Algebra Univers. (2018) 79:31

1420-8911/18/020001-21
published online April 20, 2018
https://doi.org/10.1007/s00012-018-0497-9

Algebra Universalis

Unique inclusions of maximal C-clones in maximal clones

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Abstract. C-clones are polymorphism sets of so-called clausal relations, a special type of relations on a finite domain, which first appeared in connection with constraint satisfaction problems in work by Creignou et al. from 2008. We completely describe the relationship regarding set inclusion between maximal C-clones and maximal clones. As a main result we obtain that for every maximal C-clone there exists exactly one maximal clone in which it is contained. A precise description of this unique maximal clone, as well as a corresponding completeness criterion for C-clones is given.

Mathematics Subject Classification. 08A40, 08A02, 08A99.

Keywords. Clone, C-clone, Clausal relation, Maximal C-clone, Maximal clone.

1. Introduction

Clones are sets of operations on a fixed domain that are closed under composition and contain all projections. The clones on a finite set D are precisely the Galois closed sets of operations ([7], translated in [5,6], independently [9]) with respect to the well-known *Galois connection* $\mathrm{Pol}_D - \mathrm{Inv}_D$ induced by the relation "an operation f *preserves* a relation ϱ" (see also [10,11]). In other words, every clone F on D can be described by $F = \mathrm{Pol}_D\, Q$ for some set Q of relations (cf. Section 2 for the notation).

In this paper, which is mainly based on [4], we continue the investigations from [2] and [16] concerning clones on a finite set D described by relations

Presented by M. Maróti.
The research of M. Behrisch was partially supported by the Austrian Science Fund (FWF) under Grant I836-N23 and partially by the OeAD KONTAKT Project CZ 04/2017 "Ordered structures for non-classical logics"; the one of E. Vargas-García partially by CONACYT Grant no. 207510 and partially by the Asociación Mexicana de Cultura A.C.

from a special set $C\,\mathrm{R}_D$. They are named *clausal relations* and were originally introduced in [8]. A clausal relation is the set of all tuples over D satisfying disjunctions of inequalities of the form $x \geq d$ and $x \leq d$, where x, d belong to the finite set $D = \{0, 1, \ldots, n-1\}$.

We are interested in understanding the structure of clones that are determined by sets of clausal relations, so-called *C-clones*. Their lattice has been delineated completely in Theorem 2.14 of [16] for the case that $|D| = 2$. When $|D| \geq 3$, the structure and even the cardinality of this lattice is largely unknown. In this paper we study the co-atoms in the lattice of all C-clones, the *maximal C-clones*, for an arbitrary finite set D. Since every clone on D either equals O_D (the set of all finitary operations on D) or is contained in some *maximal clone* (co-atom of the lattice of all clones) (see, e.g., [12, Hauptsatz 3.1.5, p. 80; Vollständigkeitskriterium 5.1.6, p. 123] or [15, Proposition 1.15, p. 27]), our aim is to investigate which maximal C-clones are contained in which maximal clones. We achieve a complete description in Theorem 8.2 and thereby answer the question that was left open in the pre-print [3].

Using Rosenberg's theorem (see Theorem 2.4 below), all maximal clones on D can be classified into six types. In [3] it was already established that a few of them, e.g., centralisers of prime permutations, polymorphism sets of an affine, of a central relation of arity at least three or of an h-regular relation, do not contain any maximal C-clone. We shall see that this phenomenon extends to maximal clones of monotone functions with regard to some bounded partial order whenever $|D| \geq 3$.

To our surprise, it turns out that every maximal C-clone is contained in a unique maximal clone, either given as polymorphism set of a non-trivial equivalence relation or a unary or binary central relation (vide infra for a definition of such relations). The respective details can be seen from our main result, Theorem 8.2. As a corollary we also deduce a new completeness criterion for C-clones.

We start by introducing our notation, recalling some fundamental facts about the Galois theory for clones, the characterisation of maximal clones and C-clones, respectively, and providing two basic lemmas in Section 2. After that we recollect the relevant results from the pre-print [3]. Then we devote one section each to examine possible inclusions of maximal C-clones in maximal clones of the form $\mathrm{Pol}_D\,\varrho$, where ϱ is a non-trivial unary relation, a bounded partial order relation, a non-trivial equivalence relation or an at least binary central relation. Finally, in Section 8, we deduce our main theorem from the previous results.

