Mal'tsev products of varieties, I

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Abstract. We investigate the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ of two varieties $\mathcal{V}$ and $\mathcal{W}$ of the same similarity type. While such a product is usually a quasivariety, it is not necessarily a variety. We give an equational base for the variety generated by $\mathcal{V} \circ \mathcal{W}$ in terms of identities satisfied in $\mathcal{V}$ and $\mathcal{W}$. Then the main result provides a new sufficient condition for $\mathcal{V} \circ \mathcal{W}$ to be a variety: If $\mathcal{W}$ is an idempotent variety and there are terms $f(x, y)$ and $g(x, y)$ such that $\mathcal{W}$ satisfies the identity $f(x, y) = g(x, y)$ and $\mathcal{V}$ satisfies the identities $f(x, y) = x$ and $g(x, y) = y$, then $\mathcal{V} \circ \mathcal{W}$ is a variety. We provide a number of examples and applications of this result.

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1. Introduction

In his 1967 paper [14], Mal'tsev introduced a product of classes of algebras that is now known as the Mal'tsev product. Let $\tau: \Omega \rightarrow \mathbb{N}$ be a similarity type of Ω -algebras. Let $\mathcal{B}$ and $\mathcal{C}$ be subclasses of a class $\mathcal{D}$ of Ω -algebras. Then the *Mal'tsev product* $\mathcal{B} \circ_{\mathcal{D}} \mathcal{C}$ of $\mathcal{B}$ and $\mathcal{C}$ relative to $\mathcal{D}$ consists of all algebras A in $\mathcal{D}$, with a congruence θ , such that A/θ belongs to $\mathcal{C}$ and every congruence class of θ that is a subalgebra of A belongs to $\mathcal{B}$. If $\mathcal{D}$ is the variety $\mathcal{T}$ of all Ω -algebras, then the Mal'tsev product $\mathcal{B} \circ_{\mathcal{T}} \mathcal{C}$ is called simply the *Mal'tsev product* of $\mathcal{B}$ and $\mathcal{C}$, and is denoted by $\mathcal{B} \circ \mathcal{C}$.

In the same paper, Mal'tsev proved that, if $\mathcal{Q}$ and $\mathcal{R}$ are subquasivarieties of a quasivariety $\mathcal{K}$ of Ω -algebras, then the Mal'tsev product $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is also a quasivariety, albeit under the assumption of a finite similarity type for $\mathcal{K}$. If one assumes that the second quasivariety $\mathcal{R}$ is idempotent, then there is no

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restriction on the type. This “closure” property makes the class of subquasivarieties of a given quasivariety a natural domain for consideration of Mal'tsev products. Moreover, the congruence θ in the definition of the Mal'tsev product of two quasivarieties may be taken to be the $\mathcal{R}$ -replica congruence ϱ of A , i.e. the smallest congruence of A whose induced quotient falls into the quasivariety $\mathcal{R}$. (See e.g. [20, Ch. 3].) So, in this case, the Mal'tsev product may be described formally as follows:

$$\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R} = \{A \in \mathcal{K} \mid (\forall a \in A) (a/\varrho \leq A \Rightarrow a/\varrho \in \mathcal{Q})\}. \quad (1.1)$$

Idempotent elements of algebras in $\mathcal{R}$ play a significant role in the theory of Mal'tsev products $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ of quasivarieties $\mathcal{Q}$ and $\mathcal{R}$. Recall that an element a of an Ω -algebra A is *idempotent* if $\{a\}$ is a subalgebra of A . An algebra A is *idempotent* if every element is idempotent. A class of Ω -algebras is *idempotent* if every member algebra is idempotent.

The following lemma summarizes the relationship between congruence classes that form subalgebras and idempotent elements of the quotient algebra.

Lemma 1.1. *Let A be an Ω -algebra with congruence θ , and let $a \in A$. Then a/θ is a subalgebra of A if and only if a/θ is an idempotent element of A/θ .*

Let us note that if no non-trivial $\mathcal{R}$ -algebra has idempotent elements, then $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ contains all $\mathcal{K}$ -algebras, since each of them has an $\mathcal{R}$ -replica congruence, and no corresponding congruence classes are subalgebras. So in this case $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ coincides with the class $\mathcal{K}$.

Another extreme case concerns the situation when $\mathcal{R}$ is an idempotent quasivariety. In this case, each ϱ -class of an algebra A in $\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R}$ is a subalgebra of A , and then

$$\mathcal{Q} \circ_{\mathcal{K}} \mathcal{R} = \{A \in \mathcal{K} \mid (\forall a \in A) (a/\varrho \in \mathcal{Q})\}. \quad (1.2)$$

Finally it may happen that, though $\mathcal{R}$ is not an idempotent quasivariety, all $\mathcal{R}$ -algebras have some idempotent elements. (Examples are provided by inverse semigroups.)

We usually assume that Ω -algebras considered in this paper have no nullary basic operations. Note however that this assumption is not essential. If Ω contains a symbol c of a nullary operation, one can replace it by a symbol $c(x)$ of a unary operation as follows. If $\mathcal{Q}$ is a quasivariety of Ω -algebras of a type τ containing the symbol c , then instead of $\mathcal{Q}$ one can consider the equivalent quasivariety $\mathcal{Q}'$ of the type obtained from τ by taking the symbol $c(x)$ in place of c , satisfying the quasi-identities true in $\mathcal{Q}$ and additionally the identity $c(x) = c(y)$. Hence, in particular, the unary operation is constant on all algebras of $\mathcal{Q}'$. Sometimes we have to consider varieties of algebras with at least one non-unary operation and with no nullary basic operations. Such a similarity type is called *plural* [20].

An Ω -term t such that $\mathcal{V}$ satisfies the identities

$$t \dots t \omega = t, \quad (1.3)$$

for each $\omega \in \Omega$, will be called a *term idempotent* of $\mathcal{V}$.

For varieties of algebras the following version of [5, Thm. 9] holds.

Lemma 1.2. *Let $\mathcal{V}$ be a variety of Ω -algebras of a type without nullary operation symbols. Then each algebra in $\mathcal{V}$ has idempotents if and only if the variety $\mathcal{V}$ has a term idempotent. Moreover, if $\mathcal{V}$ is of a plural type and has a term idempotent, then it has term idempotents of any arity $n \geq 1$.*

Proof. If all algebras in $\mathcal{V}$ have idempotents, then in particular also the free algebra $X\Omega$ on one-element set X has an idempotent, which is some unary term t . And as an idempotent, this term satisfies (1.3).

On the other hand, if there is an n -ary Ω -term t such that $\mathcal{V}$ satisfies (1.3), then for any algebra A in $\mathcal{V}$ and any $a \in A$, one has

$$(a \dots at) \dots (a \dots at)\omega = a \dots at$$

for each $\omega \in \Omega$, whence $a \dots at$ is an idempotent of A .

Finally note that if $x_1 \dots x_n t$ is an Ω -term satisfying (1.3) for each $\omega \in \Omega$, then the unary $x \dots x t$ also satisfies it. Moreover, for an arbitrary Ω -term u of any arity, we also have

$$(u \dots ut) \dots (u \dots ut)\omega = u \dots ut$$

for each $\omega \in \Omega$. □

A special case concerns the situation of so-called polarized varieties. A variety $\mathcal{V}$ of Ω -algebras is *polarized* if there is a unary term idempotent of $\mathcal{V}$ that is constant on every member of $\mathcal{V}$. This constant is called the *pole* of the algebra, and the algebra is called *polarized*. The pole of an algebra is unique (if it exists), and a congruence class of a polarized algebra is a subalgebra if and only if it is the congruence class of the pole. A (quasi)variety of Ω -algebras is *polarized* if all its members are polarized algebras. (See Mal'tsev [14].)

