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Finitary shadows of compact subgroups of $S(\omega)$

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Abstract. Let LF be the lattice of all subgroups of the group $SF(\omega)$ of all finitary permutations of the set of natural numbers. We consider subgroups of $SF(\omega)$ of the form $C \cap SF(\omega)$, where C is a compact subgroup of the group of all permutations. In particular, we study their distribution among elements of LF. We measure this using natural relations of orthogonality and almost containedness. We also study complexity of the corresponding families of compact subgroups of $S(\omega)$.

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1. Introduction

Let $SF(\omega)$ be the group of all finitary permutations of ω (= the set of natural numbers). This means that the elements of $SF(\omega)$ are exactly permutations $g \in S(\omega)$ with finite support. We remind the reader that $S(\omega)$ is the group of all permutations of ω and the support of $g \in S(\omega)$ is the following set $\mathrm{supp}(g) = \{x \mid g(x) \neq x\}$. The group $SF(\omega)$ appears in many parts of mathematics. We just mention the theory of locally finite groups [1] and the fact that the unique AFD von Neumann factor of type II_1 is defined as $VN(SF(\omega))$ [8]. The algebraic structure of subgroups of $SF(\omega)$ is described in [6,7].

Let LF be the lattice of all subgroups of the group $SF(\omega)$. In [5] the author studied some van Douwen invariants of LF. In the classical case these cardinals describe properties of the lattice of subsets of ω with respect to the relations of almost containedness and orthogonality associated with the ideal of finite sets (see [10]). In [5] this approach is applied to LF and the ideal IF of all finite subgroups. The appropriate invariants corresponding to $\mathfrak{a}, \mathfrak{h}, \mathfrak{p}, \mathfrak{t}, \mathfrak{r}, \mathfrak{s}$ were introduced and studied. The aim of the present paper is to apply the methods of [5] to the subsemilattice of *shadows of compact subgroups* of $S(\omega)$, i.e. groups of the form $SF(\omega) \cap C$, where C is a compact subgroup of $S(\omega)$.

It is worth noting here that the structure of compact subgroups in $S(\omega)$ is quite complicated. For example it is proved in [3] that the isomorphism relation on the family of compact subgroups of $S(\omega)$ is as complicated as the isomorphisms relation on graphs. There subgroups are considered in the natural Borel space of subgroups of $S(\omega)$, see preliminaries below. We will also use this approach.

We consider $S(\omega)$ as a complete metric space by defining

$$d(g,h) = \sum\{2^{-n} \mid g(n) \neq h(n) \text{ or } g(^{-1}(n)) \neq h(^{-1}(n))\}.$$

Let $S_{<\infty}$ denote the set of all bijections between finite substes of ω. We shall use small Greek letters δ, σ, τ to denote elements of $S_{<\infty}$. For any $\sigma \in S_{<\infty}$, let dom$[\sigma]$, rng$[\sigma]$ denote the domain and the range of σ respectively. Let

$$S^{+}_{<\infty} = \{\sigma \in S_{<\infty} \mid \text{dom}[\sigma] \text{ is an initial segment of } \omega\}.$$

Initial segments of ω will be identified with the corresponding ordinals. For every $\sigma \in S_{<\infty}$, let $\mathcal{N}_\sigma = \{f \in S_\infty \mid f \supseteq \sigma\}$. The family $\{\mathcal{N}_\sigma \mid \sigma \in S^{+}_{<\infty}\}$ is a basis of the Polish topology of $S(\omega)$. Given a subset $D \subset S(\omega)$, we define the following tree $T_D = \{\sigma \in S^{+}_{<\infty} \mid D \cap \mathcal{N}_\sigma \neq \emptyset\}$.

The *Effros structure* on $S(\omega)$ is the standard Borel space consisting of $\mathcal{F}(S(\omega))$, the set of closed subsets, together with the σ-algebra generated by the sets

$$\mathcal{C}_U = \{D \in \mathcal{F}(S(\omega)) \mid D \cap U \neq \emptyset\}, \text{where } U \text{ is open.}$$

We mention here that the set $\mathcal{U}(S(\omega))$ of all closed subgroups of $S(\omega)$ is a Borel subset of $\mathcal{F}(S(\omega))$ (see Lemma 2.5 of [3]).