2. Main notions and preliminaries

Throughout the text, D will denote the finite non-empty set $\{0, \ldots, n-1\}$ $(n > 0)$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of *natural numbers*. Put $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. Moreover, if $f\colon A \to B$ is a function, we write $\mathrm{im}(f) := \{f(x) \mid x \in A\}$ for its *image*.

Let $m \in \mathbb{N}_+$. An *m-ary relation* ϱ on D is a subset of the m-fold Cartesian product D^m. By $\mathrm{R}_D^{(m)} := \mathfrak{P}(D^m)$ we refer to the *set of all m-ary relations* on D and by $\mathrm{R}_D := \bigcup_{m\in\mathbb{N}_+} \mathrm{R}_D^{(m)}$ to the *set of all finitary relations* on the set D. Furthermore, for a binary relation $\varrho \subseteq D^2$ we denote its *inverse relation* by $\varrho^{-1} := \{(y,x) \mid (x,y) \in \varrho\}$.

We want to study clones that are determined by sets of clausal relations. Even though, for almost all results, we shall need only binary clausal relations, we define them here in full generality (the publication [8] allowed the parameters p and q to be chosen in $\mathbb{N}$ such that $p+q>0$; for compatibility with [2,16,17] we are slightly more restrictive here, by additionally requiring that $p \cdot q > 0$).

Definition 2.1. Let $p, q \in \mathbb{N}_+$. For given parameters $\mathbf{a} = (a_1,\dots,a_p) \in D^p$ and $\mathbf{b} = (b_1,\dots,b_q) \in D^q$, the *clausal relation* $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ *of arity* $p+q$ is the set of all tuples $(x_1,\dots,x_p,y_1,\dots,y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \dots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \dots \vee (y_q \leq b_q).$$

In this expression $\leq$ denotes the canonical linear order on D and $\geq$ its dual.

For $k \in \mathbb{N}_+$ we denote by $\mathrm{O}_D^{(k)} := \{f \mid f\colon D^k \to D\}$ the *set of all k-ary operations on D* and by $\mathrm{O}_D := \bigcup_{k\in\mathbb{N}_+} \mathrm{O}_D^{(k)}$ the *set of all finitary operations on D*.

Next, we consider a Galois connection between sets of operations and relations that is based on the so-called *preservation relation*. It is the most important tool for our investigations.

Definition 2.2. Let $m, k \in \mathbb{N}_+$. We say that a k-ary operation $f \in \mathrm{O}_D^{(k)}$ *preserves* an m-ary relation $\varrho \in \mathrm{R}_D^{(m)}$, denoted by $f \rhd \varrho$, if whenever

$$r_1 = (a_{11},\dots,a_{m1}) \in \varrho, \dots, r_k = (a_{1k},\dots,a_{mk}) \in \varrho,$$

it follows that also f applied to these tuples belongs to ϱ, i.e.,

$$f \circ (r_1,\dots,r_k) := (f(a_{11},\dots,a_{1k}),\dots,f(a_{m1},\dots,a_{mk})) \in \varrho.$$

Given $F \subseteq \mathrm{O}_D$, we denote by $\mathrm{Inv}_D\, F$ the set of all relations that are *invariant* for all operations $f \in F$, i.e., $\mathrm{Inv}_D\, F := \{\varrho \in \mathrm{R}_D \mid \forall f \in F\colon f \rhd \varrho\}$. Similarly, for a set $Q \subseteq \mathrm{R}_D$ of relations, $\mathrm{Pol}_D\, Q := \{f \in F \mid \forall \varrho \in Q\colon f \rhd \varrho\}$ denotes the *set of polymorphisms of* Q. Furthermore, for $k \in \mathbb{N}_+$ we abbreviate $\mathrm{Pol}_D^{(k)}\, Q := \mathrm{O}_D^{(k)} \cap \mathrm{Pol}_D\, Q$. Usually, we shall write $\mathrm{Pol}_D\, \varrho$ for $\mathrm{Pol}_D\{\varrho\}$, $\varrho \in \mathrm{R}_D$ and $\mathrm{Inv}_D\, f$ for $\mathrm{Inv}_D\{f\}$, $f \in \mathrm{O}_D$. The operators Pol_D and Inv_D define the Galois connection $\mathrm{Pol}_D - \mathrm{Inv}_D$.