Corollary 1.3. *Each algebra in $\mathcal{V}$ has precisely one idempotent if and only if the variety $\mathcal{V}$ is polarized.*

In this paper we are interested in Mal'tsev products $\mathcal{V} \circ \mathcal{W}$ of two varieties $\mathcal{V}$ and $\mathcal{W}$ of the same type $\tau: \Omega \rightarrow \mathbb{N}$. Such a product is usually a quasivariety, but need not be a variety. For some results providing sufficient conditions for the Mal'tsev product of two varieties to be a variety see e.g. [7, §57, §63], [10, 11], [14], [17]. They usually concern Mal'tsev products relative to some special varieties. The best known may be the following theorem of Mal'tsev.

Theorem 1.4 [14]. *If $\mathcal{V}$ and $\mathcal{W}$ are subvarieties of a congruence-permutable polarized variety $\mathcal{K}$, then $\mathcal{V} \circ_{\mathcal{K}} \mathcal{W}$ is a variety.*

Such products were considered earlier in the case of groups (see Hanna Neumann's book [17] on varieties of groups). The Mal'tsev product of any two varieties of groups relative to the variety of all groups is a variety. Sufficient conditions of a kind similar to Theorem 1.4 were then obtained by A.A. Iskander in [10] and [11], and some versions of Theorem 1.4 by C. Bergman in [1] and [2]. In particular, almost the same proof as the proof of Mal'tsev's Theorem 1.4 may be used to show the following, where it is sufficient to assume merely that the variety $\mathcal{W}$ is idempotent.

Theorem 1.5 [2]. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent subvarieties of a congruence permutable variety $\mathcal{K}$, then $\mathcal{V} \circ_{\mathcal{K}} \mathcal{W}$ is a variety.*

Another special case concerns the Mal'tsev product of a strongly irregular variety $\mathcal{V}$ of any plural type τ and the variety of semilattices considered as the variety of the same type τ . Recall that a *strongly irregular variety* is a variety $\mathcal{V}_t$ defined by a set of regular identities and a *strongly irregular identity* $t(x, y) = x$, where $t(x, y)$ is a binary term containing both variables x and y . The variety $\mathcal{S}$ of semilattices may be considered as the variety $\mathcal{S}_\tau$ of any plural type τ (definitionally) equivalent to $\mathcal{S}$. (See [4] for details.)

Theorem 1.6 [4]. *If $\mathcal{V}_t$ is a strongly irregular variety of a plural type τ and $\mathcal{S}_\tau$ is the variety of the same type τ equivalent to the variety of semilattices, then $\mathcal{V}_t \circ \mathcal{S}_\tau$ is a variety.*

Also in [4], it was shown how to derive an equational base for the product $\mathcal{V}_t \circ \mathcal{S}_\tau$ from equational bases for $\mathcal{V}_t$ and $\mathcal{S}_\tau$, and even more generally, how to derive all identities true in the product $\mathcal{V} \circ \mathcal{S}_\tau$, for any variety $\mathcal{V}$ of Ω -algebras of a plural type τ .

The aim of this paper is to describe the identities true in the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ of two varieties of a type τ without symbols of nullary operations, in terms of the identities satisfied by its components, and then to extend Theorem 1.6.

The identities satisfied in $\mathcal{V} \circ \mathcal{W}$ are described in Section 2. They are obtained from the identities true in $\mathcal{V}$ by substituting for their variables $\mathcal{W}$ -equivalent term idempotents of $\mathcal{W}$. In the case when the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety, one then obtains its equational base.

Note that if $\mathcal{U}$ is a variety of type τ and $\mathcal{V} \circ \mathcal{W}$ is a variety, then $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W} = (\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$ is also a variety. However, it is possible that $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ is a variety but $\mathcal{V} \circ \mathcal{W}$ is not. It was shown in [4] that the Mal'tsev product $\mathcal{S} \circ \mathcal{S}$, where $\mathcal{S}$ is the variety of semilattices, is not a variety. But since $\mathcal{S} \subseteq \mathcal{S} \circ \mathcal{S}$, it follows that $\mathcal{S} \circ_{\mathcal{S}} \mathcal{S}$ is the variety $\mathcal{S}$.

Theorem 3.3 provides a new sufficient condition for $\mathcal{V} \circ \mathcal{W}$ to be a variety. We show that if $\mathcal{W}$ is idempotent and there are terms $f(x, y)$ and $g(x, y)$ such that $\mathcal{V}$ satisfies the identities $f(x, y) = x$ and $g(x, y) = y$, and $\mathcal{W}$ satisfies the identity $f(x, y) = g(x, y)$, then $\mathcal{V} \circ \mathcal{W}$ is a variety. Some consequences of this theorem are derived. Section 4 discusses the case when the variety $\mathcal{W}$ is equivalent to a variety of bands.

We use notation and conventions similar to those of [4], [19, 20]. For details and further information concerning quasivarieties and Mal'tsev product of quasivarieties, we refer the reader to [14, 15], and then also to [2] and [20, Ch. 2]; for universal algebra, see [3] and [19, 20].

2. Identities true in Mal'tsev products

Throughout the paper, we assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of the same type τ : $\Omega \rightarrow \mathbb{N}$ without symbols of nullary operations. Let $\text{Id}(\mathcal{V})$ and $\text{Id}(\mathcal{W})$

denote the sets of identities true in $\mathcal{V}$ and $\mathcal{W}$ respectively. Members of the absolutely free Ω -algebra $X\Omega$ over a countably infinite set X , i.e. Ω -terms, are denoted $x_1 \dots x_k u, x_1 \dots x_k v$, etc. or briefly $\mathbf{x}u, \mathbf{x}v$, etc. or just u, v , etc. Note that $\mathbf{x}u$ may not actually contain a particular variable from $x_1, \dots, x_k$. We use a similar notation for elements of an Ω -algebra A . If $a_1, \dots, a_k$ are members of A , we write $\mathbf{a}u$, when u is applied to elements $a_1, \dots, a_k$, even if a particular variable x_i does not appear in u . If the identity $u = v$ holds in a variety $\mathcal{W}$, then the terms u and v will sometimes be called $\mathcal{W}$ -equivalent.

Definition 2.1. Let σ be an identity

$$x_1 \dots x_k u = x_1 \dots x_k v$$

true in $\mathcal{V}$. Define

$$\begin{aligned} \sigma^{\mathcal{W}} := \{ & r_1 \dots r_k u = r_1 \dots r_k v \mid \\ & r_1, \dots, r_k \in X\Omega, \\ & \forall 1 \leq i, j \leq k, \mathcal{W} \models r_i = r_j, \\ & \forall \omega \in \Omega, \mathcal{W} \models r_1 \dots r_1 \omega = r_1 \}. \end{aligned}$$

For a subset Σ of $\text{Id}(\mathcal{V})$, define

$$\Sigma^{\mathcal{W}} := \bigcup_{\sigma \in \Sigma} \sigma^{\mathcal{W}}.$$

Note that the last two condition of the definition of $\sigma^{\mathcal{W}}$ imply that $\mathcal{W} \models r_i \dots r_i \omega = r_i$ for all $1 \leq i \leq k$ and $\omega \in \Omega$. If $\mathcal{W}$ is an idempotent variety, then this condition is always satisfied.

In other words, we may say that the identities $\sigma^{\mathcal{W}}$ are obtained from σ by substituting, for variables of σ , pairwise $\mathcal{W}$ -equivalent term idempotents of $\mathcal{W}$. However, by Lemma 1.2, such terms can exist only if all $\mathcal{W}$ -algebras contain idempotent elements. Otherwise the set $\Sigma^{\mathcal{W}}$ is empty.

Lemma 2.2. Let Σ be any set of identities true in $\mathcal{V}$. Then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ satisfies the identities $\Sigma^{\mathcal{W}}$.

Proof. First note that if $\mathcal{W}$ has no term idempotents, then the set $\Sigma^{\mathcal{W}}$ is empty, and $\mathcal{V} \circ \mathcal{W}$ satisfies $\Sigma^{\mathcal{W}}$ vacuously.

