Algebra Univers. (2019) 80:20

1420-8911/19/020001-17
published online May 16, 2019
https://doi.org/10.1007/s00012-019-0595-3

Algebra Universalis

Barycentric algebras and beyond

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Abstract. Barycentric algebras are fundamental for modeling convex sets, semilattices, affine spaces and related structures. This paper is part of a series examining the concept of a barycentric algebra in detail. In preceding work, threshold barycentric algebras were introduced as part of an analysis of the axiomatization of convexity. In the current paper, the concept of a threshold barycentric algebra is extended to threshold affine spaces. To within equivalence, these algebras comprise barycentric algebras, commutative idempotent entropic magmas, and affine spaces, all defined over a subfield of the field of real numbers. Many properties of threshold barycentric algebras extend to threshold affine spaces.

Mathematics Subject Classification. 08A99, 52A01.

Keywords. Entropic algebra, Barycentric algebra, Convex set, Semilattice, Self-distributive, Convexity.

1. Introduction

This paper is part of a series that is devoted to extensions of the concept of barycentric algebras. Barycentric algebras form a variety generated by convex sets, when the latter are viewed as algebras equipped with the set of binary convex combinations indexed by the open real unit interval I°, subject to the hyperidentities of idempotence, skew-commutativity, and skew-associativity.

The first extension examined barycentric algebras over subfields of $\mathbb{R}$ [25, §5.8]. Next, certain subrings of $\mathbb{R}$ were brought into consideration [2,3,4], in particular the ring $\mathbb{Z}[1/2]$ of dyadic numbers [12,13,14,15], and other principal ideal subdomains of $\mathbb{R}$ [22]. In [18], it was observed that when the closed unit real interval was used to index the basic operations of a barycentric algebra, then the interval naturally assumed the structure of an LΠ-algebra, one of the

Presented by K. Kearnes.

A. Komorowski research was supported by the Warsaw University of Technology under Grant Number 504/03340/1120.

three main types of algebra in fuzzy logic [6]. Thus barycentric algebras were recognized as two-sorted algebras, where one of the two sorts is an LΠ-algebra.

More recently, threshold barycentric algebras were introduced in [9], in order to analyze the axiomatization of convexity, and for applications in mathematical biology. The open real unit interval of convex combinations is replaced by a possibly shorter symmetric subinterval containing $1/2$; the remaining operations are trivial left or right projections. Using such algebras, it was shown (in answer to a question posed by the late Klaus Keimel) that skew-associativity, which is one of the three basic hyperidentities defining barycentric algebras, cannot be replaced by entropicity in the axiomatization. It was shown that each shorter (non-trivial) subinterval generates all the operations of barycentric algebras. As a consequence, threshold barycentric algebras are equivalent either to barycentric algebras, or to commutative binary modes (algebras with a single binary commutative, idempotent and entropic operation). Threshold algebras have many interesting properties, and provide a common framework for a wide range of algebras, from the usual barycentric algebras at one end, to commutative binary modes at the other.

In this paper, the concept of a threshold algebra is extended to a yet broader spectrum of algebras, embracing affine spaces over subfields F of the field $\mathbb{R}$. Affine spaces are taken as algebras with binary affine combinations indexed by the elements of F. In threshold affine spaces, the basic set of binary operations is replaced by a set of operations indexed by a symmetrical interval J containing $1/2$, declaring the remaining operations to be trivial left or right projections. Many results concerning threshold barycentric algebras also hold for threshold affine spaces. In particular, we show that if the closed unit interval I of F is a proper subset of J, then the operations from J generate all the affine space operations. As a consequence, it transpires that threshold affine spaces over F are equivalent to one of three types of algebras: affine spaces, barycentric algebras, or commutative binary modes. This shows that threshold affine spaces provide a common framework for a larger spectrum of algebras than threshold barycentric algebras. As in the case of barycentric algebras, an analysis of the identities holding in threshold affine spaces reveals dependencies between the axioms of affine spaces, and between the axioms of affine spaces and barycentric algebras defined over the base field F.

The paper is organized as follows. Section 2 offers a short introduction to the theory of barycentric algebras and affine spaces. Section 3 provides an analysis of basic identities holding in affine spaces and barycentric algebras, showing in particular the meaning of skew-associativity for affine spaces. Relations between the axioms of affine spaces and the axioms of barycentric algebras defined on a fixed subfield of the field $\mathbb{R}$ are obtained. In Section 4, certain binary reducts of affine spaces are considered, determined by closed intervals of F of the form $[t, t' = 1 - t]$ for any $t \in F$. Up to equivalence, these intervals of operations are sufficient to describe three types of related algebras: affine spaces, barycentric algebras, and commutative binary modes. Threshold affine spaces are introduced in Section 5, where the main results concerning such algebras are proved. The final section describes meets and joins of varieties of threshold affine spaces over the field F. In particular, Theorem 6.4 provides a correction to the comparable results of [9, §12], where one case was omitted.

From this paper and [9], it is clear that all (symmetrical) intervals of F containing $1/2$ may be used to define affine spaces or barycentric algebras. However, they do not all play an equally important role in defining such algebras. The most fundamental intervals are the unit interval and the whole line F, which both carry the structure of a dual monoid with involution in the sense of [8]. Investigations of further extensions will concentrate on finding the appropriate detailed algebraic structure of such intervals, and then using this structure as a source of basic operations for further extension of the barycentric algebra concept.

Background facts concerning convex sets, barycentric algebras and affine spaces are summarized in Section 2. Readers may also consult the references at the end of this paper, and a newer survey provided in [21]. For additional information on such algebras, and modes in general, see the monographs [23, 25]. Notation and conventions generally follow those of the cited monographs and [27].

2. Modes, affine spaces and barycentric algebras

In the sense of [23,25], *modes* are defined as algebras where each element forms a singleton subalgebra, and where each operation is a homomorphism. For algebras (A, Ω) of a given type $\tau : \Omega \to \mathbb{N}$, these two properties are equivalent to satisfaction of the identity

$$x \dots x\psi = x \tag{2.1}$$

of *idempotence* for each operator ψ in Ω, and the identity

$$\begin{aligned}(x_{1,1} \dots x_{1,\psi\tau}\psi) \dots (x_{\phi\tau,1} \dots x_{\phi\tau,\psi\tau}\psi)\phi \\ = (x_{1,1} \dots x_{\phi\tau,1}\phi) \dots (x_{1,\psi\tau} \dots x_{\phi\tau,\psi\tau}\phi)\psi\end{aligned} \tag{2.2}$$

of *entropicity* for each pair ψ, ϕ of operators in Ω.

