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Exponential lower bounds of lattice counts by vertical sum and 2-sum

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Abstract. We consider the problem of finding lower bounds on the number of unlabeled n-element lattices in some lattice family. We show that if the family is closed under vertical sum, exponential lower bounds can be obtained from vertical sums of small lattices whose numbers are known. We demonstrate this approach by establishing that the number of modular lattices is at least 2.2726^n for n large enough. We also present an analogous method for finding lower bounds on the number of vertically *indecomposable* lattices in some family. For this purpose we define a new kind of sum, the *vertical 2-sum*, which combines lattices at two common elements. As an application we prove that the numbers of vertically indecomposable modular and semimodular lattices are at least 2.1562^n and 2.6797^n for n large enough.

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1. Introduction

One of the most elementary questions regarding a family of combinatorial objects is: how many are they? For various lattice families this question has been approached in two ways. Small lattices have been generated by computation, and counted exactly. Numbers of large lattices have been lower and upper bounded by assorted methods.

The purpose of this note is to demonstrate that in some lattice families, useful exponential lower bounds are obtained from vertical compositions of small lattices, which have been counted by computation. By exponential we mean c^n, where c is a constant and n is the number of elements.

Presented by P. Jipsen.

We consider two kinds of vertical composition. First we show how the ordinary *vertical sum* leads to exponential lower bounds in families that are closed under vertical sum. As an application, we establish that the number of unlabeled modular lattices is at least 2.2726^n for n large enough. This improves upon the previous bound 2^{n-3} by Jipsen and Lawless [4]. Our bound is derived from the counts of vertically indecomposable modular lattices of $n \leq 30$ elements, computed by the author in [7]. Further computations are likely to yield improved lower bounds.

Secondly we target the numbers of vertically *indecomposable* lattices, which may be more interesting. To this end we define the *vertical 2-sum*, and show that it yields exponential lower bounds on vertically indecomposable lattices. As an application, we establish that the numbers of unlabeled, vertically indecomposable modular and semimodular lattices are at least 2.1562^n and 2.6797^n for n large enough.

2. Vertical sum

All lattices in this work are finite, nonempty and unlabeled. If L and U are lattices, their *vertical sum* $L + U$ is defined by identifying the top element of L with the bottom element of U. The vertical sum is associative, and the vertical sum of several lattices is defined in the obvious way. In fact, lattices with vertical sum are a monoid, with the singleton lattice as its neutral element. For completeness we define the empty vertical sum to be the singleton.

A lattice X is *vertically decomposable* if it contains a *knot*, that is, an element distinct from top and bottom and comparable to all elements. One can then decompose X at the knot into two non-singleton lattices L and U, whose vertical sum is X. A lattice that has no knot is *vertically indecomposable*, or a *vi-lattice*. It is well known that every finite nonempty lattice has a unique vertical decomposition, that is, a representation as a vertical sum of non-singleton vi-lattices [1].

Remark 2.1. The literature is varied on whether the singleton lattice is defined as vertically indecomposable. In any case it needs some special treatment to ensure that vertical decompositions are unique. Erné *et al.* [1] define the singleton to be vertically *decomposable*. Although this feels odd, it is analogous to the now standard practice of excluding 1 from primes to make prime factorization unique. Some other authors tacitly include the singleton among vi-lattices [3,4]. We define it as a vi-lattice but exclude it explicitly when necessary.

Notation 2.2. We will generally write $f(n)$ for the number of n-element lattices in some family, and $f_{\mathrm{vi}}(n)$ for the corresponding number of vi-lattices. For the numbers of modular lattices and modular vi-lattices, we write $m(n)$ and $m_{\mathrm{vi}}(n)$. For semimodulars we write $s(n)$ and $s_{\mathrm{vi}}(n)$.

Vertical sum and decomposition have become standard tools in lattice counting, due to the following observation (cf. [3, Equation 1]).