Now assume that $\mathcal{W}$ has term idempotents. Let A be in $\mathcal{V} \circ \mathcal{W}$, with $\mathcal{W}$ -replica congruence ϱ . Let $x_1 \dots x_k u = x_1 \dots x_k v$ be an identity of Σ , and let $y_1 \dots y_n r_1, \dots, y_1 \dots y_n r_k$ be pairwise $\mathcal{W}$ -equivalent term idempotents of $\mathcal{W}$. Since $A/\varrho \in \mathcal{W}$, it follows that for any $\mathbf{a} \in A^n$, we have

$$(\mathbf{a}r_i)/\varrho = (\mathbf{a}r_j)/\varrho \tag{2.1}$$

for every $1 \leq i < j \leq k$. Thus all of the elements $\mathbf{a}r_i$, for $i = 1, \dots, k$, lie in the same congruence class of ϱ .

Since all r_i are term idempotents of $\mathcal{W}$, it follows that

$$(\mathbf{a}r_1)/\varrho \dots (\mathbf{a}r_1)/\varrho \omega = (\mathbf{a}r_1)/\varrho$$

for each $\omega \in \Omega$, so $(\mathbf{a}r_1)/\varrho$ is an idempotent of A/ϱ . By Lemma 1.1, $(\mathbf{a}r_1)/\varrho$ is a subalgebra of A . Consequently, since $A \in \mathcal{V} \circ \mathcal{W}$, we have $(\mathbf{a}r_1)/\varrho \in \mathcal{V}$.

Since $\mathbf{a}r_1, \dots, \mathbf{a}r_k \in (\mathbf{a}r_1)/\varrho$, it follows that $\mathbf{a}r_1 \dots \mathbf{a}r_k u = \mathbf{a}r_1 \dots \mathbf{a}r_k v$. Thus the identity

$$r_1 \dots r_k u = r_1 \dots r_k v$$

is satisfied in A . □

Note that Lemma 2.2 holds for any equational base Σ of $\mathcal{V}$. In particular, it holds for $\Sigma = \text{Id}(\mathcal{V})$.

Corollary 2.3. *For any equational base Σ of $\mathcal{V}$, the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is contained in the variety $\mathcal{U}$ defined by $\Sigma^{\mathcal{W}}$:*

$$\mathcal{V} \circ \mathcal{W} \subseteq \mathcal{U}. \quad (2.2)$$

The following simple examples illustrate how the set $\Sigma^{\mathcal{W}}$ is built from the set Σ .

Example 2.4. Let $\mathcal{V}$ be the variety $\mathcal{LZ}$ of left-zero semigroups defined by the identity $x \cdot y = x$, and let $\mathcal{W}$ be the variety $\mathcal{S}$ of semilattices. It is known that $r = s$ is an identity true in $\mathcal{S}$ precisely if the terms r and s have the same sets of variables. As the set Σ , we take the one-element set $\{\sigma\}$, where σ is the single identity defining $\mathcal{LZ}$. Since semilattices are idempotent, the set $\Sigma^{\mathcal{S}}$ consists of all identities of the form

$$x_1 \dots x_m r \cdot x_1 \dots x_m s = x_1 \dots x_m r,$$

where both r and s contain precisely the same variables $x_1, \dots, x_m$ and $m = 1, 2, \dots$. Note that $\Sigma^{\mathcal{S}}$ contains the idempotent law $x \cdot x = x$, but neither the commutative law nor the associative law are contained in $\Sigma^{\mathcal{S}}$. By Lemma 2.2, the Mal'tsev product $\mathcal{LZ} \circ \mathcal{S}$ satisfies the identities $\Sigma^{\mathcal{S}}$. It follows by results of [4] that in fact these identities define $\mathcal{LZ} \circ \mathcal{S}$.

Example 2.5. Let $\mathcal{V}$ be the variety $\mathcal{LZ}$ and let $\mathcal{W}$ be the variety $\mathcal{CS}$ of constant semigroups defined by the identity $x \cdot y = z \cdot t$. It is easy to see that $r = s$ is a (non-trivial) identity true in $\mathcal{CS}$ precisely if neither the term r nor the term s is a variable, or in other words, if each is the product of two groupoid terms, i.e.

$$r = r_1 \cdot r_2 \quad \text{and} \quad s = s_1 \cdot s_2.$$

As the set Σ we take the one-element set $\{\sigma\}$, where σ is the identity defining $\mathcal{LZ}$. To describe $\Sigma^{\mathcal{CS}}$, note that for $r = r_1 \cdot r_2$,

$$\mathcal{CS} \models r \cdot r = r_1 \cdot r_2 = r.$$

Consequently the last condition of the definition of $\Sigma^{\mathcal{CS}}$ is satisfied. It follows that the set $\Sigma^{\mathcal{CS}}$ consists of the identities of the form

$$(r_1 \cdot r_2) \cdot (s_1 \cdot s_2) = r_1 \cdot r_2,$$

where r_1, r_2, s_1, s_2 are any groupoid terms. If we define $c := x \cdot x$, then $\Sigma^{\mathcal{CS}}$ may be written as:

$$\Sigma^{\mathcal{CS}} = \{r \cdot s = r \mid \mathcal{CS} \models r = s = c\}.$$

Note that $\Sigma^{\mathcal{CS}}$ contains neither the idempotent nor the associative nor the commutative law. By Lemma 2.2, the Mal'tsev product $\mathcal{LZ} \circ \mathcal{CS}$ satisfies the identities $\Sigma^{\mathcal{CS}}$.

Example 2.6. In this example $\mathcal{V}$ will be again the variety $\mathcal{LZ}$ with σ defined as in the previous examples. And $\mathcal{W}$ will be the variety $\mathcal{GP}$ of groups, considered as algebras $(G, \cdot, ^{-1})$ with one binary and one unary operation, where the group identity e is defined as $x \cdot x^{-1}$. (See e.g. [4].) Left-zero bands are considered as algebras of the same type, with $x^{-1} := x$. Then Σ consists of the identity σ and the identity $x^{-1} = x$. Let r and s be groupoid terms. Then $r = s$ is an identity satisfied in $\mathcal{GP}$ if r and s have the same reduced (or canonical) form. (See e.g. [9].) Recall that each group has only one idempotent, namely the identity e . Hence $\sigma^{\mathcal{GP}}$ consists of the identities of the form

$$r \cdot s = r,$$

where $r = s$ is an identity satisfied in $\mathcal{GP}$ such that $r \cdot r = r$ and $r^{-1} = r$. However, the last two conditions are satisfied precisely if $r = e$. It follows that

$$\Sigma^{\mathcal{GP}} = \{r \cdot s = r, r^{-1} = r \mid \mathcal{GP} \models r = s = e\}.$$

By Lemma 2.2, the Mal'tsev product $\mathcal{LZ} \circ \mathcal{GP}$ satisfies the identities $\Sigma^{\mathcal{GP}}$.

In the next example, algebras in $\mathcal{W}$ have more than one idempotent.

Example 2.7. In this example $\mathcal{V}$ and Σ are defined as in Example 2.6. The variety $\mathcal{W}$ is the *regularisation* $\widetilde{\mathcal{GP}}$ of the variety $\mathcal{GP}$ of groups, defined by the regular identities true in $\mathcal{GP}$. (See e.g. [20, Ch. 4] and [4].) This class consists precisely of the Płonka sums of groups (in other words: strong semilattices of groups or Clifford semigroups), and forms a subvariety of the variety of inverse semigroups. (We refer the reader to [18] and [20, Ch. 4] for properties of Płonka sums, and to [13] for properties of inverse semigroups.) It is clear that $\widetilde{\mathcal{GP}}$ is contained in $\mathcal{GP} \circ \mathcal{S}$. Let $A = \sum_{i \in I} A_i$ be the Płonka sum of groups A_i . Then each A_i contains one idempotent, the identity e_i of A_i . We will show that two elements $a, b \in A$ belong to one summand of A precisely if $aa^{-1} = bb^{-1}$. First note that if $a \in A_i$, then a lies in the same summand as $aa^{-1} = e_i$. Let $a \in A_i$ and $b \in A_j$. Then $aa^{-1} = bb^{-1}$ precisely if $e_i = e_j$, i.e. $i = j$.

