



Topologizable structures and Zariski topology

Joanna Dutka and Aleksander Ivanov

Abstract. In this paper we study topologizability of structures. We extend the method of Kotov of topologizability of countable algebras to uncountable structures. We also show that in the case of topologizable relational countable structures the topology can be made metrizable.

Mathematics Subject Classification. 22A30, 08A45.

Keywords. Topologizable structures, Zariski topology.

1. Introduction

Let $L = \{ \mathcal{F}, \mathcal{R} \}$ be a countable language, where \mathcal{F} is a family of functional symbols and \mathcal{R} is a family of relational symbols. We denote the arity of a function $f \in \mathcal{F}$ by n_f and the arity of a relation $R \in \mathcal{R}$ by n_R respectively. We admit that n_f may be zero; this is the case of a constant function. Let $\mathcal{A} = \langle A, L \rangle$ be an L -structure. We will always assume that relations on A are not empty and have non-empty complements. Atomic formulas over \mathcal{A} are of the form

$$t_1(\bar{x}_1, \bar{a}_1) = t_2(\bar{x}_2, \bar{a}_2) \text{ and } R(t_1(\bar{x}_1, \bar{a}_1), \dots, t_{n_R}(\bar{x}_{n_R}, \bar{a}_{n_R})),$$

where \bar{x}_i are variables and \bar{a}_i are tuples from \mathcal{A} . If $\varphi(\bar{x}, \bar{a})$ is a formula, then $\varphi(\mathcal{A}, \bar{a})$ is the set of all realizations of $\varphi(\bar{x}, \bar{a})$ in \mathcal{A} .

Definition 1.1. The *Zariski topology* $\mathfrak{Z}_{\mathcal{A}}$ on \mathcal{A} is defined by a subbase of closed sets which is the collection of all sets of the form $\varphi(\mathcal{A}, \bar{a})$ where $\varphi(x, \bar{a})$ is an atomic formula and depends on a single variable x .

Thus, the family of all sets of the form $\neg\varphi(\mathcal{A}, \bar{a})$ is a subbase of open sets of $\mathfrak{Z}_{\mathcal{A}}$. We will denote it by $\mathcal{S}_{\mathcal{A}}$.

Kotov has proved in [4], that a countable algebra \mathcal{A} is topologizable if and only if the Zariski topology of \mathcal{A} is not discrete. This is a confirmation of the claim of A.D. Taimanov from [11]. An easy modification of Kotov's proof gives the same statement in the case of arbitrary countable language (i.e. together with relations), see Theorem 3.2 below. We remind the reader that a structure \mathcal{A} is *topologizable* if it admits a non-discrete Hausdorff topology such that all operations of \mathcal{A} are continuous and all relations are closed in the corresponding powers of \mathcal{A} .

The problem of topologizability of algebras was initiated by A. Markov who proved in [5] a topologizability criterion for countable groups. In the sixties and seventies the problem was considered for rings (Arnautov [1]), groupoids, semigroups and skew fields (Handson and Taimanov). At the moment this topic has become well established, having well-known achievements (for example [6, 8]). Paper [2, 4] have nice descriptions of the topic and rich bibliography.

Let (G, \cdot) be a group. For every finite system of equations $W(x_1, \dots, x_n)$ without parameters we introduce the corresponding relational symbol R_W and interpret it in G as the set of all solutions of $W(\bar{x})$. We consider only equations having solutions and non-solutions. The structure

$$G_{\text{REL}} = (G, \{ R_W \mid W \text{ is a finite system of parameter-free equations} \})$$

has the same Zariski topology with (G, \cdot) . Thus if the group G is countable it is topologizable if and only if so is the structure G_{REL} . In particular if G is a non-topologizable group (for example the group found in [6]), then G_{REL} is non-topologizable too. Does this argument work in the uncountable case? This motivates us to generalize the topologizability criterion for uncountable structures with countable language. Note that in the case G_{REL} the language is still countable even if we assume that G is uncountable.

It looks likely that structures G_{REL} can have Hausdorff non-trivial topologies which do not topologize the group G (even when it is countable and topologized). Can these topologies be chosen with some additional properties, for example metrizability? Are there generalizations of this approach to the uncountable case? These questions are central in our paper.

In Section 3 we prove Theorem 3.2 which generalizes (and corrects the proof of) the main result of [4]. Our generalization concerns the cardinality of the structure and the presence of relations in the language. In Section 4 we prove Proposition 4.1 which shows that in the case of topologized relational countable structures the topology can be made totally disconnected.

2. Zariski topology

The following lemma is a folklore fact with an obvious proof. It generalizes Lemma 3 of [4] to the case structures with relations in the language.

Lemma 2.1. *Let $\mathcal{A} = \langle A, L \rangle$ be an arbitrary structure. The Zariski topology $\mathfrak{Z}_{\mathcal{A}}$ is not discrete if and only if there exists an element $d \in A$ such that any finite system of negations of atomic formulas, whose solution is d , has more than one solution.*

It is worth noting that following the general theory of algebraic geometry in universal algebra, Kotov defines the Zariski topology by the subbase of all sets of the form $\bigcup_{i \in I} \neg \varphi_i(\mathcal{A}, \bar{a})$, where I can be infinite. This is an equivalent definition.

