



Axiomatizations of universal classes through infinitary logic

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To the memory of Bjarni Jónsson.

Abstract. We present a scheme for providing axiomatizations of universal classes. We use infinitary sentences there. New proofs of Birkhoff's HSP-theorem and Mal'cev's SPP_U -theorem are derived. In total, we present 75 facts of this sort.

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1. Unary class operators

In this paper we present an easy and unified way for proving preservation theorems for various types of universal classes. We start with a basic observation. It is a formalization of a trick which may be encountered in various proofs of preservation theorems. In order to do so, we first need to clarify the notion of unary class operator.

Let L be a fixed first-order language, i.e., a set of functional and relational symbols of finite arities. Let $\text{Struc}(L)$ be the class of all structures in L (called also L -structures). By a *unary class operator* we mean an assignment O which with each structure A in $\text{Struc}(L)$ associate a certain class $O(A) \subseteq \text{Struc}(L)$ such that the following two requirements are fulfilled

- $A \in O(A)$;
- If C is isomorphic to B and $B \in O(A)$, then $C \in O(A)$.

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(The question whether such objects exist is a set theoretical issue. However, as it is commonly done in set theory (see e.g. [12, Chapter 8]), we may assume that there is a first-order formula $\Phi(x, y)$ in the language of set theory such that $O(\mathbf{A}) = \{\mathbf{B} \mid \mathbf{B} \in \text{Struc}(L) \text{ and } \Phi(\mathbf{A}, \mathbf{B})\}$, and then identify O with Φ .) Note that we do not impose the idempotency condition $OO = O$.

We may compose unary class operators. So $C \in OO'(\mathbf{A})$ if there exists \mathbf{B} such that $C \in O(\mathbf{B})$ and $\mathbf{B} \in O'(A)$.

In this paper we deal only with two class operators which are not unary: the direct product P and the ultraproduct P_U class operators. Thus we do not need to define this notion. We do it just for the sake of completeness. By a *class operator* we mean an assignment O which with each subset \mathcal{S} of $\text{Struc}(L)$ associate a certain class $O(\mathcal{S})$ such that $\mathcal{S} \subseteq O(\mathcal{S})$ and $\mathbf{A} \in O(\mathcal{S})$ yields $\mathbf{B} \in O(\mathcal{S})$ for every structure \mathbf{B} isomorphic to \mathbf{A} . Optionally, one may also add the monotonicity conditions: if $\mathcal{S} \subseteq \mathcal{S}'$, then $O(\mathcal{S}) \subseteq O(\mathcal{S}')$. Clearly, every unary class operator O may be considered as a (monotone) class operator by putting $O(\mathcal{S}) = \bigcup\{O(\mathbf{A}) \mid \mathbf{A} \in \mathcal{S}\}$.

A class $\mathcal{C} \subseteq \text{Struc}(L)$ is closed under a class operator O (or just O -closed) provided $O(\mathcal{S}) \subseteq \mathcal{C}$ for every subset \mathcal{S} of \mathcal{C} . If O, O' are class operators and \mathcal{S} is a set of L -structures, then $\mathcal{S} \subseteq O(\mathcal{S}) \cap O'(\mathcal{S})$. Hence a class is OO' -closed iff it is O and O' -closed. It does not mean that $OO' = O'O$. For instance, in general the equality $HS(\mathbf{A}) = SH(\mathbf{A})$ does not hold. Here H denotes the homomorphic image and S denotes substructure unary class operators. Moreover, SH is not idempotent.

Let O be a unary class operator. Assume that with each structure $\mathbf{A} \in \text{Struc}(L)$ we may associate a sentences $\chi_{\mathbf{A}}^O$, say in $L_{\infty, \infty}$ (see e.g. [3] for definitions in infinitary logic), such that

$$\mathbf{B} \models \chi_{\mathbf{A}}^O \quad \text{iff} \quad \mathbf{A} \in O(\mathbf{B}).$$

Then we say that the sentences $\chi_{\mathbf{A}}^O$ are *characteristic* for O .

Basic Observation. *Let $\chi_{\mathbf{A}}^O$ be characteristic sentences for a unary class operator O and let \mathcal{C} be a O -closed class of structures. Then \mathcal{C} is axiomatizable by the class of sentences $\{\neg\chi_{\mathbf{A}}^O \mid \mathbf{A} \notin \mathcal{C}\}$.*

Proof. Let $\mathbf{B} \in \mathcal{C}$ and $\mathbf{A} \notin \mathcal{C}$. Since \mathcal{C} is O -closed, $\mathbf{A} \notin O(\mathbf{B})$. Hence $\mathbf{B} \not\models \chi_{\mathbf{A}}^O$ and, since $\chi_{\mathbf{A}}^O$ is a sentence, $\mathbf{B} \models \neg\chi_{\mathbf{A}}^O$.