Recall that throughout the paper LF denotes the lattice of all subgroups of $SF(\omega)$ and IF - the ideal of all finite subgroups. We say that G_1 and G_2 from $LF \setminus IF$ are *orthogonal* if their intersection is in IF. The group G_1 is *almost contained* in G_2 ($G_1 \leq_a G_2$) if G_1 is a subgroup of a group finitely generated over G_2 by elements of $SF(\omega)$. These relations were introduced in [5]. The van Douwen invariants for LF mentioned above were defined with respect to them.

The main results of the paper concern the structure of shadows of compact subgroups with respect to orthogonality and almost containedness. In this way, we extend several results from [5] concerning chains, antichains and reaping families. We assume that the reader knows some basic set theory, [4], [9], for example Martin's Axiom. For basic material on Polish spaces used in the paper, see [2]. Cardinal numbers which we sometimes mention can be found in [10].

2. Compact subgroups of $SF(\omega)$ and characteristics

The following description of compact subsets/subgroups of $S(\omega)$ is a well-known fact with an easy proof. It implies that the subset of $\mathcal{F}(S(\omega))$ consisting of all *compact subgroups* of $S(\omega)$ is Borel (see [3]). We will denote it by $\mathfrak{C}$.

Lemma 2.1. (1) *A closed subset $D \subset S(\omega)$ is compact if and only if the tree T_D is finite at each level.*

(2) *A subgroup $G \leq S(\omega)$ has compact closure if and only if all of its orbits on ω are finite.*

Let $\omega = \dot{\bigcup}\{Y_i \mid i \in \omega\}$ be a partition of ω into pairwise disjoint nonempty finite subsets. For every $i \in \omega$, let S_i be a subgroup of $\mathrm{Sym}(Y_i)$. We consider S_i as a subgroup of $S(\omega)$ extending its action to $\omega \setminus Y_i$ by identity. We allow that some pairs (S_i, Y_i) can be of the form $(\{id\}, \{n\})$, where $n \in \omega$. It is clear that the group $\langle \bigcup\{S_i \mid i \in \omega\}\rangle$ is a subgroup of $SF(\omega)$ and its closure in $S(\omega)$ is the compact group $\prod\{S_i \mid i \in \omega\}$ with the natural action on ω.

Lemma 2.2. (i) *Any compact subgroup of $S(\omega)$ is a closed subgroup of a group of the form $\prod\{S_i \mid i \in \omega\}$ as above.*

(ii) *Any finitary shadow of a compact subgroup of $S(\omega)$ is a subgroup of a group of the form $\langle \bigcup\{S_i \mid i \in \omega\}\rangle$ as above.*

Proof. Let G be a compact subgroup of $S(\omega)$ and let $\dot{\bigcup}\{Y_i \mid i \in \omega\}$ be a partition of ω into G-orbits. By Lemma 2.1 all Y_i are finite. Let S_i be the subgroup of $\mathrm{Sym}(Y_i)$ induced by G. Then $G \leq \prod\{S_i \mid i \in \omega\}$ and

$$SF(\omega) \cap \prod\{S_i \mid i \in \omega\} = \langle \bigcup\{S_i \mid i \in \omega\}\rangle.$$

The rest is clear. □

2.1. Between LF and $\mathfrak{C}$

The following lemma is a paraphrase of Lemma 2.4 from [3].