Let τ be a topology on a set X . We remind the reader that the pseudocharacter of τ at $p \in X$ is defined as follows:

$$\psi(p, \tau) = \min \left\{ |\mathcal{U}| \mid \mathcal{U} \subset \tau, \bigcap \mathcal{U} = \{p\} \right\}.$$

We will use some extension of this definition, see [2].

Definition 2.2. Let X be non-empty set, $\mathcal{V} \subset \mathcal{P}(X)$, $p \in X$ and $\mathcal{V}_p = \{X\} \cup \{V \in \mathcal{V} \mid p \in V\}$. The *pseudocharacter* of \mathcal{V} at p is the cardinal number

$$\psi(p, \mathcal{V}) = \min \left\{ |\mathcal{U}| \mid \mathcal{U} \subset \mathcal{V}_p, \bigcap \mathcal{U} = \bigcap \mathcal{V}_p \right\}.$$

The following lemma is a straightforward generalization of some standard arguments, see [2, 7, 9].

Lemma 2.3. (1) *Let τ be a topology on X defined by a subbase \mathcal{S} and let $p \in X$ be a non-isolated point with respect to τ . Then the pseudocharacter of the topology τ at p coincides with the pseudocharacter of \mathcal{S} at p .*

(2) *Let $\mathcal{A} = \langle A, L \rangle$ be an arbitrary structure and $p \in A$. The pseudocharacter of the Zariski subbase at $p \in A$ equals the least cardinality of a system of negations of atomic formulas whose unique solution is p .*

Corollary 2.4. *Let $\mathcal{A} = \langle A, L \rangle$ be a structure and $p \in A$. If p is not isolated in the Zariski topology $\mathfrak{Z}_{\mathcal{A}}$, then $\psi(p, \mathcal{S}_{\mathcal{A}}) \geq \aleph_0$.*

Definition 2.5. An L -structure \mathcal{A} is called *topologizable* if there exists a nondiscrete Hausdorff topology τ on A such that all functions of L are continuous and all relations of L are closed in τ .

In the following lemma we collect some easy folklore facts.

Lemma 2.6. *Let \mathcal{A} be an L -structure.*

- (1) *If τ is a Hausdorff topology on \mathcal{A} , then any set of the form $\varphi(\mathcal{A}, \bar{a})$, where φ is atomic, is closed in τ .*
- (2) *If τ is a Hausdorff topology on \mathcal{A} , then $\mathfrak{Z}_{\mathcal{A}} \subseteq \tau \subseteq 2^A$.*
- (3) *If $\mathfrak{Z}_{\mathcal{A}}$ is discrete, then structure \mathcal{A} is not topologizable.*

3. Topologizability

To prove that a structure \mathcal{A} is topologizable we will use some strategy of defining a base of the corresponding topology. The following lemma describes this strategy. It is a generalization of Lemma 11 of [4].

Lemma 3.1. *Let $\mathcal{A} = \langle A, L \rangle$ be an L -structure, $D \subseteq A$ and $D \neq \emptyset$. Let $\kappa = |A|$. If the set of functional L -symbols is not empty let*

$$\mathfrak{d} = \left\{ (f_{\gamma}, \bar{a}_{\gamma}, R_{\gamma}, \bar{a}'_{\gamma}) \right\}_{\gamma < \kappa}$$

be a sequence of all tuples of the form

$$(f, \bar{a}, R, \bar{a}'), \text{ where } f \in \mathcal{F}, \bar{a} \in A^{n_f}, R \in \mathcal{R}, \bar{a}' \in A^{n_R} \setminus R.$$

We assume that every tuple occurs cofinally in this sequence. We enumerate

$$\bar{a}_\gamma = (a_{\gamma,1}, a_{\gamma,2}, \dots, a_{\gamma,n_f}) \text{ and } \bar{a}'_\gamma = (a_{\gamma,n_f+1}, a_{\gamma,n_f+2}, \dots, a_{\gamma,n_f+n_R}).$$

If the language L is relational, then let \mathfrak{d} be a sequence of all tuples of the form $(a_{\gamma,1}, R_\gamma, \bar{a}'_\gamma)$, where $a_{\gamma,1} \in A$, $R_\gamma \in \mathcal{R}$, $\bar{a}'_\gamma \in A^{n_R} \setminus R_\gamma$ and $\bar{a}'_\gamma = (a_{\gamma,2}, a_{\gamma,3}, \dots, a_{\gamma,n_R+1})$. In this case we also assume that every tuple occurs cofinally in \mathfrak{d} .