Now assume that $\mathbf{B} \notin \mathcal{C}$. Since $\mathbf{B} \in O(\mathbf{B})$, $\mathbf{B} \models \chi_{\mathbf{B}}^O$. Consequently, $\mathbf{B} \not\models \neg\chi_{\mathbf{B}}^O$. □

2. Classical applications

By a *disjunctive universal sentence* we mean a sentence in $L_{\infty, \infty}$ of the form

$$\sigma = \forall X \bigvee_{i \in I} \neg\varphi_i \vee \bigvee_{j \in J} \psi_j,$$

where all formulas φ_i and ψ_j are atomic and all its variables are in X . The subformula $\bigvee_{i \in I} \neg\varphi_i$ is called the *negative part* of σ and $\bigvee_{j \in J} \psi_j$ is called the

positive part of σ . We say that σ is positive/negative if its negative/positive part vanishes. A class is definable by universal sentences in $L_{\infty,\infty}$ iff it is definable by disjunctive universal sentences. This fact may be deduced from Theorem 2.2 as well as from the complete distributivity of Boolean algebras of power sets [9, Chapter 8].

Every disjunctive universal sentence may be written in the form

$$\sigma = \forall X \bigwedge_{i \in I} \varphi_i \rightarrow \bigvee_{J \in J} \psi_j$$

Thus, if the positive part consists of one atomic formula, then we speak about *implications*. A first-order implication, i.e., in $L_{\omega,\omega}$, is called a *quasi-identity*. And an *identity* is a positive disjunctive universal sentence with exactly one disjunct.

A general scheme of theorems in this paper is as follows: A class of L -structures is closed under a particular class operator \mathbf{O} if and only if it is definable by sentences of a special form. The proofs split into two parts. We should check that if σ is of the special form, $\mathbf{A} \models \sigma$ and $\mathbf{B} \in \mathbf{O}(\mathbf{A})$, then $\mathbf{B} \models \sigma$. Secondly, we find characteristic sentences for \mathbf{O} which are of the considered form. Usually, the first part is easy. Hence, in most cases we just argue for the second part.

Let \mathbf{A} be an L -structure and A be its carrier. Let X_A be a set of variables for which there is a bijection $\pi_A: A \rightarrow X_A$. For a set X of variables let $\text{At}(X)$ be the set of atomic formulas with all variables in X . Define

$$\begin{aligned} \text{Diag}^+(\mathbf{A}) &= \{\psi(\pi_A(\bar{a})) \in \text{At}(X_A) \mid \mathbf{A} \models \psi(\bar{a})\}, \\ \text{Diag}^-(\mathbf{A}) &= \{\varphi(\pi_A(\bar{a})) \in \text{At}(X_A) \mid \mathbf{A} \not\models \varphi(\bar{a})\}. \end{aligned}$$

For $\mathbf{A} \in \text{Struc}(L)$ let $\mathbf{S}(\mathbf{A})$ be the class of structures isomorphic to substructures of \mathbf{A} and let $\mathbf{H}(\mathbf{A})$ be the class of homomorphic images of \mathbf{A} . Define

$$\begin{aligned} \chi_{\mathbf{A}}^{\text{HS}} &= \exists X_A \bigwedge_{\varphi \in \text{Diag}^-(\mathbf{A})} \neg\varphi, \\ \chi_{\mathbf{A}}^{\text{S}} &= \exists X_A \bigwedge_{\varphi \in \text{Diag}^-(\mathbf{A})} \neg\varphi \wedge \bigwedge_{\psi \in \text{Diag}^+(\mathbf{A})} \psi. \end{aligned}$$

Then we have the following facts.

Fact 2.1. *The sentences $\chi_{\mathbf{A}}^{\text{HS}}$ are characteristic for HS. The sentences $\chi_{\mathbf{A}}^{\text{S}}$ are characteristic for S.*

We immediately obtain the following fact.

Theorem 2.2. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is*

- (1) *S-closed if and only if it is definable by disjunctive universal sentences;*
- (2) *HS-closed if and only if it is definable by positive disjunctive universal sentences.*

We are mainly interested in first-order axiomatizations. We use the following known facts. (For (1) see e.g. [13, Theorem 7.5.2]. Point (2) may be

proved using the same construction as in Mal'cev's proof of compactness theorem [13, theorem 8.3.1].) Recall that \mathbf{P} denotes the direct product and \mathbf{P}_U the ultraproduct class operators.

Lemma 2.3. *Let $\sigma = \forall X \bigwedge_{i \in I} \varphi_i \rightarrow \bigvee_{j \in J} \psi_j$ be a disjunctive universal sentence. Assume that the class \mathcal{C} satisfies σ . If \mathcal{C} is*

- (1) \mathbf{P} -closed, then $\mathcal{C} \models \forall X \bigwedge_{i \in I} \varphi_i \rightarrow \psi_j$ for some $j \in J$,
- (2) \mathbf{P}_U -closed, then $\mathcal{C} \models \forall X \bigwedge_{i \in I_0} \varphi_i \rightarrow \bigvee_{j \in J_0} \psi_j$ for some finite subsets $I_0 \subseteq I$ and $J_0 \subseteq J$.