Lemma 2.3. *For every $X \subseteq SF(\omega)$, the following relation is Borel.*

$$\mathfrak{A}(X) = \{(Y, Z) \in \mathfrak{C}^2 \mid X \subseteq YZ\}$$

Proof. Notice that $X \subseteq YZ$ is equivalent to the Borel condition

$$(\forall \alpha \in T_X)(\exists \beta \in T_Y)(\exists \gamma \in T_Z)(\alpha = \beta \circ \gamma).$$

For the nontrivial implication, assume $f \in X$. Then to every $n \in \omega$ we can assign a pair $(\beta_n, \gamma_n) \in T_Y \times T_Z$ such that $f|_n = \beta_n \circ \gamma_n$ or equivalently a pair $(g_n, h_n) \in Y \times Z$ such that $f|_n = (g_n h_n)|_n$. Since $Y \times Z$ is compact as a product of compact sets, then passing to some converging subsequence if necessary, we can assume that (g_n, h_n) converges to some $(g, h) \in Y \times Z$ such that $\lim_{n\to\infty} g_n h_n = gh$. Therefore $f = gh$ is an element of YZ. □

For every permutation $g \in S(\omega)$, denote by $\mathcal{C}_g$ the family of all compact $C < S(\omega)$ such that $g \in C$. Note that it is a Borel set, because it can be expressed as the following intersection

$$\mathcal{C}_g = \bigcap_{n\in\mathbb{N}} \{C \in \mathfrak{C} \mid C \cap \mathcal{N}_{g|_n} \neq \emptyset\}.$$

The following proposition will be very helpful below.

Proposition 2.4. *Let $G \in LF$. The following sets are Borel.*

(1) *The set of all compact subgroups C of $S(\omega)$ such that $C \cap SF(\omega) = G$.*
(2) *The set of all compact subgroups C of $S(\omega)$ such that G is almost contained in C.*
(3) *The set of all compact subgroups C of $S(\omega)$ such that $C \cap SF(\omega)$ is almost contained in G.*
(4) *The set of all compact subgroups C of $S(\omega)$ such that G is orthogonal to C.*

(1) This is a direct consequence of the following equality:

$$\{C \mid C \cap SF(\omega) = G\} = \bigcap_{g \in G} \mathcal{C}_g \setminus \bigcup_{g \in SF(\omega) \setminus G} \mathcal{C}_g.$$

(2) For every finite $D \subset \omega$, let $S(D) = \{g \in SF(\omega) \mid \mathrm{supp}(g) \subseteq D\}$. Then, $G \leq_a C$ if and only if $G \subseteq \langle C \cup S(D) \rangle$, for some finite $D \subseteq \omega$. On the other hand, if D is a union of finitely many orbits of C, then $\langle C \cup S(D) \rangle = S(D)C$.

Summarizing the above, we have

$$G \leq_a C \;\Leftrightarrow\; \exists D \; (\; G \subseteq S(D)C \wedge (D \text{ is finite })).$$

Now we apply Lemma 2.3 to conclude that for given G, the set $\{C \mid G \leq_a C\}$ is equal to the union $\bigcup \{\mathfrak{A}(G)|_{S(D)} \mid D \text{ is a finite subset of } \omega\}$, where $\mathfrak{A}(G)|_{S(D)}$ denotes the section of $\mathfrak{A}(G)$ with respect to the first coordinate $Y = S(D)$.

(3) First, observe that for given $H < SF(\omega)$, the set below is Borel.

$$\mathfrak{C}_H = \{C \in \mathfrak{C} \mid C \cap SF(\omega) \subseteq H\}.$$

Indeed, we have

$$\mathfrak{C}_H = \mathfrak{C} \setminus \bigcup_{g \in SF(\omega) \setminus H} \mathcal{C}_g.$$

Hence the following set is Borel:

$$\{C \mid C \cap SF(\omega) \leq_a G\} = \bigcup \{\mathfrak{C}_{\langle A \cup G \rangle} \mid A < SF(\omega), A \text{ finite}\}.$$

(4) C is orthogonal to G if and only if $g \notin C$, for all but finitely many $g \in G$. Therefore

$$\{C \in \mathfrak{C} \mid C \perp G\} = \bigcup \{(\bigcap_{g \in G \setminus A} (\mathfrak{C} \setminus \mathcal{C}_g)) \mid A \subseteq G, A \text{ is finite}\}. \quad \square$$

2.2. Characteristics

It is clear that two finitary groups with finite orbits can generate a subgroup of $S(\omega)$ with infinite orbits. Thus by Lemma 2.2 (ii), finitary shadows of compact subgroups do not form a sublattice of LF. On the other hand it is obviously a meet-subsemilattice of LF. It is also worth noting that the set $LF \setminus IF$ does not have minimal elements. This follows from the following lemma.