One of the main families of examples of modes is given by affine spaces over a commutative unital ring R (*affine R-spaces*), or, more generally, by subreducts (subalgebras of reducts) of affine spaces. Here, affine spaces are considered as Mal'tsev modes, as explained in the monographs [23,25]. In particular, if 2 is invertible in R, an affine R-space may be considered as the reduct $(A, \underline{R})$ of an R-module $(A, +, R)$, where $\underline{R}$ is the family of binary operations

$$\underline{r} : A^2 \to A; (x_1, x_2) \mapsto x_1x_2\underline{r} = x_1(1 - r) + x_2r$$

for each $r \in R$. The class of all affine R-spaces is a variety [1], denoted by $\underline{\underline{R}}$. If 2 is invertible in R, then the variety $\underline{\underline{R}}$ is defined by the idempotent and entropic laws, together with the *trivial laws*

$$xy\underline{0} = x, \quad xy\underline{1} = y \tag{2.3}$$

and the *affine laws*

$$[xy\underline{p}]\,[xy\underline{q}]\,\underline{r} = xy\,\underline{pqr} \tag{2.4}$$

for all $p, q, r \in R$.

An important class of subreducts of affine spaces is given by convex sets, defined as subreducts of affine $\mathbb{R}$-spaces. *Convex sets* are characterized as subsets of a real affine space closed under the operations $\underline{r}$ of weighted means taken from the open real unit interval $I^\circ =]0, 1[$. Thus a convex set contains, along with any two of its points, the line segment joining them. The class $\mathcal{C}$ of convex sets, considered as algebras $(C, \underline{I}^\circ)$, generates the variety $\mathcal{B}$ of *barycentric algebras*, and forms a subquasivariety of $\mathcal{B}$ [17].

The definitions of convex sets and barycentric algebras may be readily extended to the case of a subfield F of $\mathbb{R}$, with its own unit interval $I^\circ = \{s \in F \mid 0 < s < 1\}$ [25, Chapters 5, 7]. For $p, q \in I^\circ$, set $p' := 1 - p$, and define the *dual product* $p \circ q := (p'q')' = p + q - pq$. Then the variety $\mathcal{B}$ of barycentric algebras over F is defined by the identities

$$xx\,\underline{p} = x \tag{2.5}$$

of *idempotence* for each p in I°, the identities

$$xy\,\underline{p} = yx\,\underline{1-p} \tag{2.6}$$

of *skew-commutativity* for each p in I°, and the identities

$$[xy\,\underline{p}]\ z\,\underline{q} = x\ [yz\,\underline{q/(p \circ q)}]\ \underline{p \circ q} \tag{2.7}$$

of *skew-associativity* for each p, q in I° [25, Section 5.8]. It is worthy of note that idempotence, skew-commutativity, and skew-associativity may all be construed as hyperidentities of algebras $(A, \underline{I}^\circ)$ (in the sense of [16]).

As reducts of affine spaces, barycentric algebras are entropic. They comprise convex sets, so-called *stammered semilattices* where $p = q$ for all $p, q \in I^\circ$, and certain sums of convex sets over (stammered) semilattices.

Barycentric algebras may also be axiomatized as *extended barycentric algebras* $(A, \underline{I})$, where I is the closed unit interval $\{s \in F \mid 0 \leq s \leq 1\}$ of F, with the operations $\underline{0}$ and $\underline{1}$ defined by

$$xy\underline{0} = x \quad \text{and} \quad xy\underline{1} = y \tag{2.8}$$

as respective left and right projections. Note that skew-associativity in the form (2.7) is then no longer a hyperidentity of $(A, \underline{I})$, since $0 \circ 0 = 0$, and $q/(p \circ q)$ is not defined for $p = q = 0$. The class $\overline{\mathcal{B}}$ of extended barycentric algebras is a variety, specified by the identities (2.5)–(2.7) defining $\mathcal{B}$ together with the additional identities (2.8). For more information about barycentric algebras, see [5, 7, 18, 19, 22, 24, 26, 28, 29].

3. Affine spaces and barycentric operations

We now extend the concept of a barycentric algebra by enlarging the set of basic operations, (possibly) weakening the algebraic structure of that set, while retaining as many key properties of barycentric algebras as possible. Since skew-associativity is essential to the definition of barycentric algebras [9], we will discuss this identity in the more general setting of affine spaces.

We first introduce some operations defined on any field F, recalling that division is not an operation in the sense of universal algebra. For $p, q \in F$, a binary operation of *implication* is defined by

$$p \to q = \begin{cases} 1 & \text{if } p = 0; \\ q/p & \text{otherwise.} \end{cases} \tag{3.1}$$

When F is the two-element field $\mathsf{GF}(2) = \{0, 1\}$, the implication (3.1) becomes the usual Boolean implication, while the dual product $p \circ q$ becomes the usual Boolean disjunction.

Consider the operation

$$\triangleright\colon F \times F \to F; (p, q) \mapsto (p \circ q \to q) \tag{3.2}$$

of (*dual*) *division* on the field F. Note that

$$p \triangleright q = p \circ q \to q = \begin{cases} 1 & \text{if } p \circ q = 0; \\ q/(p \circ q) = q/(p'q')' & \text{otherwise.} \end{cases}$$

Another operation of interest is

$$\triangleleft\colon F \times F \to F; (p, q) \mapsto (p' \circ q' \to p'q). \tag{3.3}$$

Note that

$$p \triangleleft q = (p' \circ q' \to p'q) = (pq)' \to p'q = \begin{cases} 1 & \text{if } pq = 1; \\ p'q/(pq)' & \text{otherwise.} \end{cases}$$

If F is a subfield of $\mathbb{R}$, and I° is the open unit interval of F, then for $p, q \in I^\circ$ one has $q < p \circ q$, whence $q/(p \circ q) \in I^\circ$ and the skew-associativity may be written as

$$xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{p \circ q \to q}\;\underline{p \circ q}. \tag{3.4}$$

or

$$xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{p \triangleright q}\;\underline{p \circ q}. \tag{3.5}$$

This form of skew-associativity will be called *right skew-associativity*. As shown in [18], the identity also holds for all p, q in the closed unit interval $I = [0, 1]$ of F. In fact, it may be observed directly that the original skew-associativity (2.7) holds if at least one of p, q is 1, and if precisely one of p, q is 0. The only critical value of p and q is $p = q = 0$. The identities (2.5), (2.6), and (3.4), together with (2.8), provide an axiomatization for extended barycentric algebras.