Suppose that $\{V_{\gamma,i}\}_{\gamma < \kappa, 1 \leq i \leq n_{f_\gamma} + n_{R_\gamma}}$ is a family of subsets of A satisfying the following properties:

- (1) $a_{\gamma,i} \in V_{\gamma,i}$,
- (2) if $\beta < \gamma$ and $a_{\gamma,i} \in V_{\beta,j}$, then $V_{\gamma,i} \subseteq V_{\beta,j}$,
- (3) if $\beta < \gamma$ and $f_\beta(\bar{a}_\beta) \in V_{\beta,j}$, then $f_\beta(V_{\gamma,1} \times V_{\gamma,2} \times \dots \times V_{\gamma,n_{f_\beta}}) \subseteq V_{\beta,j}$,
- (4) $V_{\gamma,n_{f_\gamma}+1} \times V_{\gamma,n_{f_\gamma}+2} \times \dots \times V_{\gamma,n_{f_\gamma}+n_{R_\gamma}} \cap R_\gamma = \emptyset$,
- (5) if $a_{\gamma,1} \in D$, then $|V_{\gamma,1}| > 1$,
- (6) if $\alpha < \beta < \gamma$, $a_{\alpha,1} = a_{\beta,1}$ and $a_{\gamma,1} \neq a_{\beta,1}$, then $V_{\beta,1} \cap V_{\gamma,1} = \emptyset$.

Then A is topologizable and the pseudocharacter of this topology at any $d \in D$ equals $\text{cf}(\kappa)$.

Proof. Assumption (1) implies $\bigcup_i V_{\gamma,i} = A$. Take any $V_{\beta,i}, V_{\gamma,j}$ and let $b \in V_{\beta,i} \cap V_{\gamma,j}$. Choose α such that $\alpha > \beta$, $\alpha > \gamma$ and $a_{\alpha,1} = b$. It follows from (1) that $b \in V_{\alpha,1}$. Then (2) implies that $V_{\alpha,1} \subseteq V_{\beta,i}$ and $V_{\alpha,1} \subseteq V_{\gamma,j}$. We have $b \in V_{\alpha,1} \subseteq V_{\beta,i} \cap V_{\gamma,j}$, so the family $(V_{\gamma,i})$ is a basis of a topology on A , say τ .

Now we show that in the case $\mathcal{F} \neq \emptyset$ all L -functions are τ -continuous. Let $f \in \mathcal{F}$, $\bar{a} \in A^{n_f}$ and $f(\bar{a}) \in V_{\beta,i}$. Choose γ such that $\gamma > \beta$, $f = f_\gamma$ and $\bar{a} = \bar{a}_\gamma$. Since $\bar{a}_\gamma \in V_{\gamma,1} \times V_{\gamma,2} \times \dots \times V_{\gamma,n_{f_\gamma}}$ it follows from (3) that

$$f(V_{\gamma,1} \times V_{\gamma,2} \times \dots \times V_{\gamma,n_{f_\gamma}}) \subseteq V_{\beta,i}.$$

We prove that all relations are closed in this topology. Take any $R \in \mathcal{R}$ and let $\bar{a} \in A^{n_R} \setminus R$. Choose γ such that $R_\gamma = R$ and $\bar{a}'_\gamma = \bar{a}$. Then by (1) we have $\bar{a}'_\gamma \in V_{\gamma,n_{f_\gamma}+1} \times V_{\gamma,n_{f_\gamma}+2} \times \dots \times V_{\gamma,n_{f_\gamma}+n_{R_\gamma}}$ and by (4)

$$V_{\gamma,n_{f_\gamma}+1} \times V_{\gamma,n_{f_\gamma}+2} \times \dots \times V_{\gamma,n_{f_\gamma}+n_{R_\gamma}} \cap R = \emptyset.$$

This topology is Hausdorff. Indeed, let $b_1 \neq b_2$. Choose γ, β, α such that $\gamma > \beta > \alpha$, $a_{\gamma,1} = a_{\alpha,1} = b_1$ and $a_{\beta,1} = b_2$. It follows from (1) that $b_1 \in V_{\gamma,1}$ and $b_2 \in V_{\beta,1}$. Then (6) implies that $V_{\gamma,1} \cap V_{\beta,1} = \emptyset$.

Every element $d \in D$ occurs cofinally as $a_{\gamma,1}$. By (2) there is a cofinal subsequence $\{V_{\gamma_\delta,1}\}_{\delta < \text{cf}(\kappa)}$ of the corresponding γ 's. As we already know it is a descending sequence of neighbourhoods of d . By (5) each $V_{\gamma_\delta,1}$ is not a singleton. Since τ is T_2 , we have

$$\bigcap_{\delta < \text{cf}(\kappa)} V_{\gamma_\delta,1} = \{d\}.$$

If \mathcal{U} is a family of τ -open sets such that $|\mathcal{U}| < \text{cf}(\kappa)$ and $\bigcap \mathcal{U} = \{d\}$, then we may assume that \mathcal{U} consists of sets $V_{\gamma,1}$. Thus, we can form a descending

sequence of elements of \mathcal{U} . Since $|\mathcal{U}| < \text{cf}(\kappa)$, there is γ_0 such that $V_{\gamma_0,1} \subseteq \bigcap \mathcal{U}$. Since by (5) $|V_{\gamma_0,1}| > 1$, we have a contradiction. Therefore, $\psi(d, \tau) = \text{cf}(\kappa)$. \square

The following theorem is generalization of Lemma 12 from [4] to the case of structures having relations in the language. Contrary to the Kotov's paper we do not assume that our structures are countable. This makes the construction slightly more complicated.