Lemma 2.5. *Let $G \in LF \setminus IF$ and $m \in \omega$. Then:*

(i) *there exists a non-trivial $g \in G$ such that* $\mathrm{supp}(g) \cap m = \emptyset$*,*

(ii) *moreover, for any $H \subset \mathrm{Sym}(m)$ and any sequence $G_0, G_1, \ldots, G_n \in LF \setminus IF$ of groups orthogonal to G, the above g can be chosen such that additionally $\langle H, g\rangle \cap G_i = \langle H\rangle \cap G_i$, for $i \leq n$.*

The proof may be found in [5], Lemma 2.3. By statement (i) any infinite $G < SF(\omega)$ contains a sequence (g_n) of permutations with pairwise disjoint supports. Any proper subsequence of (g_n) generates a proper subgroup of G. Thus, G cannot be minimal. This supports the claim made before the lemma.

Moreover, taking appropriate powers we can additionally require that every g_i consists of cycles of the same prime length greater than 1 (which can differ for distinct elements of (g_i)). Indeed if $p_1^{\alpha_1} p_2^{\alpha_1} \ldots p_s^{\alpha_s}$ is the factorization of the order of g_i into prime factors with non-trivial α_j, $j \leq s$, then the order of g_i^k is prime, where $k = p_1^{\alpha_1 - 1} p_2^{\alpha_1} \ldots p_s^{\alpha_s}$.

Hence we may assume that the subgroup $\langle g_i\rangle$ has the closure which is contained in a group of the form described in Lemma 2.2(i), where each S_i is cyclic. Since these shadows will play the main role in our arguments we introduce a notion characterizing them.

Definition 2.6. Let (g_n) be a sequence of permutations from $SF(\omega)$ with pairwise disjoint supports such that every g_i consists of cycles of the same length greater than 1. We say that a function $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ is the *characteristic* of this sequence if for every $i > 1$, the value $f(i)$ is the number of elements of order i in the sequence and $f(1)$ is the number of points fixed by all g_i.

We say that $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ is a *potential characteristic* if it is the characteristic of some sequence as above.

Note that the characteristic of (g_n) is already determined by the subgroup $\langle g_0, g_1, \ldots\rangle$. This follows from the fact that any sequence of generators of this subgroup of the form as above is equal to some $(g_0^{k_0}, g_1^{k_1}, \ldots)$ where k_i is prime to the order of g_i. Moreover the family of potential characteristics can be described as follows.

- A function $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ is a potential characteristic if and only f has infinitely many values or there is $i > 1$ such that $f(i) = \infty$.

Definition 2.7. Let f be a potential characteristic and G be a subgroup of $S(\omega)$. We say that G *accepts* f if there is a sequence $(g_0, g_1, \ldots)$ of permutations as in Definition 2.6 such that G contains all g_i and f is the characteristic of $(g_0, g_1, \ldots)$.

Lemma 2.8. *Let $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ be a potential characteristic. Then the set $\{C \in \mathfrak{C} \mid C$ accepts $f\}$ is Borel.*

Proof. Fix an arbitrary sequence (r_n) of natural numbers such that each $i \in \omega \setminus \{0\}$ appears in (r_n) exactly $f(i)$-many times. For every $n \in \omega$, let $\mathcal{S}_n$ be the countable family of infinite sequences $(g_{n,k}) \in (SF(\omega))^\omega$ satisfying the following conditions:

(i) $(\forall k,l)(k \neq l \ \Rightarrow \ \mathrm{supp}(g_{n,k}) \cap \mathrm{supp}(g_{n,l}) = \emptyset)$;
(ii) $(\forall k < n)$(the order of $g_{n,k}$ is r_k);
(iii) $(\forall k \geq n)(g_{n,k} = id_\omega)$.