The elements of the form aa^{-1} form all the idempotents of the semigroup A . Note also the following identities true in all Clifford semigroups:

$$(xx^{-1}) \cdot (xx^{-1}) = xx^{-1} \quad \text{and} \quad (xx^{-1})^{-1} = xx^{-1}.$$

It follows that if σ is the identity $x \cdot y = x$, then $\sigma^{\widetilde{\mathcal{GP}}}$ consists of the identities of the form

$$r \cdot s = r,$$

where $r = s$ are identities satisfied in $\widetilde{\mathcal{GP}}$ such that $r \cdot r = r$ and $r^{-1} = r$. Identities satisfied in $\widetilde{\mathcal{GP}}$ are described by Płonka's Theorem (see [20, Ch. 4] and [4]), whereas the identities $r \cdot r = r$ and $r^{-1} = r$ hold precisely if $r = rr^{-1} = ss^{-1} = s$. Hence

$$\Sigma^{\widetilde{\mathcal{GP}}} = \{r \cdot s = r, r^{-1} = r \mid \widetilde{\mathcal{GP}} \models r = rr^{-1} = ss^{-1} = s\}.$$

By Lemma 2.2, the Mal'tsev product $\mathcal{LZ} \circ \widetilde{\mathcal{GP}}$ satisfies the identities $\Sigma^{\widetilde{\mathcal{GP}}}$.

Example 2.8. Consider the variety $\mathcal{MU}$ of monounary algebras (A, u) . Recall that all terms of monounary type have the form $u^m(x)$ for $m \in \mathbb{N}$. There are two types of proper subvarieties of $\mathcal{MU}$: the varieties $\mathcal{U}_k$, for $k \in \mathbb{N}$, each defined by the identity $u^k(x) = u^k(y)$, and the varieties $\mathcal{U}_{n,k}$, each defined by the identity $u^{n+k}(x) = u^k(x)$, for $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. (See e.g. [12, §9.1].) Note that an idempotent unary operation of an algebra A is just the identity mapping on A . The only idempotent subvariety of $\mathcal{MU}$ is the variety $\mathcal{U}_{1,0}$ defined by the identity $u(x) = x$. A term idempotent $u^m(x)$ of a subvariety of $\mathcal{MU}$ must satisfy $u^{1+m}(x) = u^m(x)$. The only subvarieties of $\mathcal{MU}$ having term idempotents are the varieties $\mathcal{U}_k$ and $\mathcal{U}_{1,k}$ for $k \in \mathbb{N}$. Term idempotents have the form $u^{k+l}(x)$ for $k, l \in \mathbb{N}$. (Again, see [12].) Now let $v = w$ be the unique identity defining a proper subvariety $\mathcal{V}$ of $\mathcal{MU}$. As the set Σ we take the one element set consisting of this identity. Let $\mathcal{W}$ be any of the varieties $\mathcal{U}_k$ or $\mathcal{U}_{1,k}$. Then the identities of $\Sigma^{\mathcal{W}}$ are obtained from $v = w$ by substituting variables in $v = w$ with $\mathcal{W}$ -equivalent term idempotents of $\mathcal{W}$. Finally, note that the varieties $\mathcal{U}_{n,k}$ for $n > 1$ have no term idempotents, so algebras of these varieties have no idempotent elements, and the set $\Sigma^{\mathcal{W}}$ is empty. In this case the variety $\mathbf{H}(\mathcal{V} \circ \mathcal{U}_{n,k})$ generated by the Mal'tsev product $\mathcal{V} \circ \mathcal{U}_{n,k}$ coincides with $\mathcal{MU}$.

If $\mathcal{U}$ is the variety defined by $\Sigma^{\mathcal{W}}$ for any equational base Σ of $\mathcal{V}$, then

$$\mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W} \subseteq \mathcal{U}.$$

By basic properties of free algebras in quasivarieties (see e.g. [20, §3.3]), it is known that if ϱ is the $\mathcal{W}$ -replica congruence of the free $\mathcal{U}$ -algebra $X\mathcal{U}$ over a set X , then

$$X\mathcal{U}/\varrho \cong X\mathcal{W},$$

where $X\mathcal{W}$ is the free $\mathcal{W}$ -algebra over X .

Proposition 2.9. *Let $\mathcal{U}$ be the variety defined by $\Sigma^{\mathcal{W}}$, where Σ is any equational base of $\mathcal{V}$. Let $X\mathcal{U}$ be the free $\mathcal{U}$ -algebra over a set X . Then*

$$X\mathcal{U} \in \mathcal{V} \circ \mathcal{W}.$$

Proof. We already know that the $\mathcal{W}$ -replica ϱ of $F := X\mathcal{U}$ is isomorphic to $X\mathcal{W}$. Let $a \in F$ be such that the congruence class $B := a/\varrho$ is a subalgebra of F . We will show that B belongs to $\mathcal{V}$.

Let σ be an identity $x_1 \dots x_k u = x_1 \dots x_k v$ from Σ and let $r_1, \dots, r_k \in B$. Since B is a single ϱ -class, it follows that $r_i/\varrho = r_j/\varrho$. As F/ϱ is free in $\mathcal{W}$, we get that $\mathcal{W}$ satisfies the identities $r_i = r_j$. Similarly, since B is a subalgebra of F , we conclude that $(r_1 \dots r_1 \omega)/\varrho = r_1/\varrho$ for each $\omega \in \Omega$, whence $\mathcal{W}$ satisfies the identities $r_1 \dots r_1 \omega = r_1$. Thus the identity $r_1 \dots r_k u = r_1 \dots r_k v$ belongs to $\sigma^{\mathcal{W}} \subseteq \Sigma^{\mathcal{W}}$. So it is satisfied in $\mathcal{U}$. In particular in the free $\mathcal{U}$ -algebra F we have the equality

$$r_1 \dots r_k u = r_1 \dots r_k v.$$

This shows that B satisfies the identities of Σ , so it belongs to $\mathcal{V}$. $\square$

Theorem 2.10. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same similarity type, and let Σ be an equational base for $\mathcal{V}$. Then the variety $\mathbf{H}(\mathcal{V} \circ \mathcal{W})$ generated by the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is defined by the identities $\Sigma^{\mathcal{W}}$.*

Proof. Let $\mathcal{U}$ be the variety defined by $\Sigma^{\mathcal{W}}$. By Corollary 2.3, we know that $\mathbf{H}(\mathcal{V} \circ \mathcal{W}) \subseteq \mathcal{U}$, and by Proposition 2.9, that $X\mathcal{U} \in \mathcal{V} \circ \mathcal{W} \subseteq \mathbf{H}(\mathcal{V} \circ \mathcal{W})$. Thus the varieties $\mathbf{H}(\mathcal{V} \circ \mathcal{W})$ and $\mathcal{U}$ share the same free algebras. Since free algebras are uniquely defined and determine the variety, it follows that $\mathbf{H}(\mathcal{V} \circ \mathcal{W}) = \mathcal{U}$. $\square$

Corollary 2.11. *If not all $\mathcal{W}$ -algebras contain idempotent elements, then the variety $\mathbf{H}(\mathcal{V} \circ \mathcal{W})$ does not depend on $\mathcal{V}$, and coincides with the variety $\mathcal{T}$ of all Ω -algebras.*

Corollary 2.12. *If Σ is an equational base for $\mathcal{V}$ and the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety, then $\Sigma^{\mathcal{W}}$ is an equational base for $\mathcal{V} \circ \mathcal{W}$.*

Let us recall that if $\mathcal{W}$ is an idempotent variety, then all blocks of congruences of algebras in $\mathcal{V} \circ \mathcal{W}$ are subalgebras, and the last condition in the definition of $\sigma^{\mathcal{W}}$ is redundant. Note, as well, that if $\mathcal{W}$ is idempotent, then each algebra A in $\mathcal{V} \circ \mathcal{W}$ decomposes as a disjoint union of subalgebras belonging to $\mathcal{V}$ over the $\mathcal{W}$ -replica of A . Following the convention adopted in [4] we may call such algebras $\mathcal{W}$ -sums of $\mathcal{V}$ -algebras.