Lemma 2.3. *Let $\mathcal{F}$ be a lattice family that is closed under vertical sum and vertical decomposition, and contains the singleton lattice. Let $f(n)$ and $f_{\mathrm{vi}}(n)$ be the numbers of n-element lattices and vi-lattices in $\mathcal{F}$, respectively. Then f and f_{vi} are related by*

$$f(n) = \sum_{k=2}^{n} f_{\mathrm{vi}}(k)\, f(n-k+1), \quad \textit{for} \quad n \geq 2. \tag{2.1}$$

Proof. Let $n \geq 2$. Each n-element lattice $X \in \mathcal{F}$ can be uniquely represented as a vertical sum $X = L+U$, where L is vertically indecomposable and $|L| \geq 2$. Because $\mathcal{F}$ is closed under vertical decomposition, we have $L, U \in \mathcal{F}$. Note that if X is vertically indecomposable, we still have $X = L+U$, where $L = X$, and U is the singleton.

The sum (2.1) counts such vertical sums, with k iterating over the possible cardinalities of L. For each value of k, there are $f_{\mathrm{vi}}(k)$ choices for L, and $f(n-k+1)$ choices for U. In the boundary case $k = n$ we have $f(n-k+1) = f(1) = 1$ as we assumed that the singleton is in $\mathcal{F}$. Also, each such vertical sum gives a lattice in $\mathcal{F}$, because $\mathcal{F}$ is closed under vertical sum. □

Modular, semimodular, distributive, and graded lattices are examples of families where Lemma 2.3 applies. It is well known that (2.1) can be used to reduce the workload when counting small lattices by exhaustive generation. The idea is to generate only the vi-lattices in $\mathcal{F}$ up to some maximum size N, thus obtaining the values $f_{\mathrm{vi}}(2), \ldots, f_{\mathrm{vi}}(N)$, and then to calculate $f(2), \ldots, f(N)$ by the recurrence. This method has been used with various lattice families [1,2,3,4,7].

We must point out that Lemma 2.3 requires the family to be closed *both* under vertical sum *and* under vertical decomposition. Being closed under vertical sum is not enough: as a counterexample, consider the family "graded lattices of even rank". It contains lattices such as the 5-element chain that are not accounted for by the sum (2.1), as their vi-components fall out of the family. But being closed under vertical sum suffices for the *inequality*

$$f(n) \geq \sum_{k=2}^{n} f_{\mathrm{vi}}(k)\, f(n-k+1), \quad \text{for} \quad n \geq 2, \tag{2.2}$$

which is enough for proving lower bounds on $f(n)$.

We now proceed to demonstrate that besides exact counting of small lattices, vertical sums are also useful for exponential lower bounds on $f(n)$, that is, bounds of the form

$$f(n) \geq c^n$$

with some constant c. The simplest way is to take vertical sums of constant-size lattices; we begin with this method to illustrate its ease. (But we will prove stronger bounds later.)

Theorem 2.4. *Let $\mathcal{F}$ be a lattice family that is closed under vertical sum, and contains the 2-element chain. Let $f(n)$ be the number of n-element lattices in $\mathcal{F}$, and $N \geq 2$ an integer constant. Then $f(n) \geq \Omega(c^n)$, where $c = f(N)^{1/(N-1)}$.*

Proof. Let c be as stated. We first prove the case when $n = (N-1)h+1$, where $h \geq 1$ is an integer. Consider h-tuples $(L_1, L_2, \ldots, L_h)$ of N-element lattices in $\mathcal{F}$. There are $f(N)^h$ such tuples; each gives rise to a vertical sum

$$L_1 + L_2 + \cdots + L_h = X,$$

which is a lattice of $n = (N-1)h+1$ elements and belongs to $\mathcal{F}$ by assumption. Different tuples give rise to different lattices, because for each such X there is only one way of breaking X into a vertical sum of h components of N elements each. Thus the number of n-element lattices in $\mathcal{F}$ is lower bounded by the number of the tuples:

$$f(n) \geq f(N)^h = f(N)^{(n-1)/(N-1)} = c^{n-1}.$$

For arbitrary $n \geq N$ we round n down to the nearest value where the previous case applies. More precisely, let n' be the largest integer of the form $n' = (N-1)h+1$ such that $n' \leq n$ and $h \geq 1$ is an integer. Note that $n' \geq n - N + 2$. Because $\mathcal{F}$ contains the 2-element chain, f is nondecreasing (any n-element lattice can be extended to $n+1$ elements by adding the 2-element chain on top). Thus

$$f(n) \geq f(n') \geq c^{n'-1} \geq bc^n,$$

where $b = c^{1-N}$ is a constant. This holds for all $n \geq N$, so $f(n) \geq \Omega(c^n)$. □

Corollary 2.5. $m(n) \geq \Omega(2.1332^n)$.