Another form of skew-associativity for $p, q \in I^\circ$ is given by the identity

$$x\;yz\underline{p}\,\underline{q} = xy\underline{[(q - pq)/(1 - pq)]}\;z\,\underline{pq}\,, \tag{3.6}$$

which may also be written as

$$x\;yz\underline{p}\,\underline{q} = xy\underline{[(1 - pq) \to (q - pq)]}\;z\,\underline{pq} \tag{3.7}$$

or

$$x\;yz\underline{p}\,\underline{q} = xy\underline{(p \triangleleft q)}\;z\,\underline{pq}, \tag{3.8}$$

using (3.3). This form of skew-associativity will be described as *left skew-associativity.* As in the case of right skew-associativity, it may be verified that left skew-associativity holds for all s, t in the closed interval I. While for right skew-associativity the only critical values of p, q were $p = q = 0$, in the present case they are $p = q = 1$.

Lemma 3.1. *Let A be a nontrivial affine F-space over a subfield F of $\mathbb{R}$, and let $p, q \in F$.*

(a) *A satisfies the right skew-associativity (3.5) iff $p \circ q \neq 0$ or $p = q = 0$.*
(b) *A satisfies the left skew-associativity (3.8) iff $pq \neq 1$ or $p = q = 1$.*

Proof. If $p \circ q \neq 0$, then the proof of the right skew-associativity (3.5) proceeds as in the usual case $p, q \in I^\circ$ for barycentric algebras [25, §5.8]. If $p = q = 0$, then the proof proceeds as discussed above for extended barycentric algebras [18].

Conversely, suppose $p \circ q = 0$ and $(p, q) \neq (0, 0)$. Note that $q = 1$ implies $q' = 0$ and $p \circ q = (p'q')' = 0' = 1$, so $q \neq 1$ under the current assumptions, and $p = q/(q-1)$. Since A is nontrivial, it contains distinct points a and b. Then

$$[a\, b\underline{p}]\, a\underline{q} = [a\, b\underline{(q/(q-1))}]\, a\underline{q} = ab\underline{(-q)}\,.$$

On the other hand, $(p \circ q \to q) = (0 \to q) = 1$ and

$$a\,[b\, a\underline{(p \circ q \to q)}]\, \underline{p \circ q} = a\,[b\, a\underline{(0 \to q)}]\, \underline{0} = a\,[b\, a\, \underline{1}]\, \underline{0} = a\, a\, \underline{0} = a\,.$$

Since $q \neq 0$, we have $ab\underline{q} \neq a$. It follows that the right skew-associativity does not hold in this case. The treatment of the left skew-associativity is dual. ☐

The concluding observations of this section are concerned with other relations between axioms of barycentric algebras over F and affine F-spaces.

Lemma 3.2. *The affine identities 2.4, together with the trivial identities 2.3 of affine F-spaces, imply skew-commutativity for all $p \in F$.*

Proof. By the trivial and affine identities, $yx\underline{p'} = [xy\underline{1}]\, [xy\underline{0}]\, \underline{p'} = xy\, \underline{1 0 \underline{p'}} = xy\underline{p}$. ☐

Lemma 3.3. *If a skew-associativity is defined for some p and r in the field F, then it implies the affine law $[xy\underline{p}]\, [xy\underline{q}]\, \underline{r} = xy\, \underline{pqr}$ for the same p, r and any $q \in F$.*

Proof. In the skew-associative law of the form $[xy\, \underline{p}]\, z\, \underline{r} = x\, [yz\, \underline{r/(p \circ r)}]\, \underline{p \circ r}$, substitute $xy\underline{q}$ for z. Then

$$\begin{aligned} [xy\, \underline{p}][xy\, \underline{q}]\, \underline{r} &= x\, \Big[y[xy\, \underline{q}]\, \underline{r/(p \circ r)}\Big]\, \underline{p \circ r} \\ &= x(1 - p + pr - qr) + y(p - pr + qr) = xy\underline{p - pr + qr} = xy\, \underline{pqr} \end{aligned}$$

as required. ☐

Corollary 3.4. *Within extended barycentric algebras, the trivial and affine identities of affine F-spaces also hold for operators $p, q, r \in I$.*

4. Binary reducts of affine F-spaces

In this section, we consider affine F-spaces over a fixed subfield F of $\mathbb{R}$, with the open and closed unit intervals of F denoted respectively by I° and I.

Recall that a convex subset C of an affine F-space A is just a subreduct of A with respect to the operations belonging to $\underline{I}^\circ$. We will replace the interval I° by an open interval $]q, q'[$, where q is any member of F not exceeding $1/2$, and $q' = 1 - q$. Then $\underline{]q, q'[}$ will denote the set of operations $\{\underline{r} \mid r \in]q, q'[\}$.

Definition 4.1. Let $q \in F$ with $q \leq 1/2$. A subalgebra $\big(C, \underline{]q, q'[}\big)$ of the reduct $\big(A, \underline{]q, q'[}\big)$ of an affine F-space $(A, \underline{F})$ is called *q-convex*.

The subreducts of a given type of algebras in a given (quasi)variety form a quasivariety [11, §11]. In particular, the class $\mathcal{C}_q$ of all q-convex subsets of affine F-spaces is a quasivariety.

Definition 4.2. Let $q \in F$ with $q \leq 1/2$. Then the variety $\mathcal{B}_q$ generated by the quasivariety $\mathcal{C}_q$ is called the variety of *q-barycentric algebras*.