For the special case of $\mathcal{W} = \mathcal{S}_\tau$, where $\mathcal{S}_\tau$ is the variety equivalent to the variety $\mathcal{S}$ of semilattices, one obtains Corollary 3.3 of [4]. Recall that two terms are $\mathcal{S}_\tau$ -equivalent if and only if they contain exactly the same variables.

The following example was provided by C. Bergman. It shows that, in general, one cannot replace the set $\text{Id}(\mathcal{W})$ in the definition of the identities true in $\mathcal{V} \circ \mathcal{W}$ by an equational base for $\mathcal{W}$.

Example 2.13. Consider the case where $\mathcal{V} = \mathcal{W} = \mathcal{LZ}$, the variety of left-zero bands. It is defined by the unique identity $x \cdot y = x$. Every identity of $\mathcal{LZ}$ is of the form $u = v$, where u and v are groupoid words with the same first variable. By Theorem 2.10, an equational base for $\mathbf{H}(\mathcal{LZ} \circ \mathcal{LZ})$ consists of all identities of the form $u \cdot v = u$ again, with u and v having the same first variable. This is the set $\Sigma^{\mathcal{W}}$ of Definition 2.1, if we take for Σ the unique identity $x \cdot y = x$ playing the role of σ . However, if in the definition of $\sigma^{\mathcal{W}}$ we replace $\text{Id}(\mathcal{LZ})$ by the unique identity defining $\mathcal{LZ}$, then we will only get the identities:

$$\begin{aligned} x \cdot x &= x, \\ (x \cdot y) \cdot x &= x \cdot y, \end{aligned} \tag{2.3}$$

$$\begin{aligned} x \cdot (x \cdot y) &= x, \\ (x \cdot y) \cdot (x \cdot y) &= x \cdot y. \end{aligned} \tag{2.4}$$

These are not sufficient to derive all the identities of $\mathbf{H}(\mathcal{LZ} \circ \mathcal{LZ})$. To see this let us note that a binary groupoid term may first be reduced to a shorter term using the idempotent law. If such a term is already reduced by the idempotence, then it may further be reduced to a shorter term by the identities (2.3) and (2.4) only if it has the form $uv \cdot u$ or $u \cdot uv$ for some binary terms u and

v , or else has a subterm of such a form. Now consider the term $xy \cdot (x \cdot yx)$. It is easy to see that neither xy nor $x \cdot yx$ nor $xy \cdot (x \cdot yx)$ has the form required to be reduced. On the other hand, the Mal'tsev product $\mathcal{LZ} \circ \mathcal{LZ}$ satisfies the identity $xy \cdot (x \cdot yx) = xy$.

3. A sufficient condition for $\mathcal{V} \circ \mathcal{W}$ to be a variety

Each algebra A of the same similarity type as algebras of a variety $\mathcal{W}$ has a replica A/ϱ in $\mathcal{W}$. The replica congruence ϱ is the intersection of all the congruences θ of A whose quotient A/θ lies in $\mathcal{W}$. Below, we provide another description useful in the proof of the main theorem of this section.

First, we define a binary relation ϱ^0 on A as follows:

$$\begin{aligned} (a, b) \in \varrho^0 & \text{ if and only if there are} \\ & \text{an identity } x_1 \dots x_n p = x_1 \dots x_n q \text{ true in } \mathcal{W} \text{ and} \\ & d_1, \dots, d_n \in A \text{ such that } a = d_1 \dots d_n p, b = d_1 \dots d_n q. \end{aligned}$$

Lemma 3.1. *Let α be a reflexive and symmetric binary relation on an algebra A preserving the operations of A . Then the transitive closure $\text{tr}(\alpha)$ of α is a congruence relation.*

Proof. The relation $\text{tr}(\alpha)$ is obviously an equivalence relation. So it remains to show that it preserves operations. Let $\omega \in \Omega$ be an n -ary operation and let $a_1, \dots, a_n, b_1, \dots, b_n \in A$ be such that $(a_i, b_i) \in \text{tr}(\alpha)$ for $1 \leq i \leq n$. Then for each i there are elements $d_i^1, \dots, d_i^{k_i}$, such that

$$a_i = d_i^1 \alpha d_i^2 \alpha \dots \alpha d_i^{k_i} = b_i.$$

Using reflexivity of α we can extend the sequences $d_i^1, \dots, d_i^{k_i}$ to sequences of the same length, say k , so that

$$a_i = d_i^1 \alpha d_i^2 \alpha \dots \alpha d_i^k = b_i.$$

Since α preserves operations, we have

$$a_1 \dots a_n \omega \alpha \dots \alpha d_j^1 \dots d_j^n \omega \alpha d_{j+1}^1 \dots d_{j+1}^n \omega \alpha \dots \alpha b_1 \dots b_n \omega.$$

Thus $(a_1 \dots a_n \omega, b_1 \dots b_n \omega) \in \text{tr}(\alpha)$. $\square$

Proposition 3.2. *Let $\mathcal{W}$ be a variety, and let A be an algebra of the same similarity type as algebras in $\mathcal{W}$. The $\mathcal{W}$ -replica congruence ϱ of A coincides with the transitive closure $\text{tr}(\varrho^0)$ of ϱ^0 .*

Proof. We will first prove that the relation $\text{tr}(\varrho^0)$ is a congruence relation. It is obviously reflexive and symmetric. We will show that it preserves the operations of Ω . Let $\omega \in \Omega$ be an operation of arity n . Let $a_1, \dots, a_n, b_1, \dots, b_n \in A$ be such that $a_i \varrho^0 b_i$ for each $1 \leq i \leq n$. In particular, this means that there are identities $x_1^i \dots x_{k_i}^i p^i = x_1^i \dots x_{k_i}^i q^i$ satisfied in $\mathcal{W}$, and elements $d_1^i, \dots, d_{k_i}^i \in A$, such that $a_i = d_1^i \dots d_{k_i}^i p^i$ and $b_i = d_1^i \dots d_{k_i}^i q^i$. Note that the identity

$$(x_1^1 \dots x_{k_1}^1 p^1) \dots (x_1^n \dots x_{k_n}^n p^n) \omega = (x_1^1 \dots x_{k_1}^1 q^1) \dots (x_1^n \dots x_{k_n}^n q^n) \omega$$

is also satisfied in $\mathcal{W}$. As $a_1 \dots a_n \omega = (d_1^1 \dots d_{k_1}^1 p^1) \dots (d_1^n \dots d_{k_n}^n p^n) \omega$ and $b_1 \dots b_n \omega = (d_1^1 \dots d_{k_1}^1 q^1) \dots (d_1^n \dots d_{k_n}^n q^n) \omega$, it follows that

$$a_1 \dots a_n \omega \varrho^0 b_1 \dots b_n \omega.$$

Now by Lemma 3.1, we conclude that $\text{tr}(\varrho^0)$ is a congruence relation.

It remains to show that $A/\text{tr}(\varrho^0) \in \mathcal{W}$, and that $\text{tr}(\varrho^0)$ is the smallest congruence relation with this property. However, if $x_1 \dots x_n p = x_1 \dots x_n q$ is an identity true in $\mathcal{W}$, then

$$\begin{aligned} (a_1/\text{tr}(\varrho^0)) \dots (a_n/\text{tr}(\varrho^0)) p &= (a_1 \dots a_n p)/\text{tr}(\varrho^0) \\ &= (a_1 \dots a_n q)/\text{tr}(\varrho^0) = (a_1/\text{tr}(\varrho^0)) \dots (a_n/\text{tr}(\varrho^0)) q, \end{aligned}$$

since, by definition, $(a_1 \dots a_n p, a_1 \dots a_n q) \in \varrho^0 \subseteq \text{tr}(\varrho^0)$. Thus $A/\text{tr}(\varrho^0)$ satisfies every identity true in $\mathcal{W}$, and hence $A/\text{tr}(\varrho^0) \in \mathcal{W}$.