Corollary 2.4. *Let \mathcal{C} be a class of structures. Then \mathcal{C} is*

- (1) \mathbf{S} and \mathbf{P} -closed if and only if it is definable by implications (Shafaat [14]);
- (2) \mathbf{S} and \mathbf{P}_U -closed if and only if it is definable by disjunctive universal first-order sentences (Łoś, Tarski [6, Theorem 5.2.4]);
- (3) \mathbf{HS} and \mathbf{P} -closed if and only if it is definable by identities (Birkhoff [4, Section 8]);
- (4) \mathbf{HS} and \mathbf{P}_U -closed if and only if it is definable by positive disjunctive universal first-order sentences ([6, Exercise 3.2.2]);
- (5) is \mathbf{S} , \mathbf{P} and \mathbf{P}_U -closed if and only if it is definable by quasi-identities (Mal'cev [13, Theorem 11.1.2]).

Proof. It follows by

- (1) Theorem 2.2 point (1) and Lemma 2.3 point (1);
- (2) Theorem 2.2 point (1) and Lemma 2.3 point (2);
- (3) Theorem 2.2 point (2) and Lemma 2.3 point (1);
- (4) Theorem 2.2 point (2) and Lemma 2.3 point (2);
- (5) Theorem 2.2 point (1) and Lemma 2.3 points (1) and (2). □

The above proof is non-constructive. The original proof of Birkhoff's theorem has a different character. It is longer, but also has an advantage that it connects equational theories with free algebras. Similarly for Mal'cev's theorem, there is a connection of a quasi-equational theories and finitely presented algebras. It is used in categorical generalizations of Mal'cev's theorem [1, Section 16]. Although proofs of Mal'cev's theorem presented in most textbooks are, as our, non-constructive.

3. Other applications

It appears that the closure under other unary class operators leads to various restrictions on the defining sentences.

3.1. Restrictions on the negative part

In the previous section we had an extreme situation. The unary class operator \mathbf{HS} completely eliminated the negative part. But we may consider weaker restrictions: forbidding occurrences of the equality symbol \approx or occurrences of relational symbols different than \approx , called from now on just *relational symbols*. This may be achieved by considering various types of homomorphisms.

Let L_R be the set of relational symbols in L (i.e., non-functional symbol different than \approx). For $\mathbf{A} \in \text{Struc}(L)$ and $R \in L_R$ let $R^{\mathbf{A}}$ denotes the interpretation of R in \mathbf{A} .

We say that \mathbf{A} is a *relational expansion* of \mathbf{B} if \mathbf{A} and \mathbf{B} have the same algebraic reduct and $R^{\mathbf{B}} \subseteq R^{\mathbf{A}}$ for every $R \in L_R$. Thus, informally speaking, \mathbf{A} is obtained from \mathbf{B} by adding new tuples of elements from A to relations. Let $\text{H}_E(\mathbf{A})$ be the class of structures isomorphic to relational expansions of \mathbf{A} .

We say that a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ is *strict* if $R^{\mathbf{A}} = h^{-1}(R^{\mathbf{B}})$ for every $R \in L_R$. More formally, if for every relational symbol $R \in L_R$ of arity n we have

$$\mathbf{A} \models R(a_0, \dots, a_{n-1}) \quad \text{iff} \quad \mathbf{B} \models R(h(a_0), \dots, h(a_{n-1})).$$

Strict homomorphisms appear naturally in abstract algebraic logic [7], see also [8]. Let $\text{H}_{\text{Str}}(\mathbf{A})$ be the class of all strict homomorphic images of \mathbf{A} .

Theorem 3.1. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is*

- (1) H_ES -closed if and only if it is definable by disjunctive universal sentences whose negative part has no occurrences of relational symbols;
- (2) $\text{H}_{\text{Str}}\text{S}$ -closed if and only if it is definable by disjunctive universal sentences whose negative part has no occurrences of \approx .

Proof. The reasoning here is the same as in the proof of Theorem 2.2. We only provide characteristic sentences. So the sentences

$$\chi_{\mathbf{A}}^{\text{H}_E\text{S}} = \exists X_A \bigwedge_{\varphi \in \text{Diag}^-(\mathbf{A})} \neg \varphi \quad \wedge \quad \bigwedge_{\substack{\psi \in \text{Diag}^+(\mathbf{A}) \\ \psi \text{ has no occurrences} \\ \text{of relational symbols}}} \psi$$

are characteristic for H_ES . Note that $\chi^{\text{H}_E\text{S}}$ may be reduced by adding the condition in the first big conjunct that φ has no occurrences of functional symbols. Further, the sentences

$$\chi_{\mathbf{A}}^{\text{H}_{\text{Str}}\text{S}} = \exists X_A \bigwedge_{\varphi \in \text{Diag}^-(\mathbf{A})} \neg \varphi \quad \wedge \quad \bigwedge_{\substack{\psi \in \text{Diag}^+(\mathbf{A}) \\ \psi \text{ has no occurrences} \\ \text{of } \approx}} \psi$$

are characteristic for $\text{H}_{\text{Str}}\text{S}$. □

Corollary 3.2. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is $\text{H}_E\text{S} / \text{H}_{\text{Str}}\text{S}$ and $\text{P}_U / \text{P} / (\text{P}, \text{P}_U)$ -closed if and only if it is definable by first-order disjunctive universal sentences/ implications / quasi-identities whose negative part has no occurrences of relational symbols / the equality symbol \approx .*