Note that $\mathcal{C}_0$ is the quasivariety $\mathcal{C}$ of usual convex sets, and $\mathcal{B}_0$ is the variety $\mathcal{B}$ of barycentric algebras. A special relationship between the quasivarieties $\mathcal{C}_q$ and the varieties $\mathcal{B}_q$ emerges from consequences of the following general facts.

Let R be a subring of the ring $\mathbb{R}$. Recall that, under the operations of $\underline{R}$, the affine R-space R^k (for $k \in \mathbb{N}$) is the free algebra, in the variety $\underline{\underline{R}}$ of affine R-spaces, over a finite set $X = \{x_0, \ldots, x_k\}$ of free generators. The free algebra is

$$\left\{ x_0 r_0 + \cdots + x_k r_k \;\middle|\; r_i \in R, \sum_{i=0}^{k} r_i = 1 \right\}.$$

[25, §6.3]. In particular, the line R is the free affine R-space on two free generators $x_0 = 0$ and $x_1 = 1$. Recall the following.

Proposition 4.3 [20]. *Let R be a commutative, unital ring. Let Ω be a set of affine combinations over R. Let ΩR be the quasivariety of Ω-subreducts of affine R-spaces. Let J be the free ΩR-algebra on two generators.*

(a) *The free ΩR-algebra $X\Omega R$ over a set X is isomorphic to the Ω-subreduct, generated by X, of the free affine R-space XR.*

(b) *One has*

$$X\Omega R = \left\{ x_0 a_0 + \ldots + x_n a_n \;\middle|\; a_i \in J, \sum_{i=0}^{n} a_i = 1 \right\}$$

for $X = \{x_0, \ldots, x_n\}$.

Finite-dimensional simplices are free barycentric algebras. Furthermore, the quasivariety $\mathcal{C}_q$ and the variety $\mathcal{B}_q$ have the same free algebras [11, §13].

Corollary 4.4. *The free $\mathcal{B}_q$-algebra over X is isomorphic to the $\underline{]q, q'[}$-subreduct, generated by X, of the free affine F-space XF over X.*

Lemma 4.5. *Suppose $t \in F$ and $-\infty < t < 0$. Consider the line F under the operations of $\underline{[t,t']}$. Then the interval $[(t')^{k-1}t, (t')^k]$ is contained within the subalgebra A of $\big(F, \underline{[t,t']}\big)$ generated by $\{0,1\}$.*

Proof. The proof is by induction on k. First note that $01\underline{t} = t$ and $01\underline{t'} = t'$. If $t \le r \le t'$, then $t \le 01\underline{r} \le t'$. Hence $[t,t'] \subseteq A$. Now note that $0t'\underline{t} = t't$ and $0t'\underline{t'} = (t')^2$, and if $t \le r \le t'$, then $t't \subseteq 0t'\underline{r} \subseteq (t')^2$. Hence $[t't, (t')^2] \subseteq A$. Note that $t't < t < t' < (t')^2$, and more generally

$$\cdots < (t')^{k-1}t < \cdots < t't < t < t' < (t')^2 < \cdots < (t')^k < \cdots .$$

Now assume that $[(t')^{k-1}t, (t')^k] \subseteq A$. Then, similarly as before, $0(t')^k\underline{t} = (t')^k t$ and $0(t')^k\underline{t'} = (t')^{k+1}$, which implies that $[(t')^k t, (t')^{k+1}] \subseteq A$. □

Proposition 4.6. *Let $t \in F$ and $-\infty < t < 0$. Then under the operations of $\underline{[t,t']}$, the line F is generated by $\{0,1\}$.*

Proof. It suffices to note that $F = \bigcup_{k=1}^{\infty}[(t')^{k-1}t, (t')^k]$. □

Theorem 4.7. *Let $t \in F$ and $-\infty < t < 0$. Let n be a positive integer. Then under the operations of $\underline{[t,t']}$, each F^n is generated by the free generators of the affine F-space F^n.*

Proof. The inductive proof is similar to the proof of [9, Thm. 8.5], with simplices Δ_n replaced by the affine F-spaces F^n, and the extreme points of the simplices replaced by the free generators of the space F^n. For $n = 1$, the theorem follows by Proposition 4.6. Now suppose that the result is true for a positive dimension n. Consider the affine F-space F^{n+1} with free generators $x_0, x_1, \dots, x_{n+1}$. It consists of all affine combinations of a subspace F^n generated by $n+1$ free generators of F^{n+1}, say by $x_0, x_1, \dots, x_n$, and the generator x_{n+1}. Thus an arbitrary point x, which is not on a hyperplane Π_n parallel to F^n, lies on a line ℓ going through a point p of F^n and the point x_{n+1}. By Proposition 4.6, the point x is generated under the operations of $\underline{[t,t']}$ by p and the generator x_{n+1}. If a point x belongs to the hyperplane Π_n, then it lies on a line ℓ' going through a point y generated by some p of F^n and x_{n+1}, and a point q of F^n. As before, the line is generated by y and q. By induction, the points p and q are generated under the operations of $\underline{[t,t']}$ by free generators of F^n, which of course are also free generators of F^{n+1}. Thus x is generated under the operations of $\underline{[t,t']}$ by the free generators of F^{n+1}. □

Corollary 4.8. *Suppose $q \in F$ with $-\infty < q < 0$. Then the variety $\mathcal{B}_q$ of* q-barycentric algebras *and the variety $\underline{\underline{F}}$ of affine F-spaces are equivalent.*

Proof. Certainly, the $]q,q'[$-reducts of affine F-spaces are $\mathcal{B}_q$-algebras.

Now consider the variety $\mathcal{B}_q$ of q-barycentric algebras. By Proposition 4.6, the operations of $\underline{[t,t']}$, for each t with $-\infty < q < t < 0$, generate all the operations of $\underline{F}$, which are all the binary operations of affine F-spaces (and also generate the Mal'tsev operation). As the variety $\mathcal{B}_q$ is generated by subalgebras of the reducts $(A, \underline{]q,q'[})$ of affine F-spaces $(A, \underline{F})$, it follows that (under the derived operations) these subalgebras are in fact also affine F-spaces. □

Remark 4.9. Note that the variety $\mathcal{B}_q$ is defined by the affine space identities applied to the binary operations derived from the convex combinations of $\underline{]q, q'[}$.