Theorem 3.2. *Let $\mathcal{A} = \langle A, L \rangle$ be an L -structure, of cardinality κ . Let D be the set of all elements of \mathcal{A} which are not isolated in $\mathfrak{Z}_{\mathcal{A}}$. Assume that $D \neq \emptyset$ and each $d \in D$ satisfies $\psi(d, \mathfrak{Z}_{\mathcal{A}}) = \text{cf}(\kappa)$.*

Then \mathcal{A} is topologizable so that the pseudocharacter of this topology at each $d \in D$ equals $\text{cf}(\kappa)$.

Proof. Let \mathfrak{d} be a sequence defined in Lemma 3.1 when $|A| = \kappa$.

Suppose that there exists a family

$$\{U_{\gamma,i,\delta} \mid \gamma < \kappa, 1 \leq i \leq n_{f_\gamma} + n_{R_\gamma}, \delta < \kappa\}$$

of subsets of A such that the following conditions hold:

- (1) $a_{\gamma,i} \in U_{\gamma,i,\delta}$,
- (2) if $\beta < \gamma \leq \delta$ and $a_{\gamma,i} \in U_{\beta,j,\delta}$, then $U_{\gamma,i,\delta} \subseteq U_{\beta,j,\delta}$,
- (3) if $\beta < \gamma \leq \delta$ and $f_\gamma(\bar{a}_\gamma) \in U_{\beta,i,\delta}$, then $f_\gamma(U_{\gamma,1,\delta} \times \dots \times U_{\gamma,n_{f_\gamma},\delta}) \subseteq U_{\beta,i,\delta}$,
- (4) if $\gamma \leq \delta$, then $(U_{\gamma,n_{f_\gamma}+1,\delta} \times \dots \times U_{\gamma,n_{f_\gamma}+n_{R_\gamma},\delta}) \cap R_\gamma = \emptyset$,
- (5) if $\gamma \leq \delta$ and $a_{\gamma,1} \in D$, then $|U_{\gamma,1,\delta}| > 1$,
- (6) if $\alpha < \beta < \gamma \leq \delta$, $a_{\gamma,1} = a_{\alpha,1}$ and $a_{\gamma,1} \neq a_{\beta,1}$, then $U_{\beta,1,\delta} \cap U_{\gamma,1,\delta} = \emptyset$,
- (7) if $\gamma < \beta \leq \delta$ and $a_{\beta,1} \neq a_{\gamma,1}$, then $a_{\gamma,1} \notin U_{\beta,1,\delta}$,
- (8) if $\delta < \gamma$, then $U_{\gamma,i,\delta} = \{a_{\gamma,i}\}$,
- (9) if $\delta < \eta$, then $U_{\gamma,i,\delta} \subseteq U_{\gamma,i,\eta}$.

Put $V_{\gamma,i} = \bigcup_{\delta < \kappa} U_{\gamma,i,\delta}$. Then it is easy to see that the family of these $V_{\gamma,i}$ satisfies all requirements of Lemma 3.1. This would prove the theorem. Let us fix a well ordering of A : $b_0, b_1, \dots, b_\alpha, \dots$, $\alpha < \kappa$. We construct the family $\{U_{\gamma,i,\delta} \mid \gamma < \kappa, 1 \leq i \leq n_{f_\gamma} + n_{R_\gamma}, \delta < \kappa\}$ by transfinite induction where we use this ordering. We apply induction on δ .

At step 0 let $U_{\gamma,i,0} = \{a_{\gamma,i}\}$ if $\gamma \neq 0$ or $i \neq 1$. Put $U_{0,1,0} = \{a_{0,1}\}$ if $a_{0,1} \notin D$. If $a_{0,1} \in D$ put $U_{0,1,0} = \{a_{0,1}, b\}$, where b is the first element of $b_0, b_1, \dots, b_\alpha, \dots$ such that $b \neq a_{0,1}$ and all $U_{0,i,0}$ satisfy condition (4) for $\delta = 0$. The condition that $a_{0,1}$ is not isolated in the Zariski topology implies the existence of such b . As a result we see that conditions (1)–(8) are satisfied for $\delta = 0$.

The table in Figure 1 illustrates the induction process for $\delta > 0$. Let δ be an ordinal greater than 0 and assume that the family $\{U_{\gamma,i,\alpha} \mid \alpha < \delta \text{ and } \gamma < \kappa\}$ has been already constructed so as to satisfy conditions (1)–(8) and $U_{\gamma,i,\alpha_1} \subseteq U_{\gamma,i,\alpha_2}$ for all γ, i and $\alpha_1 < \alpha_2 < \delta$. Let us define a family $\{U_{\gamma,i,\delta} \mid \gamma < \kappa\}$ for which (1)–(8) hold and $U_{\gamma,i,\alpha} \subseteq U_{\gamma,i,\delta}$ for all $\alpha < \delta$, appropriate i and $\gamma < \kappa$.