Proof. Apply Theorem 2.4 with $N = 30$ and $m(30) = 3\,485\,707\,007$ [7]. □

Corollary 2.6. $s(n) \geq \Omega(2.5080^n)$.

Proof. Apply Theorem 2.4 with $N = 25$ and $s(25) = 3\,838\,581\,926$ [7]. □

Stronger lower bounds are obtained by applying the recurrence (2.2). Let $N \geq 2$ be a constant, and suppose that $f_{\mathrm{vi}}(1), f_{\mathrm{vi}}(2), \ldots, f_{\mathrm{vi}}(N)$ are known. Then we can lower bound $f(n)$ by a constant-coefficient recursive sequence as follows.

Theorem 2.7. *Let $\mathcal{F}$ be a lattice family closed under vertical sum, containing the singleton. Let $f(n)$ and $f_{\mathrm{vi}}(n)$ be the numbers of n-element lattices and vi-lattices in $\mathcal{F}$. Let $\underline{f}\colon \mathbb{N}^+ \to \mathbb{N}^+$ be the sequence defined by $\underline{f}(1) = 1$,*

$$\underline{f}(n) = \sum_{k=2}^{n} f_{\mathrm{vi}}(k)\, \underline{f}(n-k+1) \tag{2.3}$$

when $n = 2, 3, \ldots, N$, and

$$\underline{f}(n) = \sum_{k=2}^{N} f_{\mathrm{vi}}(k)\, \underline{f}(n-k+1) \tag{2.4}$$

when $n \geq N+1$. Then $f(n) \geq \underline{f}(n)$ for all $n \geq 1$. Furthermore, the infinite sequence $\underline{f}$ is determined by $f_{\mathrm{vi}}(1), f_{\mathrm{vi}}(2), \ldots, f_{\mathrm{vi}}(N)$ through a homogeneous linear recurrence relation of order $N-1$.

Proof. For $n = 1$, we have $f(n) = \underline{f}(n) = 1$. For $n = 2, 3, \ldots, N$, the claim $f(n) \geq \underline{f}(n)$ holds by (2.2). For $n \geq N+1$ it holds because the right hand side of (2.4) is a truncated form of the right hand side of (2.2).

Let us substitute $i = k-1$ and write $f_{\mathrm{vi}}(i+1) = a_i$ to emphasize that these are known constants. The recurrence 2.4 now becomes

$$\underline{f}(n) = \sum_{i=1}^{N-1} a_i \, \underline{f}(n-i). \tag{2.5}$$

This is a homogeneous linear recurrence relation of order $N-1$ with constant coefficients. The values of $f_{\mathrm{vi}}(1), f_{\mathrm{vi}}(2), \ldots, f_{\mathrm{vi}}(N)$ determine both the initial values of $\underline{f}(n)$ up to $n = N$, and the coefficients of the recurrence. Thus they also determine the whole sequence $\underline{f}$. $\square$

The remaining task is to find an exponential lower bound for $\underline{f}(n)$. A standard method for solving recurrence relations such as (2.5) begins by finding the roots of the auxiliary equation

$$x^{N-1} = \sum_{i=1}^{N-1} a_i \, x^{N-1-i}. \tag{2.6}$$

We refer to [9, § 7.7] for details. Let r be the root of (2.6) whose absolute value is the largest. If r is a single root, then a solution to the recurrence (2.5) is of the form $br^n + o(r^n)$, where b is a constant. Generally we will have to find the roots numerically. In order to obtain a rigorous lower bound, one which is not subject to floating point errors, we will choose c slightly smaller than r, and then prove directly that $\underline{f}(n) \geq c^n$ for n large enough.

Proposition 2.8. *$m(n) \geq 2.2726^n$ for all n large enough.*

Proof. Let $N = 30$, and define $\underline{f}$ as in Theorem 2.7, with the values of $f_{\mathrm{vi}}(1), f_{\mathrm{vi}}(2), \ldots, f_{\mathrm{vi}}(30)$ taken from the "modular vi" column of [7, Table 1]. By Theorem 2.7 we have $m(n) \geq \underline{f}(n)$ for all $n \geq 1$.