We claim that

$$\{C \in \mathfrak{C} \mid C \text{ accepts } f\} = \bigcap_{n\in\omega} \bigcup_{(g_{n,k})\in\mathcal{S}_n} \bigcap_k C_{g_{n,k}}.$$

To justify $\subseteq$ take any compact C accepting f and let (g'_k) be any sequence of finitary permutations witnessing this, i.e. the supports of (g'_k) are pairwise disjoint and for every $i > 0$ there are $f(i)$-many elements of order i. Moreover, by permuting (g'_n) if necessary, we may assume that the order g'_k is r_k. Now, for every n, define $(g_{n,k})$ follows:

$$g_{n,k} = \begin{cases} g'_k, & k \leq n, \\ id_\omega, & k > n. \end{cases}$$

Then $(g_{n,k}) \in \mathcal{S}_n$ and $C \in \bigcap_k C_{g_{n,k}}$, for every $n \in \omega$.

For $\supseteq$, suppose that

$$C \in \bigcap_{n\in\omega} \bigcup_{(g_{n,k})\in\mathcal{S}_n} \bigcap_k C_{g_{n,k}}$$

and for every $n \in \omega$ choose an arbitrary $(g_{n,k}) \in \mathcal{S}_n$ such that for each k, $g_{n,k} \in C$. Then $((g_{n,k}) \mid n \in \omega)$ is a sequence of elements C^ω. By the Tychonoff product theorem, the latter is compact. Hence the sequence $((g_{n,k}) \mid n \in \omega)$ contains a subsequence $((g_{n_i,k}) \mid i \in \omega)$ convergent to an element $(g'_k) \in C^\omega$. Then for every $k \in \omega$, there is $i_k \geq k$ such that for all $i \geq i_k$, $g_{n_i,l} = g'_l$, for all $l \leq k$. This implies that the elements of (g'_k) have pairwise disjoint supports and satisfy the condition $(\forall k)$(the order of g'_k is r_k). This proves that C accepts f. □

3. The relation of almost containedness

Let f_∞ be defined by $f_\infty(n) = \infty$, for every $n > 0$. This is the greatest function under the natural (partial) ordering of the family of all potential characteristics (see Definition 2.6).

The following theorem shows that the subsemilattice of finitary shadows of compact groups is sufficiently dense in LF under $<_a$. Furthermore, f_∞ is the only possible greatest element in the families of potential characteristics which appear in some natural way. The theorem extends Lemma 2.4 from [5].

Theorem 3.1. *For any countable sequence $G_0 > G_1 > \cdots > G_i > \cdots$ of elements of $LF \setminus IF$ there is a potential characteristic $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ and a finitary shadow G of a compact subgroup of $S(\omega)$ accepting f such that $G \leq_a G_i, i \in \omega$ and any finite subfamily of $\{G_i \mid i \in \omega\} \cup \{G\}$ has an infinite common subgroup.*

Moreover, the class of functions f satisfying the conditions of the previous paragraph for the sequence $G_0 > G_1 > \cdots > G_i > \cdots$ has the greatest element if and only if the function f_∞ lies in this class.

Proof. Let a decreasing sequence $G_0 > G_1 > \cdots > G_i > \cdots$ of elements of $LF \setminus IF$ be given. For every $i \in \omega$ choose non-trivial $g_i \in G_i$ such that $\mathrm{supp}(g_i)$ is disjoint from the supports of $g_0, g_1, \ldots, g_{i-1}$. We can do this by Lemma 2.5(i). Replacing g_i by an appropriate g_i^k, we can ensure that each g_i consists of cycles of the same length. Let G be the group generated by all these g_i. Then by Lemma 2.1(2) G is a shadow from $LF \setminus IF$ and there is f accepted by G. Moreover $G \leq_a G_i$ for every $i \in \omega$ and the condition of $\leq$ centeredness holds, i.e. the intersection of any finite subfamily of $\{G_i : i \in \omega\} \cup \{G\}$ does not belong to IF.