The following theorem shows that the classes of subreducts $(C, \underline{]q, q'[})$ of affine F-spaces $(A, \underline{F})$ determined by subintervals $]q, q'[$ of F symmetric with respect to $1/2$ generate precisely three types of varieties.

Theorem 4.10. *Let $q \in F$ with $q \leq 1/2$. The following conditions hold:*

(a) *The variety $\mathcal{B}_q$ is equivalent to the variety $\mathcal{CBM}$ of commutative binary modes for $q = 1/2$;*

(b) *The variety $\mathcal{B}_q$ is equivalent to the variety $\mathcal{B}$ of barycentric algebras for $0 \leq q < 1/2$;*

(c) *The variety $\mathcal{B}_q$ is equivalent to the variety $\underline{\underline{F}}$ of affine F-spaces for $q < 0$.*

Proof. The first two conditions follow by results of [9, §9]. If $0 \leq q < 1/2$, then the variety $\mathcal{B}_q$ coincides with the variety $\mathcal{B}^t_{\text{mod}}$ of t-moderate barycentric algebras of [9, §9], generated by $\underline{[t, t']}$-reducts of convex sets, and hence is equivalent to the variety $\mathcal{B}$ [9, Prop. 9.3]. The third condition follows by Corollary 4.8. □

Remark 4.11. Note that F is the union of all the intervals $]q, q'[$ such that $\mathcal{B}_q$ is equivalent to $\underline{\underline{F}}$, while I° is the union of all the intervals $]q, q'[$ such that $\mathcal{B}_q$ is equivalent to $\overline{\overline{\mathcal{B}}}$. In fact, two varieties $\mathcal{B}_{q_1}$ and $\mathcal{B}_{q_2}$ are equivalent precisely if either $]q_1, q_1'[,]q_2, q_2'[\supset I^\circ$ or $]q_1, q_1'[,]q_2, q_2'[\subseteq I^\circ$.

5. Threshold algebras

Threshold barycentric algebras were introduced by the authors in [9] as a tool to analyze the axiomatization of barycentric algebras. We first recall the definition. Take a fixed element t of the interval $[0, 1/2]$ of $\mathbb{R}$ known as the *threshold*. Then the full open real interval I° of binary barycentric operations is replaced by the reduced set of barycentric operations indexed by the subinterval $[t, 1-t] \cap I^\circ$ of I°, the so-called *moderate* operations, together with so-called *extreme* operations: left projections indexed by elements of the interval $]0, t[$, and right projections indexed by elements of the interval $]1-t, 1[$. Such operations are called *threshold-t barycentric operations*, and barycentric algebras under these operations are called *threshold-t barycentric algebras*.

Threshold barycentric algebras offer an entire spectrum of algebras, ranging from the usual barycentric algebras at one end (where $t = 0$) to the (extended) commutative binary modes at the other (for $t = 1/2$). Theorem 8.5 of [9] shows that for a threshold $0 < t < 1/2$, finite-dimensional simplices (free barycentric algebras) are also generated by their vertices under the basic threshold-t barycentric operations. This implies that such threshold-t barycentric algebras are equivalent to extended barycentric algebras.

In this section, we will extend the concepts of a threshold and threshold barycentric algebras to threshold affine spaces. (The same could be done for the q-barycentric algebras of Definition 4.2. But since they are equivalent to affine spaces anyway, it will suffice to consider affine spaces alone). First note that the definitions and results of [9] remain true in the case of subfields of the field $\mathbb{R}$. As in the previous section, F will always denote a fixed subfield of $\mathbb{R}$, and I° and I will denote the respective open and closed unit intervals of F. The following definition generalizes the concept of threshold convex set.

Definition 5.1. Set a threshold t, where $t = -\infty$ or $t \in F$ with $t \leq 1/2$.

(a) For elements x, y of an affine F-space A, define

$$xy\,\underline{\underline{r}} = \begin{cases} x & \text{if } r < t; \\ xy\,\underline{r} = x(1-r) + yr & \text{if } t \leq r \leq 1-t; \\ y & \text{if } r > 1-t \end{cases} \tag{5.1}$$

for $r \in F$. Then the binary operations $\underline{\underline{r}}$ are described as *threshold-t affine combinations*.

(b) A threshold-t affine combination $\underline{\underline{r}}$ is respectively defined to be *small*, *moderate*, or *large* when r lies in the members $]-\infty, t[$, $[t, 1-t]$, or $]1-t, \infty[$ of the partition $\{]-\infty, t[, [t, 1-t],]1-t, \infty[\}$ of F. Together, small and large threshold-t affine combinations are described as *extreme*.

(c) For a given threshold t, the algebra $(A, \underline{\underline{F}})$, where $\underline{\underline{F}} = \{\underline{\underline{r}} \mid r \in F\}$, is called a *threshold-t affine F-space*.

The following proposition holds as in the case of threshold barycentric algebras [9, §4].

Proposition 5.2. *Let t be a threshold. Let A be an affine F-space. Then under the threshold-t affine combinations $\underline{\underline{r}}$ for $r \in F$, the threshold-t affine F-space $(A, \underline{\underline{F}})$ is a mode satisfying skew-commutativity.*

Definition 5.3. For a given threshold t, the class $\mathcal{A}^t$ of *threshold-t affine F-spaces* is the variety generated by the class of affine F-spaces under the threshold-t affine combinations of Definition 5.1.

By Definition 5.1, if the threshold t equals $-\infty$, there are no extreme operations, and the threshold-t affine combinations are just the usual affine operations. In particular, the variety $\mathcal{A} = \mathcal{A}^{-\infty}$ of threshold-$(-\infty)$ affine F-spaces is precisely the variety $\underline{\underline{F}}$ of affine F-spaces.

Other special cases are obtained if $t \in [0, 1/2]$. If $t = 1/2$, then the variety $\mathcal{A}^{1/2}$ of threshold-$(1/2)$ affine F-spaces is equivalent to the variety $\mathcal{B}^{1/2}$ of threshold-$(1/2)$ barycentric algebras, and hence to the variety $\overline{\mathcal{CBM}}$ of extended commutative binary modes [9, §7].