	0	...	δ'	...	δ	...
(0, 1)	$\{a_{0,1}\}$...	$U_{0,1,\delta'}$...	$U_{0,1,\delta}$...
\vdots	\vdots		\vdots		\vdots	
(β, i)	$\{a_{\beta,i}\}$...	$U_{m,i,\delta'}$...	$U_{\beta,i,\delta}$...
\vdots	\vdots		\vdots		\vdots	
(γ, j)	$\{a_{\gamma,j}\}$...	$U_{\gamma,j,\delta'}$...	$U_{\gamma,j,\delta}$...
\vdots	\vdots		\vdots		\vdots	
$(\delta, 1)$	$\{a_{\delta,1} \in D\}$...	$\{a_{\delta,1}\}$...	$\{a_{\delta,1}, b\}$...
\vdots	\vdots		\vdots		\vdots	
(δ, i)	$\{a_{\delta,i}\}$...	$\{a_{\delta,i}\}$...	$\{a_{\delta,i}\}$...
\vdots	\vdots		\vdots		\vdots	

FIGURE 1. The induction process for $\delta > 0$

Let $a_{\delta,1} \notin D$. If δ is a successor ordinal, then we put $U_{\gamma,i,\delta} = U_{\gamma,i,\delta-1}$ and if δ is a limit ordinal, then let $U_{\gamma,i,\delta} = \bigcup_{\delta' < \delta} U_{\gamma,i,\delta'}$. It is easy to verify that in both cases conditions (1)–(5) and (7)–(8) hold. If $\gamma \neq \delta$, then (6) obviously holds. Let $\alpha < \beta < \gamma = \delta$, $a_{\alpha,1} = a_{\gamma,1}$ and $a_{\beta,1} \neq a_{\delta,1}$. By (7) $a_{\delta,1} \notin U_{\beta,1,\delta}$ while $U_{\delta,1,\delta} = \{a_{\delta,1}\}$. Then $U_{\beta,1,\delta}$ and $U_{\delta,1,\delta}$ are disjoint.

Let $a_{\delta,1} = d \in D$. To satisfy that $|U_{\delta,1,\delta}| > 1$ we construct a system of negations of atomic formulas depending on a single variable x and realized by d . When we define it we put $U_{\delta,1,\delta} = \{d, b\}$, where $b \neq d$ is the minimal solution of this system in the ordering $b_0, b_1, \dots, b_\gamma, \dots$ which is outside all $U_{\alpha,i,\beta}$ with $\max(\alpha, \beta) < \delta$.

We start with a family of sets of terms (depending on x)

$$\{U_{\gamma,i,\delta}(x) \mid \gamma < \kappa, 1 \leq i \leq n_{f_\gamma} + n_{R_\gamma}\}.$$

Put $U_{\gamma,i,\delta}(x) = \{a_{\gamma,i}\}$ for $\gamma > \delta$. If $\gamma \leq \delta$ and $d \notin \bigcup_{\delta' < \delta} U_{\gamma,i,\delta'}$ then let $U_{\gamma,i,\delta}(x) = \bigcup_{\delta' < \delta} U_{\gamma,i,\delta'}$. If $a_{\delta,i} = a_{\delta,1}$ let $U_{\delta,i,\delta}(x) = \{a_{\delta,1}, x\}$.

The remaining sets are constructed as follows. For every finite sequence of ordinals $\bar{\alpha} = \{\alpha_1 < \dots < \alpha_k = \delta\}$ define $U_{\bar{\alpha},i,j}(x)$ by induction in order of decreasing $j \leq k$. Let $U_{\bar{\alpha},i,k}(x) = U_{\delta,i,\delta}(x)$. Suppose that all sets $U_{\bar{\alpha},i,j}(x)$ for $m < j \leq k$ are already defined. Put

$$\begin{aligned}
 U_{\bar{\alpha},i,m}(x) = & \left(\bigcup_{\delta' < \delta} U_{\alpha_m,i,\delta'} \right) \cup \bigcup_{\substack{m < j \leq k \\ a_{\alpha_j,t} \in \bigcup_{\delta' < \delta} U_{\alpha_m,i,\delta'}}} U_{\bar{\alpha},l,j}(x) \cup \\
 & \cup \bigcup_{\substack{m < j \leq k \\ f_{\alpha_j}(\bar{a}_{\alpha_j}) \in \bigcup_{\delta' < \delta} U_{\alpha_m,i,\delta'}}} f_{\alpha_j}(U_{\bar{\alpha},1,j}(x) \times \dots \times U_{\bar{\alpha},n_{f_{\alpha_j}},j}(x)).
 \end{aligned}$$

Let us define $U_{\gamma,i,\delta}(x)$ to be the union of all $U_{\bar{\alpha},i,j}(x)$ for all possible finite sequences $\bar{\alpha} = \{\gamma = \alpha_1 < \dots < \alpha_k = \delta\}$.