Assume that for any f realizing the statement of the theorem there is $k > 0$ such that $f(k) < \infty$. To show that such f is not the greatest element we apply the following procedure. For the first $f(k) + 1$ steps we define g_i to be arbitrary (pairwise disjoint) cycles of length k, and thereafter follow the above construction. The group generated by (g_i) fulfills the conditions of the theorem and accepts a characteristic which is not $\leq f$. □

Note. It is worth noting that when $f = f_\infty$ the Borel set of compact groups accepting f is dense in $\mathcal{F}(S(\omega))$. This follows from the definition of the topology on $\mathcal{F}(S(\omega))$ and the Baire category theorem. On the other hand it is also easy to see that for every $f \neq f_\infty$ this set is nowhere dense in $\mathcal{F}(S(\omega))$.

Proposition 3.2. *Let a countable sequence $G_0 > G_1 > \cdots$ of elements of $LF \setminus IF$ and a potential characteristic $f : \omega \setminus \{0\} \to \omega \cup \{\infty\}$ satisfy the terms of Theorem 3.1. Then the set of all compact groups C accepting f that fulfill the following conditions is Borel:*

(1) *$C \cap SF(\omega) \leq_a G_i, i \in \omega$ and*
(2) *any finite subfamily of $\{G_i \mid i \in \omega\} \cup \{C \cap SF(\omega)\}$ has an infinite common subgroup.*

Proof. Since $(G_i)_{i\in\omega}$ is decreasing, condition (2) above is equivalent to the following one:

(2′) $(\forall i \in \omega)(G_i \not\perp C)$.

Then the family of all compact sets satisfying conditions (1)–(2) above is equal to the following countable intersection:

$$\bigcap_{i\in\omega} (\{C \in \mathfrak{C} \mid C \cap SF(\omega) \leq_a G_i\} \setminus \{C \in \mathfrak{C} \mid C \perp G_i\}).$$

By Lemma 2.4(3,4) this is a Borel set. Now to finish the proof it suffices to intersect the latter with the family of compact groups accepting f, which is Borel by Lemma 2.8. □

4. The relation of orthogonality

In this section we consider the relation of orthogonality in $LF \setminus IF$. We will see that the groups accepting f_∞ are also distinguished in this context. The following theorem extends the statements of Theorem 2.6 of [5] concerning the reaping number.

Theorem 4.1. *Let f be a potential characteristic such that $f(1) = \infty$ and $f(p) = \infty$ for all primes p ($f(k)$ is arbitrary for other $k \in \omega$). If $\Psi \subset LF \setminus IF$ is countable, then there exists $G \in LF \setminus IF$ which is a finitary shadow of a compact subgroup of $S(\omega)$ accepting f, such that for every $G' \in \Psi$ the groups G, G' are not orthogonal and $G' \not\leq_a G$.*

Assuming Martin's Axiom and $f = f_\infty$, the statement above holds for any family $\Psi \subset LF \setminus IF$ of cardinality $< 2^\omega$.

Proof. Let $\{G_0, G_1, \ldots\}$ be an enumeration of Ψ. Assume that each member of Ψ occurs infinitely often in the enumeration. We construct two sequences of finitary permutations with pairwise disjoint supports

$$g_{0,0}, g_{0,1}, g_{1,0}, g_{1,1}, g_{2,0}, \ldots \quad \text{and} \quad h_0, h_1, \ldots$$

such that for all $i, j \in \omega$, $l \in \{0,1\}$ we have $\mathrm{supp}(g_{i,l}) \cap \mathrm{supp}(h_j) = \emptyset$, $g_{i,0}, h_i \in G_i$ and for any $k \in \omega \setminus \{0,1\}$ the set $\{g_{i,1} \mid i \in \omega\}$ contains $f(k)$ cycles of length k. It is easily seen that Lemma 2.5(i) yields the existence of such sequences. Replacing every $g_{i,0}$ by an appropriate $g^k_{i,0}$, we will ensure that the elements of $(g_{i,0})$ are products of cycles of the same prime length (not necessarily the same for distinct i). Let $\hat{G}_1 = \langle \{g_{i,l} \mid i \in \omega, l \in \{0,1\}\} \rangle$ and $\hat{G}_2 = \langle h_0, h_1, \ldots \rangle$. Then $\hat{G}_1$, $\hat{G}_2$ are orthogonal to each other but they are not orthogonal to any G_i (since each member of Ψ is enumerated infinitely often). Applying Lemma 2.1(2) it is easy to see that $G = \hat{G}_1$ satisfies the conclusion of the first part of the statement of the theorem.