The auxiliary equation (2.6) is now

$$\begin{aligned} x^{29} = {} & x^{28} + x^{26} + x^{25} + 2x^{24} + 3x^{23} + 7x^{22} + 12x^{21} + 28x^{20} + 54x^{19} \\ & + 127x^{18} + 266x^{17} + 614x^{16} + 1356x^{15} + 3134x^{14} + 7091x^{13} \\ & + 16482x^{12} + 37929x^{11} + 88622x^{10} + 206295x^{9} + 484445x^{8} + 1136897x^{7} \\ & + 2682451x^{6} + 6333249x^{5} + 15005945x^{4} + 35595805x^{3} + 84649515x^{2} \\ & + 201560350x + 480845007. \end{aligned}$$

Numerically we find that the root with the largest absolute value is a single real root $r \approx 2.272651$. For a lower bound, we take $c = 2.2726$ and claim that $\underline{f}(n) \geq c^n$ for $n \geq 150\,000$. We prove this by induction. Applying (2.4) recursively, we see that the claim holds for $150\,000 \leq n \leq 150\,028$, which serves as the base case. We then observe that if $\underline{f}(k) \geq 2.2726^k$ for 29 consecutive values from $k = n - 29$ to $k = n - 1$, then by applying these inequalities in (2.4) we have $\underline{f}(n) \geq 2.2726^n$. This completes the induction. $\square$

The new bound improves upon the bound $m(n) \geq 2^{n-3}$ by Jipsen and Lawless [4], but still falls short of the empirical growth rate. The ratios $\frac{m(n)}{m(n-1)}$ and $\frac{m_{\mathrm{vi}}(n)}{m_{\mathrm{vi}}(n-1)}$ for $n \leq 30$ look like $m(n)$ and $m_{\mathrm{vi}}(n)$ are growing roughly as 2.4^n [7]. If the values of m_{vi} are computed further, Proposition 2.8 is likely to yield improved lower bounds. For example, if further computations reveal that $m_{\mathrm{vi}}(31) \geq 2.35 m_{\mathrm{vi}}(30)$, which seems likely, then the constant c in our lower bound will increase by about 0.0060.

For semimodular lattices, no previous lower bound seems to be known, other than that of modulars. Using the values of $s_{\mathrm{vi}}(n)$ for $n \leq 25$ from [7], Theorem 2.7 yields a lower bound $s(n) \geq 2.6459^n$ for n large enough. We omit the details because the bound is superseded by a stronger lower bound on semimodular vi-lattices in the next section. However, even the stronger bound is only exponential. We note that the ratios of the consecutive values $s_{\mathrm{vi}}(22), s_{\mathrm{vi}}(23), s_{\mathrm{vi}}(24)$, and $s_{\mathrm{vi}}(25)$ are 3.5082, 3.5579 and 3.6057 [7]. Since the ratios are steadily increasing, we suspect that the growth of $s(n)$ may be faster than exponential.

We can try applying Theorem 2.7 to other lattice families. For distributive lattices, using the data for $n \leq 49$ by Erné et al. [1], we get a lower bound of 1.8388^n, which does not improve upon their results. For graded lattices, using the data for $n \leq 21$ by the author [7], we get a lower bound of 3.4015^n, but this is not really useful, because it is already known that their growth is faster than exponential. From Klotz and Lucht [6] and Kleitman and Winston [5] we have lower and upper bounds of the form $c^{n^{3/2}+o(n^{3/2})}$ both for graded lattices and for all lattices.

Let us conclude this section with a brief qualitative comparison. From subset relations between families, we have

$$d(n) \leq m(n) \leq s(n) \leq g(n) \leq \ell(n),$$

where $d(n)$, $g(n)$, and $\ell(n)$ are the numbers of distributive lattices, graded lattices, and all lattices of n elements. For $d(n)$, exponential lower *and* upper bounds are known [1]. For $m(n)$ we have an exponential lower bound, and the empirical growth seems exponential, but an exponential upper bound is lacking; the only known upper bound on (semi)modulars seems to be that of all lattices [4]. For $s(n)$ we have an exponential lower bound, but empirically the growth seems faster. The growths of $g(n)$ and $\ell(n)$ are *known* to be faster than exponential. It remains a topic of further study to better separate the growth rates of different lattice families.