Assume that $b \in A \setminus \{d\}$ satisfies the condition that for any finite sequence $\bar{\alpha}$ as above and any non-constant term $t(x)$ from any

$$\bigcup_{\substack{m < j \leq k \\ a_{\alpha_j, i} \in \bigcup_{\delta' < \delta} U_{\alpha_m, i, \delta'}}} U_{\bar{\alpha}, i, j}(x) \cup \bigcup_{\substack{m < j \leq k \\ f_{\alpha_j}(\bar{a}_{\alpha_j}) \in \bigcup_{\delta' < \delta} U_{\alpha_m, i, \delta'}}} f_{\alpha_j}(U_{\bar{\alpha}, 1, j}(x) \times \cdots \times U_{\bar{\alpha}, n_{f_{\alpha_j}}, j}(x))$$

if $\gamma \leq \delta$ and $t(d) \neq a_{\gamma, i}$ then $t(b) \neq a_{\gamma, i}$. Then for all $\gamma \leq \delta$ and such a b ,

$$U_{\gamma, i, \delta}(b) \cap \bigcup_j \bigcup_{\delta' < \delta} \bigcup_{\beta \leq \delta} U_{\beta, j, \delta'} = \bigcup_{\delta' < \delta} U_{\gamma, i, \delta'}.$$

To see this note that when $t(d)$ equals to some $a_{\gamma, i}$ as above, any its subterm of the form $f_{\delta}(\bar{a}_{\delta})$ belongs to some $\bigcup_{\delta' < \delta} U_{\beta, j, \delta'}$ with $\beta < \delta$ and $a_{\gamma, i}$ is the value of some f_{α} , $\alpha < \delta$, with a substituted tuple from

$$\bigcup_j \bigcup_{\delta' < \delta} \bigcup_{\beta < \delta} U_{\beta, j, \delta'}.$$

Thus the membership of $a_{\gamma, i}$ to the corresponding $V_{\beta, j}$, $\beta < \delta$, was already decided at previous steps.

We set $U_{\gamma, i, \delta} = U_{\gamma, i, \delta}(b)$ for all $\gamma \leq \delta$. It is easy to see that conditions (1)–(3), (5) and (8) hold for our $U_{\gamma, i, \delta}$'s. For example let $\beta < \gamma \leq \delta$ and $a_{\gamma, i} \in U_{\beta, j, \delta}(b)$. Note that by the choice of b

$$a_{\gamma, i} \in \bigcup_j \bigcup_{\delta' < \delta} \bigcup_{\beta \leq \beta' < \gamma} U_{\beta', j, \delta'}.$$

If $a_{\gamma, i} \in \bigcup_{\delta' < \delta} U_{\beta, j, \delta'}$, then considering finite sequences $\bar{\alpha}$ where $\alpha_1 = \beta$, $\alpha_2 = \gamma$, we see that the definition guarantees that $U_{\gamma, i, \delta}(b) \subseteq U_{\beta, j, \delta}(b)$. Similar arguments work for the case when $a_{\gamma, i} \in \bigcup_{\delta' < \delta} U_{\beta', j, \delta'}$ for some $\beta' > \beta$ with $\beta' < \gamma$.

To satisfy (4), (6), (7) and the choice of b as above we form a system of inequations S which consists of the following system:

$$\bigcup_{\substack{\alpha < \beta \leq \delta \\ a_{\alpha, 1} \neq a_{\beta, 1} \\ t(x) \in U_{\beta, 1, \delta}(x)}} \{a_{\alpha, 1} \neq t(x)\} \cup \bigcup_{\substack{\alpha < \beta < \gamma \leq \delta \\ a_{\alpha, 1} = a_{\gamma, 1} \\ a_{\alpha, 1} \neq a_{\beta, 1} \\ t_1(x) \in U_{\beta, 1, s}(x) \\ t_2(x) \in U_{\gamma, 1, s}(x)}} \{t_1(x) \neq t_2(x)\} \cup \bigcup_{\gamma \leq \delta} \neg R_{\gamma}(U_{\gamma, n_{f_{\gamma}}+1, s}(x) \times \cdots \times U_{\gamma, n_{f_{\gamma}}+n_{R_{\gamma}}, \delta}(x))$$

and

$\{t(x) \neq a_{\gamma, j} \mid \gamma \leq \delta \text{ and for some } i \text{ the term } t(x) \text{ belongs to}$

$$\bigcup_{\substack{m < j \leq k \\ a_{\alpha_j, i} \in \bigcup_{\delta' < \delta} U_{\alpha_m, i, \delta'}}} U_{\bar{\alpha}, i, j}(x) \cup$$

$$\bigcup_{f_{\alpha_j}(\bar{a}_{\alpha_j}) \in \bigcup_{\delta' < \delta} U_{\alpha_m, i, \delta'}} \bigcup_{m < j \leq k} f_{\alpha_j}(U_{\bar{\alpha}, 1, j}(x) \times \cdots \times U_{\bar{\alpha}, n_{f_{\alpha_j}}, j}(x))$$

and does not satisfy $t(d) = a_{\gamma, j}$.