If $0 < t < 1/2$, then the variety $\mathcal{A}^t$ of threshold-t affine F-spaces is equivalent to the variety $\mathcal{B}^t$ of threshold-t barycentric algebras, and hence to the variety $\overline{\mathcal{B}}$ of extended barycentric algebras [9, §6]. The variety $\mathcal{A}^0$ is also equivalent to the variety $\overline{\mathcal{B}}$ of extended barycentric algebras.

These observations may be summarized in the following classification theorem.

Theorem 5.4. *Each variety of threshold affine F-spaces is equivalent to one of the following classes:*

(a) *the variety $\underline{\underline{F}}$ of affine F-spaces;*
(b) *the variety $\overline{\mathcal{B}}$ of extended barycentric algebras; or*
(c) *the variety $\overline{\mathcal{CBM}}$ of extended commutative binary modes.*

Remark 5.5. Note that the addition of extreme operations to barycentric algebras or affine spaces has no influence on their structure and basic algebraic properties. Nevertheless, it has a major influence on their axiomatization, and on the varieties they form. Recall that the *regular identities* that are satisfied in an algebra or a class of algebras are those having the same set of variables on each side. Adding extreme operations satisfying irregular identities to the moderate operations as basic operations may violate the satisfaction of regular identities true in affine spaces or barycentric algebras. For example, as was shown in [9], skew-associativity holds in threshold-t barycentric algebras only if $t = 0$. Similarly, one may easily find instances of $p, q, r \in F$ such that the affine law (2.4), satisfied in all affine F-spaces, does not hold in a threshold-t affine F-space.

Example 5.6. Consider F as a threshold-0 affine space $(F, \underline{\underline{F}})$. Let $p = 1/2$, $q = 2$ and $r = 1/2$. Then $pq\underline{r} = (1/2)(2)\underline{\underline{1/2}} = 5/4$. For $x = 0, y = 1$ in F, we have $(01)\,\underline{\underline{pq\underline{r}}} = (01)\underline{\underline{5/4}} = 1$. On the other hand $[(01)\,\underline{\underline{p}}][(01)\,\underline{\underline{q}}]\,\underline{\underline{r}} = [(01)\,\underline{\underline{(1/2)}}]\,[(01)\underline{\underline{(2)}}]\,\underline{\underline{1/2}} = 1/4 + 1/2 = 3/4$.

The *regularization* of a given variety is the variety that is defined by the regular identities holding in the given variety. The regularization of a regular variety is that same variety. Since the variety $\mathcal{B} = \mathcal{B}^0$ of barycentric algebras is regular, its regularization $\widetilde{\mathcal{B}}$ coincides with $\mathcal{B}$. On the other hand, if $t \neq 0$, then threshold-t barycentric algebras form a (strongly) irregular variety $\mathcal{B}^t$. Its regularization $\widetilde{\mathcal{B}^t}$, consisting of Płonka sums of $\mathcal{B}^t$-algebras, does not coincide with $\mathcal{B}^t$.

Note that the variety $\mathcal{A} = \underline{\underline{F}}$ may also be defined as the variety of skew-commutative modes, of the type of $\underline{\underline{F}}$-algebras, satisfying the trivial identities (2.3), the affine identities (2.4), and the binary Mal'tsev identities

$$x[xy\underline{2^{-1}}]\underline{2} = y = x[yx\underline{2^{-1}}]\underline{2}. \tag{5.2}$$

Definition 5.7. For $t \in F$ with $t \leq 1/2$, let $\mathcal{A}^t_{\mathrm{mod}}$ be the variety generated by the reducts $\big(A, \underline{\underline{[t, t']}}\big)$ of threshold-t affine F-spaces with respect to moderate threshold combinations. Members of $\mathcal{A}^t_{\mathrm{mod}}$ are said to be *t-moderate affine F-spaces.*

The variety $\mathcal{A}^0_{\mathrm{mod}}$ coincides with the variety $\overline{\mathcal{B}}$ of extended barycentric algebras, and $\mathcal{A}^{1/2}_{\mathrm{mod}}$ coincides with the variety $\mathcal{CBM}$ of commutative binary modes. By Theorem 4.3, the free $\mathcal{A}^t_{\mathrm{mod}}$-algebra over X is isomorphic to the $\underline{\underline{[t, t']}}$-subreduct, generated by X, of the free affine F-space XF. Moreover if $t > 0$, then by [9, Thm. 8.1], the free $\mathcal{A}^t_{\mathrm{mod}}$-algebra is equivalent to the free barycentric algebra XB over X, and if $t < 0$, then by Theorem 4.7, it is equivalent to the free affine F-space XF over X. The following corollary extends Theorem 9.5 of [9].

Corollary 5.8. *Let $-\infty < t < 1/2$. Then the variety $\mathcal{A}^t$ of threshold-t affine F-spaces is defined by the following identities:*

(a) *Idempotence, skew-commutativity and entropicity for all operations of $\underline{\underline{F}}$;*
(b) *The identity $xy\,\underline{\underline{r}} = x$ for each small operation $\underline{\underline{r}}$;*
(c) *The identity $xy\,\underline{\underline{r}} = y$ for each large operation $\underline{\underline{r}}$; and*
(d) *Skew-associativity for the (derived) binary operations that are generated by the moderate operations in the case when $t > 0$, and the trivial, affine and binary Mal'tsev identities for (derived) binary operations generated by the moderate operations in the case $t < 0$.*

Remark 5.9. Both threshold barycentric algebras and threshold affine spaces fall into the following more general scheme, which may be interesting in its own right, even though we currently have no further examples.