3. Vertical 2-sum

We now turn our attention to the numbers of vertically indecomposable lattices. Our method is similar to the previous section: arbitrarily large lattices are constructed from smaller lattices, whose number is known.

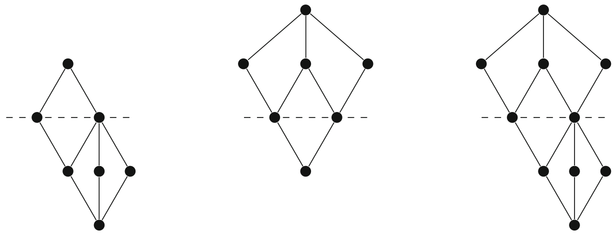

FIGURE 1. Two semimodular lattices (left and center) and their vertical 2-sum, which is also semimodular (right)

Let L and U be lattices such that L has two coatoms and U has two atoms. Then a *vertical 2-sum* of L and U is a poset obtained by removing $\hat{1}_L$ (the top of L) and $\hat{0}_U$ (the bottom of U), and identifying the coatoms of L with the atoms of U. The operation is illustrated in Figure 1.

Remark 3.1. The choice of which coatom is identified with which atom may give rise to two nonisomorphic vertical 2-sums, but we will not delve further into that issue here. For our purposes it suffices that for any L and U there is at least one vertical 2-sum, which we denote by $L +_2 U$, by a slight abuse of notation. Vertical 2-sums of several lattices can be defined in the obvious way by associativity.

Remark 3.2. $L +_2 U$ has $|L| + |U| - 4$ elements.

Lemma 3.3. *A vertical 2-sum of two lattices is a lattice.*

Proof. Let $V = L +_2 U$, and write for brevity $L' = L \backslash \hat{1}_L$ and $U' = U \backslash \hat{0}_U$. Furthermore let a, b be the two common elements of L' and U'. We claim that every pair of distinct elements $s, t \in V$ has a least upper bound. We consider three cases.

(1) Case $s, t \in U'$. The claim holds because U is a lattice.

(2) Case $s \in L'$ and $t \in U'$. If $s \leq t$, the claim is clear. Otherwise, without loss of generality, let $s \leq a \not\leq t$ and $s \not\leq b \leq t$. Now $w = a \vee t$ is an upper bound of s and t. If $u \in U'$ is an upper bound of s and t, we must have $a \leq u$ (because $s \not\leq b$), thus u is an upper bound of $a \vee t = w$. So w is the least upper bound of s and t in V.

(3) Case $s, t \in L'$. Let w be their least upper bound in L. If $w = \hat{1}_L$, then $a \vee b$ is the least upper bound of s, t in V. Now suppose $w \neq \hat{1}_L$, and let u be any upper bound of s and t. If $u \in L'$, we have $w \leq u$ because L is a lattice. Let then $u \in U'$. If $a, b \leq u$, then $w \leq u$. Otherwise, without loss of generality, let $a \leq u$ and $b \not\leq u$. Because $s, t \leq u$, we have $s, t \leq a$. Then $w \leq a \leq u$. Thus w is the least upper bound of s, t in V.

We have shown that V is a join-semilattice. Since it has a bottom element $\hat{0} = \hat{0}_L$, it is also a lattice. $\square$

Remark 3.4. A vertical 2-sum of two vi-lattices is a vi-lattice, and a vertical 2-sum of two graded lattices is graded.

Lemma 3.5. *A vertical 2-sum of two semimodular lattices is semimodular.*

Proof. Let L, U be semimodular, $V = L +_2 U$, and $L' = L \setminus \hat{1}_L$ and $U' = U \setminus \hat{0}_U$. Let $s, t \in V$ such that $s, t \succ (s \wedge t)$. Then either $s, t \in U'$ or $s, t \in L' \setminus U'$. In the first case, $s, t \prec (s \vee t)$ because U is semimodular. In the second case, $s, t \prec (s \vee t)$ because L is semimodular. □

Lemma 3.6. *A vertical 2-sum of two modular lattices is modular.*

Proof. Apply Lemma 3.5 to both the vertical 2-sum and its dual. □

For families of graded vi-lattices, vertical 2-sum leads to a recurrence analogous to Lemma 2.3. Let us first define the building blocks that we are going to use. If X is a graded lattice, we say that two elements of X are a *neck* if (1) they have the same rank, (2) they are the only elements having that rank, and (3) they are not atoms or coatoms. We say that a graded vi-lattice is a *piece* if it has two atoms, two coatoms and no neck, and its rank is at least three. It follows that a piece has at least six elements.