On a finite set D the Galois closed sets of relations [7,5,6,9] with respect to $\mathrm{Pol}_D - \mathrm{Inv}_D$ are exactly the so-called *relational clones*. These can be characterised as those sets of finitary relations on D that are closed under *primitive positively definable relations*, i.e., those arising as interpretations of first order formulæ where only predicate symbols corresponding to relations from Q, falsity, variable identifications, finite conjunctions and finite existential quantification are allowed. For a set $Q \subseteq \mathrm{R}_D$ of relations, we denote by $[Q]_{\mathrm{R}_D}$

the closure of Q with regard to such formulæ, which equals the least relational clone generated by Q, i.e., by the above, we have $[Q]_{\mathrm{R}_D} = \mathrm{Inv}_D \mathrm{Pol}_D Q$.

A relation $\varrho \in \mathrm{R}_D$ is called *trivial* if it is preserved by every function, i.e., if $\mathrm{Pol}_D \varrho = \mathrm{O}_D$, or equivalently $\varrho \in \mathrm{Inv}_D \mathrm{O}_D$. The set of trivial relations $\mathrm{Inv}_D \mathrm{O}_D$ can be characterised to contain precisely all so-called *diagonal relations* (see, e.g., [11, 3.2 Definitions (R0), p. 25] or [1, p. 5] for a definition), which are generalisations of the binary diagonal relations $\Delta = \{(x, x) \mid x \in D\}$ and $\nabla = D \times D$.

A set $F \subseteq \mathrm{O}_D$ of operations is called a *C-clone* if $F = \mathrm{Pol}_D Q$ for some set Q of clausal relations. All C-clones on D, ordered by set inclusion, form a complete lattice, whose co-atoms are called *maximal C-clones.*

From [17] we have a description of all maximal C-clones on finite sets as polymorphism sets of clausal relations $\mathrm{R}^{(a)}_{(b)} = \{(x, y) \in D^2 \mid x \geq a \vee y \leq b\}$.

Theorem 2.3 [17]. *Let $M \subseteq \mathrm{O}_D$ be a C-clone. M is maximal if and only if there are elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$ such that $M = \mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$.*

Likewise, the following characterisation of maximal clones on finite sets is well known.

Theorem 2.4 [13,14]. *A clone $F \subseteq \mathrm{O}_D$ is maximal if and only if it is of the form $\mathrm{Pol}_D \varrho$, where ϱ is a non-trivial relation belonging to one of the following classes:*

1. *The set of all partial orders with least and greatest element.*
2. *The set of all graphs of prime permutations.*
3. *The set of all non-trivial equivalence relations,* $\mathrm{Eq}(D) \setminus \{\Delta, \nabla\}$.
4. *The set of all affine relations with respect to some elementary Abelian p-group on D for some prime p.*
5. *The set of all central relations of arity h $(1 \leq h < |D|)$.*
6. *The set of all h-regular relations $(3 \leq h \leq |D|)$.*

For some sorts of relations from Theorem 2.4 we give a brief explanation.

If $s \in \mathrm{Sym}(D)$ is a permutation, by its *graph* we mean the binary relation $\mathrm{graph}\, s := \{(x, s(x)) \mid x \in D\}$. The permutation is called *prime* if, for some prime p, it has only cycles of length p. In particular such a function s cannot have cycles of length one, i.e., it has to be fixed point free.

For a prime p a group $\mathbf{G} = \langle G; +, -, o\rangle$ is called an *elementary Abelian p-group*, if $\mathbf{G}$ is a commutative group and satisfies the law $x + \cdots + x \approx o$ where the variable symbol x occurs p times in the sum. The latter means that every element in $G \setminus \{o\}$ has order p. If G is finite, then, by the fundamental theorem of finitely generated Abelian groups, $\mathbf{G}$ must be isomorphic to a finite direct power of the cyclic group of order p, so in particular the cardinality of G must be a power of p.