To prove the second part of this statement we introduce a *ccc* forcing notion $\mathbf{P}$ as follows.

- $\mathbf{P}$ consists of all pairs (H, H') where $H, H' \subset SF(\omega)$ are finite, the supports of any two elements of $H \cup H'$ have empty intersection and each permutation from H is a tuple of cycles of the same length.
- The order is defined as follows $(H, H') \leq (F, F')$ iff $F \subseteq H$ and $F' \subseteq H'$.

Let $\Psi \subset LF \setminus IF$ have cardinality $< 2^\omega$. Applying Lemma 2.5(i) for any $k, m \in \omega \setminus \{0,1\}$ and $G' \in \Psi$ we see that the arguments of the first part of the proof show that the family

$$\begin{aligned} \{(H, H') \in \mathbf{P} \mid k < \mathrm{card}(H' \cap G'), k < \mathrm{card}(H \cap G'),& \\ H \text{ contains at least } k \text{ elements of order } m\}& \end{aligned}$$

is dense in $\mathbf{P}$. Assume **MA**. For a generic Φ define $\hat{G} = \langle \bigcup \{H \mid (H, H') \in \Phi\} \rangle$. By Lemma 2.1(2) $\hat{G}$ is a shadow of a compact subgroup of $S(\omega)$. Moreover it is easily seen that the characteristic of the generating sequence of $\hat{G}$ is f_∞ and for any $G' \in \Psi$, the groups $\hat{G}$, G' are not orthogonal and G' is not contained in $\hat{G}$ under $\leq_a$. □

Concerning the condition of the theorem that $f(p) = \infty$ for all primes p note the following observation.

Proposition 4.2. *Assume that f is a potential characteristic such that for any countable $\Psi \subset LF \setminus IF$ there exists a sequence (g_i) satisfying Definition 2.6* *for f such that for every $G' \in \Psi$ the groups $G = \langle g_0, g_1, \dots, g_i, \dots\rangle$ and G' are not orthogonal and $G' \not\leq_a G$. Then $f(p) = \infty$ for all primes p.*

Proof. Suppose that $f(p) < \infty$ for some prime $p > 1$. Consider a family Ψ containing a group H generated by a sequence of p-cycles with pairwise disjoint supports. If a group G has an infinite intersection with H, then G is not generated by $g_0, g_1, \dots, g_i, \dots$ as above. $\square$

In the following proposition we implement a descriptive approach to the conclusion of Theorem 4.1.

Proposition 4.3. *Let f satisfy the assumptions of Theorem 4.1 and let a family $\Psi \subset LF \setminus IF$ be countable. Then the set of all compact $C < S(\omega)$ accepting f such that for every $G \in \Psi$, C and G are not orthogonal and $G \not\leq_a C \cap SF(\omega)$, is Borel.*

Proof. The considered set can be expressed as the following intersection

$$\{C \in \mathfrak{C} \mid C \text{ accepts } f\} \cap \bigcap_{G\in\Psi} (\{C \in \mathfrak{C} \mid C \not\perp G\} \cap \{C \in \mathfrak{C} \mid G \not\leq_a C \cap SF(\omega)\}).$$

This set is Borel by Proposition 2.4(2,4) and Lemma 2.8. $\square$

Theorem 4.5 below extends the statements of Theorem 2.6 of [5] concerning the almost disjointness number. To prove it we need the following lemma (Lemma 2.5 from [5]). We write that G_1 is a-equivalent to G_2 whenever $G_1 \leq_a G_2$ and $G_2 \leq_a G_1$.