Consider a variety $\mathcal{V}$ of Ω-algebras such that Ω is the disjoint union $\Omega_1 \cup \Omega_2$ of Ω_1 and Ω_2, and, by composition, the Ω_2-operations generate the Ω_1-operations. Let $\mathcal{W}$ be the variety generated by the Ω_2-subreducts of $\mathcal{V}$-algebras. We additionally assume that free algebras in both varieties generated by the same set of generators are the same. Then the varieties $\mathcal{V}$ and $\mathcal{W}$ are equivalent. Now replace each (n-ary) operation ω in Ω_1 by some trivial operation $\underline{\omega}$, where $x_1 \dots x_n \underline{\omega} = x_i$ for a fixed i depending on ω. Denote the set of such operations $\underline{\omega}$ by $\underline{\Omega_1}$. Let $\underline{\Omega}$ be the disjoint sum of $\underline{\Omega_1}$ and Ω_2. Call the resulting algebras $(A, \underline{\Omega})$ *threshold* $\mathcal{V}$-algebras, and denote the variety they form by $\mathcal{V}^T$. Then, as in the case of threshold barycentric algebras and affine spaces:

(a) Each $\mathcal{V}^T$-algebra $(A, \underline{\Omega})$ is equivalent to the algebra $(A, \Omega, \underline{\Omega}_1)$ obtained from (A, Ω) by adding trivial operations $\underline{\Omega_1}$.
(b) The variety $\mathcal{V}^T$ is defined by the trivial identities that are satisfied by the operations $\underline{\Omega}_1$, and all the $\mathcal{V}$-identities satisfied by the derived operations generated by the operations of Ω_2.

6. Varieties of threshold algebras

Meets and joins of varieties $\mathcal{A}^t$ of threshold affine F-spaces may be described using methods similar to those employed for threshold barycentric algebras in [9] (noting the correction of [9, Th. 12.6] originally provided in [10], and discussed below following the proof of Theorem 6.4).

Recall that for $0 \le t < 1/2$, the varieties $\mathcal{A}^t$ are equivalent to the variety $\overline{\mathcal{B}}$ of extended barycentric algebras, and for $-\infty < t < 0$, they are equivalent to the variety $\mathcal{A}$ of affine F-spaces. If $0 < t < 1/2$, then the variety $\mathcal{A}^t$ contains as a unique non-trivial subvariety the variety $\mathcal{S}^t$ of extended semilattices, where all operations from $\underline{\underline{[t, t']}}$ are equal and associative, and the extreme operations reduce to one left and one right projection. For $t < 0$, the varieties $\mathcal{A}^t$ have no non-trivial subvarieties. The proof of the following is similar to the proof of [9, Th. 12.3].

Proposition 6.1. *For distinct thresholds s, t, the meet $\mathcal{B}^s \wedge \mathcal{B}^t$ is the variety $\mathcal{T}$ of trivial algebras.*

We now consider joins of varieties $\mathcal{A}^t$. First, we will define the varieties $\mathcal{SC}^t$, in a similar way as in [9, §11], but this time for all elements t of the field F. This means that $\mathcal{SC}^t$ is the variety, of the type of $\underline{\underline{F}}$-algebras, defined by the identities of idempotence, skew-commutativity and entropicity for all operations of $\underline{\underline{F}}$, along with left-zero identities for all operations $\underline{\underline{p}}$ with $p < t$ and right-zero identities for all operations $\underline{\underline{p}}$ with $t' < p$. Each variety $\mathcal{A}^t$ is a subvariety of $\mathcal{SC}^t$. In a variety $\mathcal{SC}^t$, each word may be reduced, using the left-zero and right-zero identities and skew-commutativity, to its reduced form w^t without extreme operations, and containing only moderate operations $\underline{\underline{r}}$ belonging to $\underline{\underline{[t, t']}}$. It follows that each identity $w = v$ satisfied in $\mathcal{A}^t$ may be written in its reduced form $w^t = v^t$, only containing symbols of moderate operations, and satisfied in the variety $\mathcal{A}^t_{\text{mod}}$.

Definition 6.2. Let $s, t \in F \cup \{-\infty\}$. Set thresholds $-\infty \leq s < t \leq 1/2$. Let $\mathcal{A}^{s,t}$ be the variety of idempotent, entropic, skew-commutative algebras, of the same type as $\underline{\underline{F}}$-algebras, defined by the following identities:

(1) $xy\,\underline{\underline{p}} = x$ for all $p < s$;
(2) $xy\,\underline{\underline{p}} = y$ for all $p > s'$;
(3) all identities true in the variety $\mathcal{A}^t_{\text{mod}}$ of Definition 5.7.

Let $Id_{s,t}$ be the set of identities $w = v$ that hold in $\mathcal{A}^s_{\text{mod}}$, with all operation symbols $\underline{\underline{p}}$ belonging to $\underline{\underline{[s, t[}}$ or $\underline{\underline{]t', s']}}$. Let $\mathcal{A}_{s,t}$ be the subvariety of $\mathcal{A}^{s,t}$ defined by all the identities $w = v$ of $Id_{s,t}$, such that in $\mathcal{A}^t$ both sides of the identity are equal to the same variable.

Remark 6.3. Note that there are identities in $Id_{s,t}$ which are not satisfied in the variety $\mathcal{A}^t$. Examples are provided by some of the skew-associativity laws (2.7) satisfied in $\mathcal{A}^s$. If $s < p, q < t$ and $p \circ q$, $q/(p \circ q) \in \,]t', s'[$, then the identity (2.7) is satisfied in $\mathcal{A}^s$, while in $\mathcal{A}^t$

$$[xy\,\underline{\underline{p}}]\; z\,\underline{\underline{q}} = x \quad \text{ and } \quad x\;[yz\,\underline{\underline{q/(p\circ q)}}]\;\underline{\underline{p\circ q}} = z.$$

Take for example $s = 1/8 < p = q = 3/8 < 31/64 = t$. Then $p \circ q = 39/64$ and $q/(p \circ q) = 8/13$ both belong to $]33/64, 7/8[\, = \,]t', s'[$.

Theorem 6.4. *Let $s, t \in F \cup \{-\infty\}$. If $-\infty \leq s < t \leq 1/2$, then the join $\mathcal{A}^s \vee \mathcal{A}^t$ of the varieties $\mathcal{A}^s$ and $\mathcal{A}^t$ is equal to the variety $\mathcal{A}_{s,t}$.*

Proof. For $r < 1/2$, each $\mathcal{A}^r$-algebra satisfies the identities $xy\,\underline{\underline{p}} = x$ for all small operations $\underline{\underline{p}}$, the identities $xy\,\underline{\underline{p}} = y$ for all large operations $\underline{\underline{p}}$, and all identities true in $\bar{\mathcal{A}}^u_{\text{mod}}$, for $u \geq r$, that only involve moderate operations. Since $s < t$, it follows by Definition 6.2 that any identity true in $\mathcal{A}^{s,t}$ is satisfied in both the varieties $\mathcal{A}^s$ and $\mathcal{A}^t$, and the same holds for the identities defining the subvariety $\mathcal{A}_{s,t}$. Hence each identity true in $\mathcal{A}_{s,t}$ holds in $\mathcal{A}^s \vee \mathcal{A}^t$. Consequently, $\mathcal{A}^s \vee \mathcal{A}^t \leq \mathcal{A}_{s,t}$.