Conditions (4),(6) and (7) are met for the family $U_{\gamma, 1, \delta}(b)$ if we take as b any solution of S which is outside all $U_{\alpha, i, \beta}$ with $\max(\alpha, \beta) < \delta$. Since for any α

$$U_{\alpha, i, \delta}(d) = \bigcup_{\delta' < \delta} U_{\alpha, i, \delta'},$$

the inductive assumptions imply that d is one of the solution of this system. Since $|S| < \text{cf}(\kappa)$ by Lemma 2.3(2) this system has a solution $b_\sigma \neq d$ with $\sigma < \kappa$ such that $b_\sigma \notin \bigcup \{U_{\alpha, i, \beta} \mid \max(\alpha, \beta) < \delta\}$. The rest of the argument is clear. □

Remark 3.3. The proof of the corresponding place presented in [4, Lemma 12] is not complete. The choice of b so that it realizes all inequalities over

$$\bigcup_j \bigcup_{\delta' \leq \delta} \bigcup_{\beta \leq \delta} U_{\beta, j, \delta'}$$

realized by d is an essential element of the proof. For example we cannot verify condition (2) at step δ without this property.

The assumption of the theorem, that the pseudocharacter of the Zariski topology is $\text{cf}(\kappa)$ at any non-isolated point is essential. Let us assume CH and consider the non-topologizable group M of cardinality 2^ω constructed by Shelah [8].

Proposition 3.4. *The pseudocharacter of any element of M in the Zariski topology of M is equal to ω .*

Proof. The pseudocharacter of 1 in the Zariski topology of M is infinite. This follows from non-topologizability of M and a general observation which appears in Lemma 1 of [10]. Note that for every inequation $W(x) \neq 1$ with solution 1 any element $a \in M$ is a solution of the inequation $W(x \cdot a^{-1}) \neq 1$. Thus the pseudocharacter of any element $a \in M$ in the Zariski topology of M is infinite and is the same with 1. If it is 2^ω , then M is *ungebunden* in the sense of Podewski and thus topologizable, see [7]. As a result we see that the pseudocharacter is equal to ω . □

Remark 3.5. Let J be the uncountable hereditarily non-topologizable group found in [3]. The term hereditarily non-topologizable means that for any $H < J$ any quotient of H is non-topologizable. It is shown in Theorem 2.5 of [3] that there is a natural number n_0 and an element $g \in J$ such that the set of non-solutions of the equation $(gg^x)^{n_0} = 1$ is finite and contains 1. Thus for any $a \in J$ the equation $(gg^{x \cdot a^{-1}})^{n_0} = 1$ is not satisfied by a . In particular any a is isolated in the Zariski topology of J . This in particular implies that the group J is not topologizable even as a relational structure J_{REL} .

4. Metrizable

In the case of a countable relational structure the construction of the previous section gives a totally disconnected space.

Proposition 4.1. *Assume that $\mathcal{A} = \langle A, L \rangle$ is a countable L -structure and the language L is relational. If the Zariski topology $\mathfrak{Z}_{\mathcal{A}}$ is not discrete, then the structure \mathcal{A} is topologizable by a topology τ which is zero dimensional, metrizable and satisfies the following property:*

if D is the set of all elements of \mathcal{A} which are not isolated in $\mathfrak{Z}_{\mathcal{A}}$, then all elements of D are not isolated with respect to τ .

Proof. We start with some observations concerning the proof of Theorem 3.2 in the case of a countable relational structure. The sequence \mathfrak{d} defined in Lemma 3.1 consists of all tuples of the form $(a_{m,1}, R_m, \vec{a}'_m)$, where $a_{m,1} \in A$, $R_m \in \mathcal{R}$, $\vec{a}'_m \in A^{n_{R_m}} \setminus R_m$ and $\vec{a}'_m = (a_{m,2}, a_{m,3}, \dots, a_{m,n_{R_m}+1})$. We may assume that any element of the tuple \vec{a}'_m appears in this sequence as $a_{s,1}$ for some $s < m$. This can be obtained by a small modification of the construction by duplications of some tuples \vec{a}'_m at some steps and introducing several steps in the beginning where we do nothing.

When $a_{s,1} \notin D$ and $a_{s,1}$ was not used before step s as some b in order to satisfy condition (5), then the construction of Theorem 3.2 ensures that all $V_{n,i}$ containing $a_{s,1}$ are singletons. Moreover note that any element $a_{s,1} \notin D$ first appearing as such a b defines singletons $V_{n,i}$ for $a_{n,i} = a_{s,1}$ from the moment of the second appearance.

Let us consider the corresponding $V_{n,i}$ for elements of D . Since L is relational the procedure of Theorem 3.2 in this case can be equivalently reformulated in a more convenient way. We now describe it.

Let $a_{s,1} = d \in D$. To satisfy that $|U_{s,1,s}| > 1$ we construct a system of negations of atomic formulas depending on a single variable x and realized by d . When we define it we put $U_{s,1,s} = \{d, b\}$, where $b \neq d$ is the minimal solution of this system in the enumeration $b_0, b_1, \dots, b_n, \dots$ which is outside all $U_{k,i,l}$ with $\max(k, l) < s$. The assumptions of the beginning of the proof guarantee that b does not belong to the set $\{a_{s,i} \mid i \leq n_{R_s} + 1\}$.