Proof. It follows by Theorem 3.1 and Lemma 2.3. □

In what follows, every theorem would be accompanied by a statement like Corollary 3.2. They are derivable in a straightforward way. Thus we omit them.

There is a type of homomorphism commonly occurring in the literature, mostly implicitly, with the connection to congruences. We say that a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ is *strong* if $R^{\mathbf{B}} = h(R^{\mathbf{A}})$ for every $R \in L_R$. More

precisely, if $\mathbf{B} \models R(b_0, \dots, b_{n-1})$, then there are $a_0, \dots, a_{n-1} \in A$ such that $\mathbf{A} \models R(a_0, \dots, a_{n-1})$ and $h(a_i) = b_i$ for all $i < n$. Informally, if there is a tuple in a relation of \mathbf{B} , then there is a witness of it in \mathbf{A} . Note that if γ is a congruence on the algebraic reduct of \mathbf{A} , then there is a unique structure \mathbf{A}/γ such that $a \mapsto a/\gamma$ is a strong homomorphism from \mathbf{A} onto \mathbf{A}/γ . On the other side, if $h: \mathbf{A} \rightarrow \mathbf{B}$ is a surjective strong homomorphism, then \mathbf{B} is isomorphic to \mathbf{A}/γ , where γ is the kernel of h . Let $H_{\text{Sng}}(\mathbf{A})$ be the class of all strong homomorphic images of \mathbf{A} .

It appears that the closure under H_{Sng} is very restrictive. The following fact was observed in [2, Section 5, Point (3)]. Our proof falls into the presented scheme. We say that a disjunctive universal sentence is *weak* if in its negative part every variable appears at most once and there are no occurrences of \approx nor of functional symbols.

Theorem 3.3. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is $H_{\text{Sng}}\mathcal{S}$ -closed if and only if it is definable by weak disjunctive universal sentences.*

Proof. Unlike in the previous proofs we also verify the “if” part. Satisfaction of every universal sentence is preserved when taking substructures. Thus it is enough to show the preservation under H_{Sng} . Let us consider a weak disjunctive universal sentence

$$\sigma = \forall X \bigwedge_{i \in I} \varphi_i \rightarrow \bigvee_{j \in J} \psi_j$$

and a strong surjective homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$. Assume that $\mathbf{A} \models \sigma$. In order to verify that $\mathbf{B} \models \sigma$ we need to consider a valuation $\nu: X \rightarrow B$ such that $\mathbf{B} \models \bigwedge_{i \in I} \varphi_i[\nu]$, and show that $\mathbf{B} \models \bigvee_{j \in J} \psi_j[\nu]$. Let us observe that there exists a valuation $\mu: X \rightarrow A$ such that

- $\nu = h \circ \mu$,
- for every $i \in I$, we have $\mathbf{A} \models \varphi_i[\mu]$.

Indeed, we may define μ as follows. Let X_i be the set of variables occurring in φ_i . Firstly, we define $\mu_i: X_i \rightarrow A$. Assume that $\varphi_i = R(x_0, \dots, x_{n-1})$. Then $X_i = \{x_0, \dots, x_{n-1}\}$. Since $\mathbf{B} \models R(\nu(x_0), \dots, \nu(x_{n-1}))$ and h is strong, there are $a_0, \dots, a_{n-1} \in A$ such that $\mathbf{A} \models R(a_0, \dots, a_{n-1})$ and $h(a_i) = \nu(x_i)$ for $i < n$. We put $\mu_i(x_i) = a_i$. Since variables x_0, \dots, x_{n-1} are mutually distinct, the definition of μ_i is correct. Furthermore, since $X_i \cap X_{i'} = \emptyset$ for $i \neq i'$, we may define $\mu(x) = \mu_i(x)$ when $x \in X_i$ for some $i \in I$. Finally, if x is a variable which does not appear in the negative part of σ , then as $\mu(x)$ we take any element from $h^{-1}(\nu(x))$.

Since $\mathbf{A} \models \sigma$ and $\mathbf{A} \models \bigwedge_{i \in I} \varphi_i[\mu]$, we have $\mathbf{A} \models \psi_j[\mu]$ for some $j \in J$. Hence, since h is a homomorphism, we have $\mathbf{B} \models \psi_j[h \circ \mu]$. This shows that $\mathbf{B} \models \bigvee_{j \in J} \psi_j[\nu]$.