Conversely, we will show that each identity true in both $\mathcal{A}^s$ and $\mathcal{A}^t$ (and hence in $\mathcal{A}^s \vee \mathcal{A}^t$) is also satisfied in $\mathcal{A}_{s,t}$. First note that all left-zero and all right-zero identities true in $\mathcal{A}^s$ also hold in $\mathcal{A}^s \vee \mathcal{A}^t$, and in $\mathcal{A}_{s,t}$.

Now let

$$w = v \tag{6.1}$$

be an identity satisfied in $\mathcal{A}^s \vee \mathcal{A}^t$ containing some operation symbols $\underline{\underline{p}}$ for $s \leq p \leq s'$.

Suppose that all the operation symbols appearing in (6.1) belong to $\underline{\underline{[t, t']}}$. Then the identity is satisfied by all the $\mathcal{A}^t$-algebras, and hence by all the $\mathcal{A}^t_{\text{mod}}$-algebras. Consequently, it holds in all $\mathcal{A}^{s,t}$-algebras, and hence in all $\mathcal{A}_{s,t}$-algebras.

Now let (6.1) be an identity, true in the variety $\mathcal{A}^s$, that contains both extreme and moderate operations in the type of $\mathcal{A}^s$-algebras. Then by [9, §11], the identity is equivalent to the identity $w^s = v^s$ true in $\mathcal{A}^s_{\text{mod}}$ containing only operation symbols from $\underline{\underline{[s, s']}}$.

Now assume that (6.1) contains operation symbols $\underline{\underline{p}}$ only in the range $s \leq p \leq s'$, with some of the $\underline{\underline{p}}$ in the set $\underline{\underline{[s, t[}} \cup \underline{\underline{]t', s']}}$. The identity also holds in the variety $\mathcal{A}^t$ precisely in two cases: either all its operation symbols belong to $\underline{\underline{[s, t[}} \cup \underline{\underline{]t', s']}}$, and then both sides are equal to the same variable, or there are operation symbols in (6.1) belonging to $\underline{\underline{[t, t']}}$, and then the identity reduces to the identity $w^t = v^t$ true in $\mathcal{A}^t_{\text{mod}}$. It follows that the identities true in both $\mathcal{A}^s$ and $\mathcal{A}^t$ satisfy the conditions of Definition 6.2. Hence they hold in $\mathcal{A}_{s,t}$, and $\mathcal{A}_{s,t} \leq \mathcal{A}^s \vee \mathcal{A}^t$. □

Note that a small change in Definition 6.2 and Theorem 6.4 will provide a correction to Theorem 12.6 of [9], where one case was lost [10]. It is sufficient to assume that $F = \mathbb{R}$ and $0 \leq s < t \leq 1/2$. Recall that in this case the varieties $\mathcal{A}^t$ and $\mathcal{B}^t$ are equivalent.

Note that the variety $\mathcal{A}_{s,t}$ is a proper subvariety of the variety $\mathcal{A}^{s,t}$. This is shown by Example 6.5 below. First observe that the algebra $(F, \underline{\underline{F}})$, with appropriately defined operations, may be considered as a member of each of the varieties $\mathcal{A}^t$ and $\mathcal{A}^{s,t}$. As a member F^t of $\mathcal{A}^t$, it satisfies the identities that define $\mathcal{A}^t$, and as a member $F^{s,t}$ of $\mathcal{A}^{s,t}$, it satisfies the identities that define $\mathcal{A}^{s,t}$.

Example 6.5. Let $-\infty < s < 1/5$ and $2/5 < t < 1/2$. Let $p = 1/4$ and $q = 1/5$. Then $p \circ q = 2/5$ and $q/(p \circ q) = 1/2$. Since $s < p, q, p \circ q < t$, it follows that the variety $\mathcal{A}^s$ satisfies skew-associativity for $p = 1/4$ and $q = 1/5$. On the other hand, the same identity holds in $\mathcal{A}^t$, since in this case both of its sides are equal to x. It follows that the identity holds in $\mathcal{A}_{s,t}$.

Now consider the algebra $F^{s,t}$ but satisfying additionally the following conditions: $xy\underline{\underline{1/5}} = y$ and $xy\underline{\underline{2/5}} = x$, and moreover $xy\underline{\underline{4/5}} = x$ and $xy\underline{\underline{3/5}} = y$. It is easy to see that this algebra is a member of $\mathcal{A}^{s,t}$. However, it does not belong to $\mathcal{A}_{s,t}$. The left-hand side of skew-associativity for $p = 1/4$ and $q = 1/5$ equals z, whereas the right-hand side equals x.

The following example indicates some relations between the variety $\mathcal{A} = \mathcal{A}^{-\infty} = \underline{\underline{F}}$ of affine F-spaces and the variety $\mathcal{A}^0$ equivalent to the variety $\overline{\mathcal{B}}$ of extended barycentric algebras.

Example 6.6. Note that the varieties $\mathcal{A}$, $\mathcal{A}^0$, $\mathcal{S}^0$, the trivial variety $\mathcal{T}$ and the variety $\mathcal{A}_{-\infty,0}$ form a lattice isomorphic to N_5. The variety $\mathcal{A}_{-\infty,0}$ satisfies all identities of $\mathcal{A}^0_{\mathrm{mod}}$, in particular all the identities defining the variety $\mathcal{B}$ of barycentric algebras (since affine spaces satisfy all of them), then all the identities true in affine F-spaces which in $\mathcal{A}^0$-algebras have both sides equal to the same variable. Among them are some (but not all) skew-associative identities, and some affine identities containing only operation symbols $\underline{\underline{p}}$ for $p < 0$ or $p > 1$. No binary Mal'tsev identities are satisfied in $\mathcal{A}_{-\infty,0}$.

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Received: 18 September 2018.
Accepted: 26 March 2019.