We can now state the recurrence. For simplicity we state it as a lower bound only; in particular, this implies that we need not separate the cases where there are two nonisomorphic vertical 2-sums.

Theorem 3.7. *Let $\mathcal{F}$ be a family of graded vi-lattices that is closed under vertical 2-sum. Let $f_{\mathrm{vi}}(n)$ and $f_{\mathrm{pc}}(n)$ be the numbers of n-element lattices and pieces in $\mathcal{F}$, respectively. Let $N \geq 6$ be an integer constant, and let $\underline{f} \colon \mathbb{N}^+ \to \mathbb{N}^+$ be the sequence defined by*

$$\underline{f}(n) = f_{\mathrm{pc}}(n)$$

when $1 \leq n \leq 6$,

$$\underline{f}(n) = f_{\mathrm{pc}}(n) + \sum_{k=6}^{n-1} f_{\mathrm{pc}}(k)\underline{f}(n-k+4), \tag{3.1}$$

when $7 \leq n \leq N$, and

$$\underline{f}(n) = \sum_{k=6}^{N} f_{\mathrm{pc}}(k)\underline{f}(n-k+4) \tag{3.2}$$

when $n \geq N+1$. Then $f_{\mathrm{vi}}(n) \geq \underline{f}(n)$ for all $n \geq 1$.

Proof. We prove by induction a stronger claim: that $\mathcal{F}$ contains at least $\underline{f}(n)$ lattices of n elements that have two coatoms and two atoms. For $n \leq 6$ the claim is clear.

Let then $n \geq 7$. For each k such that $6 \leq k \leq n-1$, there are $f_{\mathrm{pc}}(k)$ ways to choose a k-element piece $L \in \mathcal{F}$ and, by the induction assumption, at least $\underline{f}(n-k+4)$ ways to choose an $(n-k+4)$-element lattice $U \in \mathcal{F}$ that has two coatoms and two atoms. For each choice of L and U, if $X = L +_2 U$, then X is in $\mathcal{F}$, has two coatoms and two atoms, and has n elements.

We claim that different choices of the pair (L, U) cannot yield the same lattice X. Suppose that $X = L +_2 U$, with L and U chosen as above. Let

a and b be the neck of lowest rank in X. The only way to represent X as $X = L' +_2 U'$, so that L' is a piece, is that $L' = L$ and $U' = U$.

Adding up the choices, and including the $f_{\mathrm{pc}}(n)$ pieces of n elements, we observe that in $\mathcal{F}$ there are at least

$$f_{\mathrm{pc}}(n) + \sum_{k=6}^{n-1} f_{\mathrm{pc}}(k)\underline{f}(n-k+4)$$

n-element lattices that have two coatoms and two atoms. For $n > N$ this can be further lower bounded by leaving out the first term and stopping the sum at $k = N$. This concludes the induction. □

Since $\underline{f}$ in Theorem 3.7 is defined by a homogeneous linear recurrence with constant terms, it can be lower bounded by the same method as in the previous section, if f_{pc} is known up to $f_{\mathrm{pc}}(N)$. Modular vi-lattices of $n \leq 30$ elements and semimodular vi-lattices of $n \leq 25$ elements were generated in [7], and the listings are available in [8]. With a short program we can check which of those vi-lattices are pieces (as defined above), and count them. From the counts we obtain the following results.