Let us move to the “only if” part. As previously, we just present characteristic sentences. Let

$$A^\circ = A \dot{\cup} \{(R, a_0, \dots, a_{n-1}, i) \mid R \in L_R \text{ of arity } n, a_0, \dots, a_{n-1} \in A, \\ \mathbf{A} \models R(a_0, \dots, a_{n-1}), i < n\},$$

where $\dot{\cup}$ denotes disjoint union. Let X_A° be a superset of X_A of variables of cardinality $|A^\circ|$. Let us extend the mapping π_A to a bijection $\pi_A^\circ: A^\circ \rightarrow X_A^\circ$. Let $\xi_A: A^\circ \rightarrow A$ be the mapping given by $\xi_A((R, a_0, \dots, a_{n-1}, i)) = a_i$ and $\xi_A(a) = a$ for $a \in A$. Let $\rho_A: X_A^\circ \rightarrow A$ be given by $\rho_A = \xi_A \circ (\pi_A^\circ)^{-1}$. Then the sentences

$$\begin{aligned} \chi_{\mathbf{A}}^{\mathbf{H}_{\text{Sng}}\mathbf{S}} = & \exists X_A^\circ \bigwedge_{\substack{\varphi(\bar{x}) \in \text{At}(X_A^\circ) \\ \mathbf{A} \not\models \varphi(\rho_A(\bar{x}))}} \neg \varphi \\ & \wedge \bigwedge_{\mathbf{A} \models R(\bar{a})} R(\pi_A^\circ((R, \bar{a}, 0)), \dots, \pi_A^\circ((R, \bar{a}, n-1))) \end{aligned}$$

are characteristic for $\mathbf{H}_{\text{Sng}}\mathbf{S}$. □

Let us argue that the closure under $\mathbf{H}_{\text{Sng}}\mathbf{S}$ is a restrictive condition. Assume that our fixed language L is finite and has only relational symbols. Then there are only finitely many subvarieties (HSP-closed classes, i.e., classes defined by identities) of $\text{Struc}(L)$. Indeed, there are, up to logical equivalence, only finitely many identities in L . Hence considering varieties for relational structures is not interesting. What we gain when we shift to $\mathbf{H}_{\text{Sng}}\mathbf{SP}$ -closed classes? Actually, not too much. There are still only finitely many such classes.

In order to see this let us recall the notion of relative subdirect irreducibility. Let \mathcal{C} be a \mathbf{S} -closed subclass of $\text{Struc}(L)$ and \mathbf{S} be a structure in \mathcal{C} . We say that \mathbf{S} is \mathcal{C} -subdirectly irreducible if for every embedding $e: \mathbf{S} \rightarrow \prod_{i \in I} \mathbf{A}_i$, where $\mathbf{A}_i \in \mathcal{C}$, there is a projection $\pi_k: \prod_{i \in I} \mathbf{A}_i \rightarrow \mathbf{A}_k$ such that the composition $\pi_k \circ e: \mathbf{S} \rightarrow \mathbf{A}_k$ is an embedding. We say that \mathbf{S} is \mathbf{H}_{Sng} -subdirectly irreducible if \mathbf{S} is $\mathbf{H}_{\text{Sng}}\mathbf{SP}(\mathbf{S})$ -subdirectly irreducible. Then \mathbf{S} is \mathcal{C} -subdirectly irreducible for some $\mathbf{H}_{\text{Sng}}\mathbf{SP}$ -closed class \mathcal{C} iff it is \mathbf{H}_{Sng} -subdirectly irreducible. Recall also a known fact that if \mathcal{C} is a quasivariety ($\mathbf{SPP}_{\mathbf{U}}$ -closed class) then every algebra in \mathcal{C} embeds into a product of \mathcal{C} -subdirectly irreducible algebras [10, Theorem 3.1.1]. Every \mathbf{H}_{Sng} and \mathbf{P} -closed class is $\mathbf{P}_{\mathbf{U}}$ -closed. Hence every $\mathbf{H}_{\text{Sng}}\mathbf{SP}$ -closed class \mathcal{C} is a quasivariety and the recalled fact may be applied to \mathcal{C} . We thus infer that every structure embeds into a product of \mathbf{H}_{Sng} -subdirectly irreducible structures.

The next proposition shows that if L is finite and has no functional symbols, then there are, up to isomorphism, only finitely many \mathbf{H}_{Sng} -subdirectly irreducible L -structures. Consequently, by the conclusion from the previous paragraph, there are only finitely many $\mathbf{H}_{\text{Sng}}\mathbf{SP}$ -closed classes for such L .