On the other hand, if θ is a congruence of A such that $A/\theta \in \mathcal{W}$, and $a_1, \dots, a_n \in A$, then $(a_1/\theta) \dots (a_n/\theta) p = (a_1/\theta) \dots (a_n/\theta) q$, and hence $a_1 \dots a_n p \theta a_1 \dots a_n q$. Thus $\varrho^0 \subseteq \theta$, and we have $\text{tr}(\varrho^0) \subseteq \text{tr}(\theta) = \theta$. $\square$

Theorem 3.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same similarity type without nullary operation symbols, and let $\mathcal{W}$ be idempotent. If there exist terms $f(x, y)$ and $g(x, y)$ such that*

- (a) $\mathcal{V} \models f(x, y) = x$ and $\mathcal{V} \models g(x, y) = y$,
- (b) $\mathcal{W} \models f(x, y) = g(x, y)$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof. Let A be a quotient of an algebra belonging to the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$, i.e. $A \in \mathbf{H}(\mathcal{V} \circ \mathcal{W})$. Our aim is to prove that A itself belongs to $\mathcal{V} \circ \mathcal{W}$.

Recall that the algebra A has $\mathcal{W}$ -replica A/ϱ , and by Proposition 3.2, $\varrho = \text{tr}(\varrho^0)$. First, we will show that the relation ϱ^0 is transitive, which implies that $\varrho = \varrho^0$. Let $a, b, c \in A$, and $a \varrho^0 b \varrho^0 c$. This means that there are identities $x_1 \dots x_n p_1 = x_1 \dots x_n q_1$ and $y_1 \dots y_m p_2 = y_1 \dots y_m q_2$ satisfied in $\mathcal{W}$, and elements $d_1, \dots, d_n, e_1, \dots, e_m \in A$, such that

$$\begin{aligned} a &= d_1 \dots d_n p_1, & b &= d_1 \dots d_n q_1, \\ b &= e_1 \dots e_m p_2, & c &= e_1 \dots e_m q_2. \end{aligned}$$

Now consider the identity

$$f(x_1 \dots x_n p_1, y_1 \dots y_m p_2) = g(x_1 \dots x_n q_1, y_1 \dots y_m q_2). \quad (3.1)$$

It is clear that (3.1) is satisfied in $\mathcal{W}$. On the other hand, since $\mathcal{W}$ is idempotent, it follows by Theorem 2.10 that A satisfies the identities

$$f(p_1, q_1) = p_1 \quad \text{and} \quad g(p_2, q_2) = q_2. \quad (3.2)$$

Thus

$$\begin{aligned} f(d_1 \dots d_n p_1, e_1 \dots e_m p_2) &= f(d_1 \dots d_n p_1, b) \\ &= f(d_1 \dots d_n p_1, d_1 \dots d_n q_1) = d_1 \dots d_n p_1 = a, \end{aligned}$$

and similarly

$$\begin{aligned} g(d_1 \dots d_n q_1, e_1 \dots e_m q_2) &= g(b, e_1 \dots e_m q_2) \\ &= g(e_1 \dots e_m p_2, e_1 \dots e_m q_2) = e_1 \dots e_m q_2 = c. \end{aligned}$$

Hence, for the elements $d_1, \dots, d_n, e_1, \dots, e_m$, the left-hand side of the identity (3.1) equals a and its right-hand side equals c . Consequently, $a \varrho^0 c$, whence ϱ^0 is transitive, and $\varrho = \varrho^0$.

Next, we will show that each congruence class of ϱ satisfies the identities true in $\mathcal{V}$. Since $\mathcal{W}$ is idempotent, by Lemma 1.1, all congruence classes of ϱ are subalgebras of A . Consider the identities

$$f(x, y) = x \quad \text{and} \quad g(x, y) = y, \quad (3.3)$$

true in the variety $\mathcal{V}$. If $x_1 \dots x_n p = x_1 \dots x_n q$ is satisfied in $\mathcal{W}$, then by Theorem 2.10, A satisfies the identities

$$f(p, q) = p \quad \text{and} \quad g(p, q) = q. \quad (3.4)$$

Now let $a, b \in A$ and $a \varrho b$. Since $\varrho = \varrho^0$, it follows that $a = d_1 \dots d_n p$ and $b = d_1 \dots d_n q$ for some identity $x_1 \dots x_n p = x_1 \dots x_n q$ satisfied in $\mathcal{W}$ and some $d_1, \dots, d_n \in A$. Then

$$f(a, b) = f(d_1 \dots d_n p, d_1 \dots d_n q) = d_1 \dots d_n p = a,$$

and similarly,

$$g(a, b) = g(d_1 \dots d_n p, d_1 \dots d_n q) = d_1 \dots d_n q = b.$$

Therefore $B = a/\varrho$ satisfies the identities (3.3). We will use this fact to show that B satisfies any identity $x_1 \dots x_k u = x_1 \dots x_k v$ true in $\mathcal{V}$. Define terms $t_1, \dots, t_k$ in the following way:

$$\begin{aligned} x_1 \dots x_k t_1 &= f(f(\dots f(f(f(x_1, x_2), x_3), x_4) \dots), x_{k-1}), x_k), \\ x_1 \dots x_k t_2 &= f(f(\dots f(f(g(x_1, x_2), x_3), x_4) \dots), x_{k-1}), x_k), \\ x_1 \dots x_k t_3 &= f(f(\dots f(g(f(x_1, x_2), x_3), x_4) \dots), x_{k-1}), x_k), \\ &\dots \\ x_1 \dots x_k t_k &= g(f(\dots f(f(f(x_1, x_2), x_3), x_4) \dots), x_{k-1}), x_k). \end{aligned}$$

Since $\mathcal{W}$ satisfies the identity $f(x, y) = g(x, y)$, it also satisfies the identities $t_i = t_j$, for every $1 \leq i, j \leq k$. By Theorem 2.10, it follows that A satisfies the identity

$$t_1 \dots t_k u = t_1 \dots t_k v.$$

Now let $b_1, \dots, b_k$ be elements of B . Since the identities (3.3) hold in B , it follows that $b_1 \dots b_k t_i = b_i$ for every $1 \leq i \leq k$. Thus,

$$\begin{aligned} b_1 \dots b_k u &= (b_1 \dots b_k t_1) \dots (b_1 \dots b_k t_k) u \\ &= (b_1 \dots b_k t_1) \dots (b_1 \dots b_k t_k) v = b_1 \dots b_k v. \end{aligned}$$

Consequently, B belongs to $\mathcal{V}$. □

Corollary 3.4. *Let $\mathcal{U}$ be any variety of Ω -algebras. Let $\mathcal{V}$ and $\mathcal{W}$ be subvarieties of $\mathcal{U}$ satisfying the assumptions of Theorem 3.3. Then the Mal'tsev product $\mathcal{V} \circ_{\mathcal{U}} \mathcal{W}$ is a variety.*

The main result of [4] (Theorem 1.6) follows easily as a corollary of Theorem 3.3.

Corollary 3.5 [4]. *Let $\mathcal{V}$ be a strongly irregular variety of a plural type τ and $\mathcal{S}_{\tau}$ be the variety of type τ that is equivalent to the variety of semilattices. Then $\mathcal{V} \circ \mathcal{S}_{\tau}$ is a variety.*

Proof. Let $t(x, y) = x$ be a strongly irregular identity satisfied in $\mathcal{V}$. Define terms $f(x, y) = t(x, y)$ and $g(x, y) = t(y, x)$. Since $\mathcal{S}_{\tau}$ satisfies all regular identities of type τ , it also satisfies $f(x, y) = g(x, y)$. As $\mathcal{V}$ satisfies $f(x, y) = x$ and $g(x, y) = y$, and the variety $\mathcal{S}_{\tau}$ is idempotent, it follows by Theorem 3.3 that $\mathcal{V} \circ \mathcal{S}_{\tau}$ is a variety. $\square$

Note that the satisfaction of the three identities of Theorem 3.3 implies that the variety $\mathcal{V} \cap \mathcal{W}$ is trivial. That suggests the following conjecture: If $\mathcal{V} \cap \mathcal{W}$ is trivial, then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety. The following example shows that this conjecture is false. The example was originally provided by C. Bergman, however with a different proof.