Proposition 3.8. *$m_{\mathrm{vi}}(n) \geq 2.1562^n$ for all n large enough.*

Proof. The numbers of modular n-element pieces, for $n = 6, 7, \ldots, 30$, are

$$1, 0, 0, 3, 3, 4, 15, 27, 52, 117, 259, 554, 1253, 2802, 6366, 14429, 33150,$$
$$76090, 175799, 406851, 946151, 2204246, 5153946, 12076517, 28375409.$$

Applying Theorem 3.7 with these values, we obtain a sequence $\underline{f}$ such that $m_{\mathrm{vi}}(n) \geq \underline{f}(n)$ for all $n \geq 1$. Numerically we find that the root of the auxiliary equation is a single real root $r \approx 2.156295$. For a lower bound, we take $c = 2.1562$ and claim that $\underline{f}(n) \geq c^n$ for $n \geq 150\,000$. This follows by induction as in the proof of Proposition 2.8. □

Proposition 3.9. *$s_{\mathrm{vi}}(n) \geq 2.6797^n$ for all n large enough.*

Proof. The numbers of semimodular n-element pieces, for $n = 6, 7, \ldots, 25$, are

$$1, 0, 0, 5, 6, 9, 40, 122, 323, 964, 2999, 9374, 30292, 100539,$$
$$339046, 1159101, 4018137, 14116920, 50263399, 181341142.$$

Applying Theorem 3.7 with these values, we obtain a sequence $\underline{f}$ such that $s_{\mathrm{vi}}(n) \geq \underline{f}(n)$ for all $n \geq 1$. Numerically we find that the root of the auxiliary equation is a single real root $r \approx 2.679797$. For a lower bound, we take $c = 2.6797$ and claim that $\underline{f}(n) \geq c^n$ for $n \geq 200\,000$. This follows by induction as in the proof of Proposition 2.8. □

4. Concluding remarks

This work was motivated by two empirical observations. The first is that modular vi-lattices are usually long and narrow (cf. [7, Figs. 4 and 5]). The second is that the numbers of modular (vi-)lattices exhibit a rather stable exponential growth, at least up to $n = 30$. Together these observations suggest that much of that growth could be attributed to a "Cartesian" vertical combination of independently chosen parts.

In contrast, the vertical sum and 2-sum are not likely to be very useful with lattice families whose members tend to be short and wide; for example, with graded lattices, exponential bounds are superseded by the already known bounds of the form $c^{n^{3/2}}$.

The notion of constructing vi-lattices by some kind of vertical composition bears similarity to the work of Erné *et al.* on distributive lattices [1]; however, their vertical construction is different, and seems specific to distributive lattices, as it works on finite posets that are in one-to-one correspondence with finite distributive lattices (by a theorem of Birkhoff). Our vertical 2-sum works on lattices directly, and is applicable to several lattice families.

It is tempting to extend the idea of the vertical 2-sum to lattices that have more than two atoms and coatoms, but the result may not be a lattice. Consider, for example, defining *vertical 3-sum* ($+_3$) as the obvious analogue of the vertical 2-sum. Then the analogue of Lemma 3.3 does not hold: for a counterexample, if B_3 is the Boolean lattice of order 3, then $B_3 +_3 B_3$ is not a lattice. In order to use such generalized vertical sums for counting purposes, one needs an efficient method of filtering out the non-lattices. We leave such studies for future research.

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References

[1] Erné, M., Heitzig, J., Reinhold, J.: On the number of distributive lattices. Electron. J. Combin. **9**, Article #R24 (2002)

[2] Gebhardt, V., Tawn, S.: Constructing unlabelled lattices. J. Algebra (2018). https://doi.org/10.1016/j.jalgebra.2018.10.017

[3] Heitzig, J., Reinhold, J.: Counting finite lattices. Algebra Univers. **48**, 43–53 (2002)

[4] Jipsen, P., Lawless, N.: Generating all finite modular lattices of a given size. Algebra Univers. **74**, 253–264 (2015)

[5] Kleitman, D.J., Winston, K.J.: The asymptotic number of lattices. Ann. Discr. Math. **6**, 243–249 (1980)

[6] Klotz, W., Lucht, L.: Endliche Verbände. J. Reine Angew. Math **247**, 58–68 (1971)

[7] Kohonen, J.: Generating modular lattices of up to 30 elements. Order (2018). https://doi.org/10.1007/s11083-018-9475-2

[8] Kohonen, J.: Lists of finite lattices (modular, semimodular, graded and geometric). https://b2share.eudat.eu/records/dbb096da4e364b5e9e37b982431f41de. Accessed 18 Feb 2019

[9] Slomson, A., Allenby, R.B.J.T.: How to Count: An Introduction to Combinatorics, 2nd edn. Chapman and Hall/CRC, Boca Raton (2010)

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