For any (not necessarily commutative) group $\mathbf{G} = \langle G; +, -, o\rangle$, we define the corresponding *affine relation* $\varrho_{\mathbf{G}} := \{(x, y, u, v) \in G^4 \mid x + y = u + v\}$. In case $\mathbf{G}$ is Abelian, $\varrho_{\mathbf{G}}$ is given by permuting the middle two variables of the graph of the Mal'cev operation $(x, y, u) \mapsto x - y + u$, hence its name.

If $h \in \mathbb{N}_{\geq 3}$ let $\iota_h := \{(a_1, \ldots, a_h) \in \{0, \ldots, h-1\}^h \mid h > |\{a_1, \ldots, a_h\}|\}$. An h-ary relation $\varrho \in \mathrm{R}_D^{(h)}$ is *h-regular*, if there exists an integer $m \geq 1$ and a surjection $\varphi\colon D \to \{0, \ldots, h-1\}^m$ such that

$$\varrho = \left\{ (a_1, \ldots, a_h) \in D^h \;\middle|\; \forall j \in \{1, \ldots, m\}\colon ((\varphi(a_1))_j, \ldots, (\varphi(a_h))_j) \in \iota_h \right\}.$$

A *central relation* is a totally symmetric, totally reflexive relation having a central element and not being a diagonal relation. *Total symmetry* means closure under all permutations of entries of tuples; *total reflexivity* requires that every tuple having two identical entries has to belong to the relation. An element $c \in D$ is *central* for ϱ if any tuple containing c as an entry is a member of ϱ.

The only unary diagonal relations are $\emptyset$ and D, the binary ones are Δ and $D \times D$. Therefore, unary central relations are precisely all proper subsets $\emptyset \subsetneq \varrho \subsetneq D$. Binary central relations can be described as follows. Note that for binary relations the notions of total symmetry and total reflexivity coincide with ordinary symmetry and reflexivity, respectively. For $c \in D$ define the relations $\varrho_c := \Delta \cup (\{c\} \times D) \cup (D \times \{c\})$ and $A_c := \left\{ (x, y) \in D^2 \setminus \varrho_c \;\middle|\; x < y \right\} = \left\{ (x, y) \in (D \setminus \{c\})^2 \;\middle|\; x < y \right\}$. For every subset $S_c \subsetneq A_c$ we have a binary central relation $\varrho_{c,S_c} := \varrho_c \cup S_c \cup S_c^{-1}$, and it is easy to see that all of them arise in this way. Note that for $n = |D| = 3$ we always have $S_c = \emptyset$ as A_c contains only one pair.

Supposing $|D| \geq 3$, the goal of the following sections is to understand completely, for which parameters $a \in D \setminus \{0\}$, $b \in D \setminus \{n-1\}$ and which relations ϱ from Theorem 2.4 we have the inclusion $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \operatorname{Pol}_D \varrho$.

To do this, we may want to use unary functions $f \in \operatorname{Pol}_D^{(1)} \mathrm{R}_{(b)}^{(a)} \setminus \operatorname{Pol}_D \varrho$ as witnesses for $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \nsubseteq \operatorname{Pol}_D \varrho$, where $\operatorname{Pol}_D \varrho$ is a maximal clone. The following lemma gives a simple sufficient condition for functions $f \in \mathrm{O}_A^{(1)}$ to preserve $\mathrm{R}_{(b)}^{(a)}$.