Proposition 3.4. *Assume that L has only relational symbols and that their arities are bounded by m . Assume that \mathbf{S} is a \mathbf{H}_{Sng} -subdirectly irreducible structure in L . Then $|S| \leq \max(m + 1, 2)$.*

Proof. Let \mathbf{A} be a structure in L . Then every equivalence relation γ on A induces the strong homomorphism $\eta_\gamma: \mathbf{A} \rightarrow \mathbf{A}/\gamma$; $a \mapsto a/\gamma$. For every pair a, b of distinct elements of A let $\gamma_{a,b}$ be an equivalence relation on A such that $(a, b) \notin \alpha$ and $|A/\alpha| = 2$. For every $R \in L_R$ of arity n and every tuple $\bar{a} = (a_0, \dots, a_{n-1})$ such that $\mathbf{A} \not\models R(a_0, \dots, a_{n-1})$ let $\gamma_{R,\bar{a}}$ be the equivalence

relation given by $(A - \{a_0, \dots, a_{n-1}\})^2 \cup \{(a_0, a_0), \dots, (a_{n-1}, a_{n-1})\}$. Then $|A/\gamma_{R,\bar{a}}| \leq n + 1$. We have

$$\mathbf{A}/\gamma_{a,b} \models a/\gamma_{a,b} \not\approx b/\gamma_{a,b} \quad \text{and} \quad \mathbf{A}/\gamma_{R,\bar{a}} \models \neg R(a_0/\gamma_{R,\bar{a}}, \dots, a_{n-1}/\gamma_{R,\bar{a}}).$$

This yields that \mathbf{A} embeds into the product

$$\prod_{\mathbf{A} \not\models a \approx b} \mathbf{A}/\gamma_{a,b} \quad \times \quad \prod_{\mathbf{A} \not\models R(\bar{a})} \mathbf{A}/\gamma_{R,\bar{a}}$$

of structures each of which is a strong homomorphic image of \mathbf{A} and has the carrier of size at most $\max(m + 1, 2)$. □

However, it is not difficult to see that there are denumerably many $\mathbf{H}_{\text{SngSPU}}$ -closed classes if the language has no functional neither relational symbols. Indeed for every $n > 0$ the sentence

$$\sigma_n = \forall x_0, \dots, x_n \ x_0 \approx x_1 \vee x_0 \approx x_1 \vee \dots \vee x_{n-1} \approx x_n$$

expresses “there are at most n elements”. Thus they are mutually logically nonequivalent. Also, there are continuum many quasivarieties generated by simple graphs [5, Theorem 2].

Example 3.5. Let us discuss transitivity and anti-reflexivity for binary relations. Both properties are commonly expressed by quasi-identities without occurrences of \approx in the negative part. In order to see that it is impossible to express them by weak quasi-identities, let us consider a structure $\mathbf{A} = (A, R)$ where $A = \{a, b, c, d\}$ and $R = \{(a, b), (c, d)\}$. Let $\gamma = \{(b, c), (c, b)\} \cup D$ and $\delta = \{(a, d), (d, a), (b, c), (c, b)\} \cup D$, where D is the diagonal in A^2 . Then the strong homomorphic image \mathbf{A}/γ is not transitive and \mathbf{A}/δ is not antisymmetric.

We have the same situation in case of ordered algebras. Let us consider ordered semigroups. Let $\mathbf{N} = (\mathbb{N}, +, \leq)$ be the structure of natural numbers with the standard addition and order. Then for a congruence γ on $(\mathbb{N}, +)$ the strong homomorphic image \mathbf{N}/γ is transitive but it does not have to be antisymmetric. Indeed, it is antisymmetric iff γ has at most one nontrivial class. Also transitivity is not preserved by strong homomorphisms for ordered semigroup. Let $\mathbf{S} = (\mathbb{N}^4, +, \sqsubseteq)$ be a structure, where $+$ is the standard componentwise addition and be the order \sqsubseteq is given in the following way: $(r_0, r_1, r_2, r_3) \sqsubseteq (s_0, s_1, s_2, s_3)$ iff $r_0 + r_1 = s_0 + s_1$, $r_2 + r_3 = s_2 + s_3$ and $r_1 \leq s_1$, $r_3 \leq s_3$. It is a least order on \mathbb{N}^4 which is compatible with the addition and such that $e_0 \leq e_1$, $e_2 \leq e_3$, where e_i is the quadruple with three 0s and with 1 in the i 's coordinate. Let γ be the congruence on $(\mathbb{N}^4, +)$ generated by (e_1, e_2) . Then in \mathbf{S}/γ we have $e_0/\gamma \sqsubseteq e_1/\gamma = e_2/\gamma \sqsubseteq e_3/\gamma$. But, since $e_0/\gamma = \{e_0\}$, $e_3/\gamma = \{e_3\}$ and not $e_0 \sqsubseteq e_3$, we do not have $e_0/\gamma \sqsubseteq e_3/\gamma$.

The restrictive character of \mathbf{H}_{SngS} leads us to the following question. Is there a notion weaker than strict homomorphism and stronger than strong homomorphism which corresponds to disjunctive universal sentences without occurrences of \approx nor of functional symbols in the negative part but allowing

repetitions of variables? In particular, what condition on strong homomorphisms ensures that transitivity and antisymmetry are preserved? The answer is provided by the next theorem.