The sets $\{U_{m,i,s}(x) \mid m \in \mathbb{N}, 1 \leq i \leq n_{R_m} + 1\}$ are constructed according the following rules. Put $U_{m,i,s}(x) = U_{m,i,s}$ for $m > s$. If $m \leq s$ and $d \notin U_{m,i,s-1}$ then $U_{m,i,s} = U_{m,i,s-1}$. If $d \in U_{m,i,s-1}$ then let $U_{m,i,s}(x) = U_{m,i,s-1} \cup \{x\}$. It is easy to see that these rules agree with the corresponding procedure of Theorem 3.2.

Let us consider the following system:

$$S = \bigcup_{m \leq s} \neg R_s(U_{m,2,s}(x) \times \dots \times U_{m,n_{R_s}+1,s}(x)).$$

Since $U_{m,i,s}(d) = U_{m,i,s-1}$, the inductive assumptions imply that d is one of the solution of this system. By Lemma 2.3(2) this system has a solution $b \neq d$. Any solution $b \neq d$ which is outside all $U_{m,i,k}$ with $\max(m, k) < s$ realizes all conditions (1), (2), (4)–(8) for the corresponding $U_{m,i,s}(b)$.

As a result we see that in the case $a_{s,1} \in D$ the procedure of step s only extends the sets $U_{m,i,s-1}$ which contain $a_{s,1}$. Moreover these sets are extended by the same element which is outside all $U_{m,i,k}$ with $m \leq s$ and $k < s$.

Applying easy induction we obtain that for any (m, i, k) and (n, j, k) with $m < n$ and $k \leq s$ if the intersection $U_{m,i,k} \cap U_{n,j,k}$ is not empty, then $U_{m,i,k}$ contains $U_{n,j,k}$. This implies the following claim.

Claim. For any (m, i) and (n, j) with $m < n$ if the intersection $V_{m,i} \cap V_{n,j}$ is not empty, then $V_{m,i}$ contains $V_{n,j}$.

To see that the topology is zero dimensional it suffices to show that all sets of the form $V_{n,1}$ are clopen. Let F be the complement of some $V_{n,1}$. Any $a \in F$ coincides with some $a_{m,1}$ with $m > n$. By our claim the corresponding $V_{m,1}$ has empty intersection with $V_{n,1}$.

Thus the set F can be presented as a union of basic open sets, i.e. F is clopen.

Since the family of all $V_{m,i}$ is countable, by the Urysohn's metrization theorem we have that the topology τ is metrizable. The rest of the formulation follows from Theorem 3.2. \square

Let us mention that in the situation when G is a countable topological group we cannot state that the topology obtained for G_{REL} by the method of Proposition 4.1 topologizes the group G . This is because the construction from the proof does not take care of continuity of multiplication.

Acknowledgements

The authors are grateful to the referee for very helpful remarks.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Arnautov, V.: Nondiscrete topologizability of countable rings. Dokl. AN SSSR **191**, 747–750 (1970). (Russian)
- [2] Banach, T., Protasov, I., Sipacheva, O.: Topologization of a set endowed with an action of a monoid. Topol. Appl. **169**, 161–174 (2014)
- [3] Klyachko, A., Olshanskii, A., Osin, D.: On topologizable and non-topologizable groups. Topol. Appl. **160**, 2104–2120 (2013)
- [4] Kotov, M.V.: On the topologizability of countable equationally Noetherian algebras. Algebra Log. **52**, 105–115 (2013)

- [5] Markov, A.: On unconditionally closed sets. *Mat. Sb.* **18**, 3–28 (1946). (Russian)
- [6] Olshanski, A.: A remark on a countable non-topologizable group. *Vestn. Mosk. Gos. Univ. Mat. Mekh.* **3**, 103 (1980). (Russian)
- [7] Podewski, K.-P.: Topologisierung algebraischer Strukturen. *Rev. Roum. Math. Pures Appl.* **22**(9), 1283–1290 (1977)
- [8] Shelah, S.: On a problem of Kurosh, Jonsson groups, and applications. In: *Word Problems, II, Conf. on Decision Problems in Algebra*, Oxford, 1976. *Stud. Logic Found. Math.*, vol. 95, pp. 373–394. North-Holland, Amsterdam, New York (1980)
- [9] Sipacheva, O.: Unconditionally τ -closed and τ -algebraic sets in groups. *Topol. Appl.* **155**, 335–341 (2008)
- [10] Sipacheva, O.: Consistent solution of Markov’s problem about algebraic sets, [arxiv:math/0605558v2](https://arxiv.org/abs/math/0605558v2)
- [11] Taimanov, A.D.: Topologization of countable algebras. *Dokl. Akad. Nauk. SSSR* **243**, 284–286 (1978). (Russian)

Joanna Dutka and Aleksander Ivanov
Department of Applied Mathematics
Silesian University of Technology
44-100 Gliwice
Poland
e-mail [J. Dutka]: Joanna.Dutka@polsl.pl
e-mail [A. Ivanov]: iwanowaleksander@gmail.com

Received: 20 December 2016.

Accepted: 16 June 2018.