Example 3.6. Let $\mathcal{V}$ be any variety of commutative groupoids, and let $\mathcal{LZ}$ be the variety of left-zero bands. Then the intersection $\mathcal{V} \cap \mathcal{LZ}$ is the trivial variety $\mathcal{E}$.

Let A be the groupoid whose multiplication table is given below.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	2	2	2	2
3	2	3	2	3

The $\mathcal{LZ}$ -replica congruence of A is the congruence ϱ with the congruence classes $\{0, 1\}$ and $\{2, 3\}$ which are semilattices. Hence A is a member of $\mathcal{V} \circ \mathcal{LZ}$.

The groupoid A also has a congruence θ with three congruence classes $\{0, 2\}, \{1\}, \{3\}$. The quotient $B = A/\theta$ has only one proper non-trivial congruence α with the classes $\{\{0, 2\}, \{3\}\}$ and $\{\{1\}\}$. The groupoid B is neither commutative nor a left-zero band, and the quotient B/α is not a left-zero-band. Hence B is not a member of $\mathcal{V} \circ \mathcal{LZ}$.

The following corollaries show that triviality of $\mathcal{V} \cap \mathcal{W}$, combined with some types of congruence permutability, is sufficient for the existence of the two terms required by Theorem 3.3. First recall that two congruences α and β are 3-permutable (or 3-permute) if

$$\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta.$$

An algebra is 3-permutable if any two of its congruences 3-permute, and a variety is 3-permutable if all of its algebras are 3-permutable. Note also the following.

Lemma 3.7 [16, Lemma 4.66]. *Let α and β be congruences of an algebra A . Then $\alpha \circ \beta \circ \alpha \supseteq \beta \circ \alpha \circ \beta$ if and only if $\alpha \vee \beta = \alpha \circ \beta \circ \alpha$.*

We will also need one more lemma.

Lemma 3.8. *Let $\theta, \theta_1, \theta_2$ be congruences of an Ω -algebra A such that $\theta \subseteq \theta_1, \theta_2$. If the quotient A/θ is congruence 3-permutable, then the congruences θ_1 and θ_2 are 3-permutable.*

Proof. Let $a, b \in A$ and $a \theta_1 \circ \theta_2 \circ \theta_1 b$. Then

$$a/\theta \quad (\theta_1/\theta) \circ (\theta_2/\theta) \circ (\theta_1/\theta) \quad b/\theta.$$

By 3-permutability of A/θ , we also have

$$a/\theta \quad (\theta_2/\theta) \circ (\theta_1/\theta) \circ (\theta_2/\theta) \quad b/\theta.$$

So there are elements $c, d \in A$ such that

$$a/\theta \quad (\theta_2/\theta) \quad c/\theta \quad (\theta_1/\theta) \quad d/\theta \quad (\theta_2/\theta) \quad b/\theta.$$

Thus $a \theta_2 c \theta_1 d \theta_2 b$, whence finally

$$\theta_1 \circ \theta_2 \circ \theta_1 \subseteq \theta_2 \circ \theta_1 \circ \theta_2.$$

The opposite inclusion is derived analogously. $\square$

Corollary 3.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be subvarieties of a variety $\mathcal{U}$ of Ω -algebras, and let $\mathcal{W}$ be idempotent. If $\mathcal{V} \cap \mathcal{W}$ is the trivial variety and $\mathcal{U}$ is congruence 3-permutable, then $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Consider the $\mathcal{V}$ -replica congruence $\varrho_{\mathcal{V}}$, the $\mathcal{W}$ -replica congruence $\varrho_{\mathcal{W}}$ and the $\mathcal{U}$ -replica congruence $\varrho_{\mathcal{U}}$ of the free algebra $X\Omega$ for $X = \{x, y\}$. Then $\varrho_{\mathcal{U}} \subseteq \varrho_{\mathcal{V}}, \varrho_{\mathcal{W}}$. By assumption $X\Omega/\varrho_{\mathcal{U}}$ is congruence 3-permutable. Hence, by Lemma 3.8, the congruences $\varrho_{\mathcal{V}}$ and $\varrho_{\mathcal{W}}$ are 3-permutable. Thus

$$\varrho_{\mathcal{V}} \vee \varrho_{\mathcal{W}} = \varrho_{\mathcal{V}} \circ \varrho_{\mathcal{W}} \circ \varrho_{\mathcal{V}}.$$

The quotient $X\Omega/(\varrho_{\mathcal{V}} \vee \varrho_{\mathcal{W}})$ lies in $\mathcal{V} \cap \mathcal{W}$, so it is a trivial algebra. It follows that all the elements of $X\Omega$ form one congruence class. In particular,

$$x \varrho_{\mathcal{V}} \circ \varrho_{\mathcal{W}} \circ \varrho_{\mathcal{V}} y.$$

This means that there are terms $f(x, y)$ and $g(x, y)$ such that

$$x \varrho_{\mathcal{V}} f(x, y) \varrho_{\mathcal{W}} g(x, y) \varrho_{\mathcal{V}} y.$$

By definition of the congruences $\varrho_{\mathcal{V}}$ and $\varrho_{\mathcal{W}}$, the variety $\mathcal{W}$ satisfies the identity $f(x, y) = g(x, y)$ and the variety $\mathcal{V}$ satisfies the identities $f(x, y) = x$ and $g(x, y) = y$. By Theorem 3.3, $\mathcal{V} \circ \mathcal{W}$ is a variety. $\square$

In the proof of the next corollary, we need Theorem 7.3(2) of [6]. We state it below as a lemma.

Lemma 3.10. *Let A be an algebra with a congruence θ . If $P(x, y, z)$ is a term satisfying $P(y, y, x) = x = P(x, y, y)$ on each congruence class of θ , then for any other congruence ψ of A ,*

$$\theta \circ \psi \circ \theta \subseteq \psi \circ \theta \circ \psi.$$

Corollary 3.11. *Let $\mathcal{V}$ be a congruence permutable variety, and let $\mathcal{W}$ be an idempotent variety. If $\mathcal{V} \cap \mathcal{W}$ is the trivial variety, then $\mathcal{V} \circ \mathcal{W}$ is a variety.*

Proof. Let $\mathcal{U} = \mathbf{H}(\mathcal{V} \circ \mathcal{W})$. Define $X = \{x, y\}$. Suppose that congruences $\varrho_{\mathcal{V}}$, $\varrho_{\mathcal{W}}$ and $\varrho_{\mathcal{U}}$ of the free algebra $X\Omega$ are defined the same way as in the proof of Corollary 3.9. By Proposition 2.9 and Theorem 2.10, it follows that the free $\mathcal{U}$ -algebra $X\mathcal{U}$ on X belongs to $\mathcal{V} \circ \mathcal{W}$. Recall that $X\mathcal{U}$ is isomorphic to $X\Omega/\varrho_{\mathcal{U}}$. The $\mathcal{W}$ -replica of $X\mathcal{U}$ is idempotent, so the congruence classes of the $\mathcal{W}$ -replica congruence $\varrho_{\mathcal{W}}/\varrho_{\mathcal{U}}$ of $X\mathcal{U}$ are subalgebras of $X\mathcal{U}$ belonging to the congruence permutable variety $\mathcal{V}$. Thus there exists a term that is Mal'tsev on each congruence class of $\varrho_{\mathcal{W}}/\varrho_{\mathcal{U}}$. Then by Lemma 3.10,

$$\varrho_{\mathcal{W}}/\varrho_{\mathcal{U}} \circ \varrho_{\mathcal{V}}/\varrho_{\mathcal{U}} \circ \varrho_{\mathcal{W}}/\varrho_{\mathcal{U}} \subseteq \varrho_{\mathcal{V}}/\varrho_{\mathcal{U}} \circ \varrho_{\mathcal{W}}/\varrho_{\mathcal{U}} \circ \varrho_{\mathcal{V}}/\varrho_{\mathcal{U}}.$$

This implies that

$$\varrho_{\mathcal{W}} \circ \varrho_{\mathcal{V}} \circ \varrho_{\mathcal{W}} \subseteq \varrho_{\mathcal{V}} \circ \varrho_{\mathcal{W}} \circ \varrho_{\mathcal{V}}.$$

Therefore, by Lemma 3.7,

$$\varrho_{\mathcal{V}} \vee \varrho_{\mathcal{W}} = \varrho_{\mathcal{V}} \circ \varrho_{\mathcal{W}} \circ \varrho_{\mathcal{V}}.$$

The remaining part of the proof proceeds exactly as in the last part of the proof of Corollary 3.9. $\square$

4. Band sums of algebras

In this section we provide some applications of Theorem 3.3. We first consider Ω -algebras of a plural type τ with binary and possibly unary operations. Recall that a *band* $(S, \cdot)$ is an idempotent semigroup. (For properties of bands, see [8].) A band can be considered as an algebra of the same type τ by defining $xy\omega := x \cdot y$ if ω is binary, and $x\omega = x$ if ω is unary, for each $\omega \in \Omega$.