We say that a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ is *uniformly strong* if there is an embedding $e: \mathbf{B}^{rel} \rightarrow \mathbf{A}^{rel}$ of the relational reduct of \mathbf{B} into the relational reduct of \mathbf{A} such that $h \circ e$ is the identity mapping on B . The intuition behind this definition is that for a uniformly strong homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ for all tuples in the relations of \mathbf{B} there are witnesses for them given in a uniform way. More formally, if $\mathbf{B} \models R(b_0, \dots, b_{n-1})$, then $\mathbf{A} \models R(e(b_0), \dots, e(b_{n-1}))$. Note that for relational structures uniformly strong homomorphisms are just retractions. Hence $H_{uSng}S = S$ and no new theorem is obtained in this case. Let $H_{uSng}(\mathbf{A})$ be the class of uniformly strong homomorphic images of \mathbf{A} . As in Theorems 2.2 and 3.1, we obtain the following fact.

Theorem 3.6. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is $H_{uSng}S$ -closed if and only if it is definable by disjunctive universal sentences without occurrences of \approx and functional symbols in their negative part.*

3.2. Restrictions on the positive part and on both parts

Note that the situation is not symmetric in the following sense. For every disjunctive universal sentence there is a logically equivalent disjunctive universal sentence without functional symbols and repetitions of variables in the positive part and without equations of the form $x \approx y$ in the negative part. Thus, considering restrictions on the positive part, we only provide analogs of Theorems 2.2 and 3.1.

If O is a unary class operator, then O^{-1} is the unary class operator given by: $\mathbf{A} \in O^{-1}(\mathbf{B})$ iff $\mathbf{B} \in O(\mathbf{A})$. For instance, $H_E^{-1}S$ is a very common operation in graph theory. It is there the operation of taking (not necessarily induced) subgraphs.

Theorem 3.7. *Let \mathcal{C} be a class of L -structures. Then \mathcal{C} is*

- (1) $H^{-1}S$ -closed if and only if it is definable by negative disjunctive universal sentences (see Gorbunov, Kravchenko [11, Theorem 1.1] for the first-order case);
- (2) $H_E^{-1}S$ -closed if and only if it is definable by disjunctive universal sentences whose positive part is a disjunction of equations of variables (see [15, Appendix] for such quasi-identities).
- (3) $H_{Str^{-1}}S$ -closed if and only if it is definable by disjunctive universal sentences without occurrences of \approx and functional symbols in their positive part;

Proof. We provide characteristic sentences for Points (2) and (3). So the sentences

$$\chi_{\mathbf{A}}^{H_E^{-1}S} = \exists X_A \bigwedge_{\mathbf{A} \not\models a \approx b} \neg \pi_A(a) \approx \pi_A(b) \wedge \bigwedge_{\psi \in \text{Diag}^+(\mathbf{A})} \psi$$

are characteristic for $H_E^{-1}S$ and

$$\chi_{\mathbf{A}}^{H_{Str}^{-1}S} = \exists X_A \bigwedge_{\mathbf{A} \neq R(a_0, \dots, a_{n-1})} \neg R(\pi_A(a_0), \dots, \pi_A(a_{n-1})) \wedge \bigwedge_{\psi \in \text{Diag}^+(\mathbf{A})} \psi$$

are characteristic for $H_{Str}^{-1}S$. (The definition of π_A in on Page 3.) □

We may also put restrictions on both sides. The following theorem summarizes such results, together with theorems presented till now. For a unary class operator O let $R(O)$ be given as in the table below. Here l is the isomorphic image class operator.

O	$R(O)$
l	\emptyset
H_E	relational symbols
H_{Str}	\approx
H_{uSng}	\approx , functional symbols
H_{Sng}	\approx , functional symbols, repetitions of variables
H	any symbols

Theorem 3.8. *Let $O_1 \in \{l, H_E, H_{Str}, H\}$ and $O_2 \in \{l, H_E, H_{Str}, H_{uSng}, H_{Sng}, H\}$. Then a class of L -structures $O_1^{-1}O_2S$ -closed if and only if it is definable by disjunctive universal sentences without occurrences of $R(O_1)$ in the positive part and without occurrences of $R(O_2)$ in the negative part.*

With every pair of unary class operators O_1, O_2 in Theorem 3.8 in the table there are connected additional statements analogical to Corollary 3.2 which concern the closure under P and P_U . Thus in total, we obtain $(4 \times 6 - 1) \times 4 - 17 = 75$ statements. We subtract 1 since the case when $O_1 = O_2 = H$ is trivial (we obtain the class of all L -structures). We subtract 17 since whenever a class is H_{Sng} and P -closed, then it is P_U -closed, and whenever it is H^{-1} -closed, then it is P -closed (for example a $H_{Sng}H^{-1}$ -closed class is PP_U -closed).