Lemma 4.4. *Let $G_0, \dots, G_{n-1}$ be a sequence of infinite groups from LF not a-equivalent to $SF(\omega)$. Then for any $k, m \in \omega$, $k > 0$ and $H \subseteq \mathrm{Sym}(m)$, there is a non-trivial finitary permutation ρ consisting of $(k+1)$-cycles such that $\mathrm{supp}(\rho) \subset \omega \setminus m$ and for every $i < n$,*

$$\langle H, \rho\rangle \cap G_i = \langle H\rangle \cap G_i.$$

Theorem 4.5. *Let $f\colon \omega \setminus \{0\} \to \omega \cup \{\infty\}$ be a potential characteristic with infinitely many fixed points. Let $\Psi \subset LF \setminus SF(\omega)_{IF}$ be countable. Then there is a shadow $G \in LF \setminus IF$ accepting f which is orthogonal to every element of Ψ.*

Under Martin's Axiom this is true for $f = f_\infty$ and any Ψ of cardinality $< 2^\omega$.

Proof. For the first part of this statement we construct G by induction. Fix an enumeration of Ψ: $G_0, G_1, \dots$. Let H_{n-1} be the set of the elements ρ_i constructed at the first $n-1$ steps. At the n-th step we choose a permutation ρ_n satysfying the thesis of Lemma 4.4 with respect to $G_0, \dots, G_n$, H_{n-1} and $m_{n-1} = \sup(\bigcup\{\mathrm{supp}(\rho_i) : i < n\})$. Since we can choose ρ_n of arbitrary order, we do it so that the characteristic of sequence (ρ_n) is f. Then the group $H = \langle \bigcup H_n\rangle$ is orthogonal to any group from Ψ.

To prove the second part of the statement, given an infinite set $\Delta \subset SF(\omega)$ and an infinite family $\Psi \subset LF$ of infinite groups, define a forcing notion $\mathbf{P}_{\Delta,\Psi}$ as follows. Let $\mathbf{P}_{\Delta,\Psi}$ be the set of all pairs (H, F) where F is a finite subset of Ψ and H is a finite set of permutations from Δ such that their supports are pairwise disjoint. We define $(H,F) \leq (H',F')$ iff $H' \subset H, F' \subset F$ and each $h \in \langle H\rangle \setminus \langle H'\rangle$ is not contained in any $G \in F'$. It is easily verified that $\mathbf{P}_{\Delta,\Psi}$ is a *ccc* forcing notion.

Assume **MA**. Let Δ be the set of all permutations consisting of cycles of the same length k, for all $k \in \omega \setminus \{0,1\}$. Consider $\mathbf{P}_{\Delta,\Psi}$ with respect to Δ and $\Psi \subset LF$ of cardinality $< 2^{\omega}$. It is easy to see that the following sets are dense in $\mathbf{P}_{\Delta,\Psi}$ (in the latter case apply Lemma 4.4):

- $\Sigma_G = \{(H,F) \mid G \in F\}, G \in \Psi$,
- $\Sigma_{l,k} = \{(H,F) \mid$ the number of the elements of H of order k is $> l\}$, for $l \in \omega, k \in \omega \setminus \{0,1\}$.

By **MA** we have a filter $\Phi \subset \mathbf{P}_{\Delta,\Psi}$ meeting all these Σ's. It is easy to see that the group $\hat{G} = \langle \bigcup\{H \mid (H,F) \in \Phi\}\rangle$ is orthogonal to any group from Ψ. If f is the characteristic of the generating sequence of $\hat{G}$, then $f(k) = f_{\infty}(k)$ for $1 < k$. Omitting appropriate generators of $\hat{G}$ we additionally get $f(1) = \infty$. $\square$

The following observation easily follows from Lemmas 2.4 and 2.8.

Proposition 4.6. *Let $f\colon \omega\setminus\{0\} \to \omega\cup\{\infty\}$ satisfy the assumptions of Theorem 4.5. Let $\Psi \subset LF\setminus SF(\omega)_{IF}$ be countable. Then the set of all compact $C < S(\omega)$ accepting f and orthogonal to any group from Ψ is Borel.*

The set under the claim is Borel by Proposition 2.4(4) and Lemma 2.8 since it can be expressed as the following intersection.

$$\{C \in \mathfrak{C} \mid C \text{ accepts } f\} \cap \bigcap_{G\in\Phi} \{C \in \mathfrak{C} \mid C \perp G\}. \quad \square$$

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