In all the following examples, $f(x, y)$ and $g(x, y)$ denote the terms one needs in order to apply Theorem 3.3.

Example 4.1 (Lattices and bands). Let $\mathcal{L}$ be any variety of lattices $(L, +, \cdot)$. Let $\mathcal{B}$ be any variety of bands considered as algebras $(B, +, \cdot)$ satisfying $x + y = x \cdot y$. Let

$$f(x, y) = x + xy \quad \text{and} \quad g(x, y) = xy + y.$$

Then the variety of all lattices satisfies $f(x, y) = x$ and $g(x, y) = y$, and the variety of all bands satisfies $f(x, y) = x \cdot xy = xy = xy \cdot y = g(x, y)$. By Theorem 3.3, it follows that $\mathcal{L} \circ \mathcal{B}$ is a variety.

A small adjustment shows that the same holds for the Mal'tsev product of the variety of Boolean algebras and a variety of bands. This time we consider Boolean algebras as algebras $(A, +, \cdot, ')$ with 0 defined by $x \cdot x' = y \cdot y'$ and 1 defined by $x + x' = y + y'$, and bands as algebras of the same type satisfying $x + y = x \cdot y$ and $x' = x$. The same terms $f(x, y)$ and $g(x, y)$ as for lattices show that the assumptions of Theorem 3.3 are satisfied.

Example 4.2 (Quasigroups and bands). We consider quasigroups as algebras $(Q, \cdot, /, \backslash)$ with three binary operations, and bands as algebras of the same type with three equal binary band operations. Let

$$f(x, y) = (x \cdot y)/y \quad \text{and} \quad g(x, y) = x \backslash (x \cdot y).$$

The variety of quasigroups satisfies $f(x, y) = x$ and $g(x, y) = y$. The variety of bands satisfies $f(x, y) = g(x, y)$. Theorem 3.3 shows that for any variety $\mathcal{Q}$ of quasigroups and any variety $\mathcal{B}$ of bands, the Mal'tsev product $\mathcal{Q} \circ \mathcal{B}$ is a variety.

Again a small adjustment shows that the Mal'tsev product of any variety of loops and any variety of bands is a variety. Along with the empty set, loops are considered here as algebras of the same type as quasigroups, with the identity 1 defined by $x/x = y \backslash y$.

As quasigroups and loops are congruence permutable, and the intersection $\mathcal{Q} \cap \mathcal{B}$ is trivial, the statements of this example may also be proved using Corollary 3.11.

Example 4.3 (Groups and bands). Groups are loops with an associative multiplication. By Example 4.2, it follows that the Mal'tsev product of the variety of groups and the variety of bands is a variety. However, instead of algebras with three binary operations, groups (along with the empty set) may be considered as algebras $(G, \cdot, {}^{-1})$ with one binary and one unary operation, where the identity 1 is defined as $x \cdot x^{-1}$. (See e.g. [4].) Bands are considered as algebras of the same type. As terms for Theorem 3.3, we take

$$f(x, y) = x \cdot y^{-1} \cdot y \quad \text{and} \quad g(x, y) = x \cdot x^{-1} \cdot y.$$

It is easy to see that these terms satisfy the assumptions of Theorem 3.3.

A similar argument shows that if $\mathcal{V}$ is a variety of rings or modules, and bands are considered as algebras of the same type as $\mathcal{V}$, then $\mathcal{V} \circ \mathcal{B}$ is a variety.

Let us note that the terms $f(x, y)$ and $g(x, y)$ of Examples 4.1, 4.2 and 4.3 have the same first and the same last variable. Then recall that bands may be considered as Ω -bands, i.e. algebras of any plural type $\tau : \Omega \rightarrow \mathbb{N}$ with binary and possibly unary operations. These observations allow us to make the following generalization.

Theorem 4.4. *Let $\mathcal{V}$ be a variety of some plural type τ with binary and possibly unary operations, and let $\mathcal{B}$ be a variety of Ω -bands of the same type. Let $f(x, y)$ and $g(x, y)$ be terms containing both variables x and y , and such that the first variable of both terms is the same and the last variable of both terms is the same. If*

$$\mathcal{V} \models f(x, y) = x \quad \text{and} \quad \mathcal{V} \models g(x, y) = y,$$

then $\mathcal{V} \circ \mathcal{B}$ is a variety.

Proof. First recall that the free band on two generators x and y consists of the following elements: x, y, xy, yx, xyx, yxy . Hence if f and g contain both variables x and y and have the same first and the same last variable, then the idempotence of bands implies that $f(x, y) = g(x, y)$. Thus the assumptions of Theorem 3.3 are satisfied. $\square$

Note that algebras in the varieties $\mathcal{V} \circ \mathcal{B}$ of Theorem 4.4 are band sums of $\mathcal{V}$ -algebras. In particular, these varieties of band sums contain as a subvariety the variety $\mathcal{V} \circ \mathcal{S}_\tau$ of semilattice sums of $\mathcal{V}$ -algebras (provided that $\mathcal{B}$ is not trivial).

As a final remark, let us note that Theorem 4.4 may be extended to varieties of any plural type. First we show that bands may be considered as algebras of any plural type. Let $(A, \cdot)$ be any band. For a given (plural) type τ , define operations of this type on A as follows. For each n -ary $\omega \in \Omega$, define the operation ω on A as

$$x_1 \dots x_n \omega := x_1 \cdot \dots \cdot x_n.$$

In this way, one obtains an algebra (A, Ω) of type τ . The semigroup operation $\cdot$ is recovered via

$$x \cdot y = xy \dots y\omega$$

for any at least binary $\omega \in \Omega$. It satisfies the associative law, and each at least binary operation ω may be obtained by composition from this binary operation. On the other hand, let (A, Ω) be an idempotent algebra of a (plural) type τ such that for any unary ω , one has $x\omega = x$, and for any two at least binary ω and ω' in Ω , one has $xy \dots y\omega = xy \dots y\omega'$. Moreover assume that this binary operation $x \cdot y := xy \dots y\omega$ is associative and that $x_1 \dots x_n \omega = x_1 \dots x_n$. In this way, one obtains a band $(A, \cdot)$. It is easy to see that the varieties of bands and Ω -bands are (definitionally) equivalent. If $(A, \cdot)$ belongs to some special variety $\mathcal{B}$ of bands, the identities defining $\mathcal{B}$ may be “translated” into the language of Ω -algebras using the binary operation $\cdot$ derived from Ω -operations. In this way one obtains the variety of Ω -bands equivalent to the variety $\mathcal{B}$.

Then in Ω -bands, any (at least binary) term reduces to a product of variables. For example, if $\Omega = \{\omega_1, \omega_2\}$ with ternary ω_1 and binary ω_2 , and t is the term $y(xz\omega_2)x\omega_1$, then in each Ω -band, $x \cdot y = xy y\omega_2 = xy\omega_1$, and $t = y \cdot x \cdot z \cdot x$. So if $f(x, y)$ and $g(x, y)$ are binary Ω -terms containing both x and y and sharing the same first and the same last variable, then in Ω -bands they must have the form described in the proof of Theorem 4.4, and the same proof shows that Theorem 4.4 remains true in the case of Ω -bands of any plural type.

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