathscr{P}(A)}(D_1, \ldots, D_n)) \subseteq a{\downarrow}$ we conclude that $p(u) \leqslant a$ (because $u \in \overline{t_{\mathscr{P}(A)}(D_1, \ldots, D_n)}$), as was to be shown. This completes the proof. □

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