The proofs of statements which were not considered earlier fall into the same schema but have more complicated details. Let us finish the paper with an example of such reasoning.

Proof. (Sample proof) We argue for the $HH_{Str}^{-1}S$ unary class operator. We check that the sentences

$$\chi_{\mathbf{A}}^{HH_{Str}^{-1}S} = \exists X_A \bigwedge_{\substack{\varphi \in \text{Diag}^-(\mathbf{A}) \\ \varphi \text{ has no occurrences of } \approx}} \neg \varphi$$

are characteristic for $HH_{Str}^{-1}S$. So let us assume that there are homomorphisms $h: \mathbf{C} \rightarrow \mathbf{A}$ and $g: \mathbf{C} \rightarrow \mathbf{B}$ such that h is surjective and g is strict. Let $\nu: X_A \rightarrow B$ be a valuation such that $\nu(\pi_A(a)) = g(c)$, where c is any element from $h^{-1}(a)$. Then $\mathbf{B} \models \neg \varphi[\nu]$ for every $\varphi \in \text{Diag}^-(\mathbf{A})$ without occurrences of \approx . Thus $\mathbf{B} \models \chi_{\mathbf{A}}^{HH_{Str}^{-1}S}$.

For the inverse implication, assume that $\nu: X_A \rightarrow B$ is a valuation such that $\mathbf{B} \models \varphi[\nu]$ for every $\varphi \in \text{Diag}^-(\mathbf{A})$ without occurrences of \approx . We want to find homomorphisms $h: \mathbf{C} \rightarrow \mathbf{A}$ and $g: \mathbf{C} \rightarrow \mathbf{B}$ such that h is surjective and g is strict. Let the algebraic reduct \mathbf{C}^{alg} of \mathbf{C} be the algebra of terms with variables in X_A . Let $h: \mathbf{C}^{alg} \rightarrow \mathbf{A}^{alg}$ be a unique algebraic homomorphism extending $(\pi_A)^{-1}$. Similarly, let $g: \mathbf{C}^{alg} \rightarrow \mathbf{B}^{alg}$ be a unique algebraic homomorphism extending ν . We uniquely define the relational reduct of \mathbf{C} in such a way that g is a strict homomorphism. Then, by assumption on $\chi_{\mathbf{A}}^{\text{HH}_{\text{Str}}^{-1}\text{S}}$, h is a homomorphism. For the $\text{H}_{\text{Str}}\text{H}_{\text{Str}}^{-1}\text{S}$ -case one may also consult [8, Lemma 5]. \square

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References

- [1] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and Concrete Categories: The Joy of Cats. Pure and Applied Mathematics. Wiley, New York (1990)
- [2] Andréka, H., Németi, I.: Formulas and ultraproducts in categories. *Beiträge zur Algebra und Geometrie* **8**, 133–151 (1979)
- [3] Bell, J.L.: Infinitary logic. In: Zalta, E.N. (ed.) *The Stanford Encyclopedia of Philosophy*, winter, 2016th edn. Stanford University, Metaphysics Research Lab (2016)
- [4] Birkhoff, G.: On the structure of abstract algebras. *Proc. Camb. Philos. Soc.* **31**, 433–454 (1935)
- [5] Caicedo, X.: Finitely axiomatizable quasivarieties of graphs. *Algebra Universalis* **34**, 314–321 (1995)
- [6] Chang, C.C., Keisler, H.J.: *Model Theory*, 3rd edn. Studies in Logic and the Foundations of Mathematics, vol. 73. North-Holland Publishing Co., Amsterdam (1990)
- [7] Czelakowski, J.: *Protoalgebraic Logics*, Trends in Logic, Studia Logica Library, vol. 10. Kluwer Academic Publishers, Dordrecht (2001)

- [8] Dellunde, P., Jansana, R.: Some characterization theorems for infinitary universal Horn logic without equality. *J. Symbolic Logic* **61**, 1242–1260 (1996)
- [9] Givant, S., Halmos, P.: *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. Springer, New York (2009)
- [10] Gorbunov, V.A.: *Algebraic Theory of Quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York (1998)
- [11] Gorbunov, V.A., Kravchenko, A.V.: Universal Horn classes and antivarieties of algebraic systems. *Algebra Log* **39**(120), 3–22 (2000)
- [12] Just, W., Weese, M.: *Discovering Modern Set Theory. I The Basics*. Graduate Studies in Mathematics, vol. 8. American Mathematical Society, Providence (1996)
- [13] Mal'cev, A.I.: *Algebraic Systems*. Springer, New York (1973)
- [14] Shafaat, A.: On implicationally defined classes of algebras. *J. Lond. Math. Soc.* **44**, 137–140 (1969)
- [15] Stronkowski, M.M.: Quasi-equational bases for graphs of semigroups, monoids and groups. *Semigroup Forum* **82**, 296–306 (2011)

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