Lemma 2.5. *For $a, b \in D$ and every $f \in \mathrm{O}_D^{(1)}$ such that $\operatorname{im}(f) \subseteq \{0, \ldots, b\}$ or dually $\operatorname{im}(f) \subseteq \{a, \ldots, n-1\}$, we always have $f \in \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)}$.*

Proof. If $\operatorname{im}(f) \subseteq \{0, \ldots, b\}$, then we have $f(y) \leq b$ for all $(x, y) \in \mathrm{R}_{(b)}^{(a)}$ and so $f \rhd \mathrm{R}_{(b)}^{(a)}$. If $\operatorname{im}(f) \subseteq \{a, \ldots, n-1\}$, then likewise $f(x) \geq a$ for all $(x, y) \in \mathrm{R}_{(b)}^{(a)}$ and also $f \rhd \mathrm{R}_{(b)}^{(a)}$. $\square$

When constructing unary functions $f \in \operatorname{Pol}_D^{(1)} \mathrm{R}_{(b)}^{(a)} \setminus \operatorname{Pol}_D \varrho$ as witnesses for non-inclusions $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \nsubseteq \operatorname{Pol}_D \varrho$, where $\operatorname{Pol}_D \varrho$ is a maximal clone, it is helpful to know how much choice we have for f. We cannot achieve a converse to Lemma 2.5, but the following result seems to be as good as we can get in this respect.

Lemma 2.6. *For $a, b \in D$ and every $f \in \operatorname{Pol}_D^{(1)} \mathrm{R}_{(b)}^{(a)}$ the following conditions hold:*

(a) $f \rhd \{0,\dots,b\}$ *or* $\operatorname{im}(f) \subseteq \{a,\dots,n-1\}$.
(b) $f \rhd \{a,\dots,n-1\}$ *or* $\operatorname{im}(f) \subseteq \{0,\dots,b\}$.
(c) $f \rhd \{a,\dots,n-1\}$ *or* $f \rhd \{0,\dots,b\}$.

Proof. Statement (c) follows from (a) as the condition $\operatorname{im}(f) \subseteq \{a,\dots,n-1\}$ implies $f \rhd \{a,\dots,n-1\}$. The proof of statement (b) is dual to that of (a), so we only deal with the latter one. If $f \not\rhd \{0,\dots,b\}$, then there exists some $y \leq b$ such that $f(y) > b$. This means we have $(x,y) \in \mathrm{R}_{(b)}^{(a)}$ for all $x \in D$. Since $f \rhd \mathrm{R}_{(b)}^{(a)}$, we obtain $(f(x), f(y)) \in \mathrm{R}_{(b)}^{(a)}$, i.e., $f(x) \geq a$ due to $f(y) > b$. □

3. Selfdual and quasilinear functions, and such preserving central or *h*-regular relations

In this section we recall from [3] that $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \nsubseteq \operatorname{Pol}_D \varrho$ whenever ϱ is the graph of a prime permutation, an affine relation or an at least ternary central or h-regular relation.

In this regard, we begin with the case of selfdual functions. For a unary operation $s \in \mathrm{O}_D^{(1)}$ the clone $\operatorname{Pol}_D \operatorname{graph} s$ contains all functions $f \in \mathrm{O}_D$ that *commute* with s, i.e., where $s\colon \langle D; f\rangle \to \langle D; f\rangle$ is an endomorphism. For permutations $s \in \operatorname{Sym}(D)$ this condition can be expressed as the equality $f(\mathbf{x}) = s^{-1}(f(s \circ \mathbf{x}))$ for all $\mathbf{x} \in D^{\operatorname{ar} f}$, whence functions $f \in \operatorname{Pol}_D \operatorname{graph} s$ are called *s-selfdual.*

Lemma 3.1. *If $s \in \operatorname{Sym}(D)$ is a permutation without fixed points, then for all $p, q \in \mathbb{N}_+$ and $\mathbf{a} \in D^p$, $\mathbf{b} \in D^q$, we have $c_{a_1} \in (\operatorname{Pol}_D \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}) \setminus (\operatorname{Pol}_D \operatorname{graph} s)$, in which c_{a_1} denotes the unary constant with value a_1. Thus, in particular, no maximal C-clone $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)}$ is a subset of $\operatorname{Pol}_D \operatorname{graph} s$.*

Since prime permutations cannot have fixed points, this result applies in particular to maximal clones in the second case of Theorem 2.4.

Proof. It is clear that $c_{a_1} \in \operatorname{Pol}_D \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. Moreover, by the above characterisation, c_{a_1} is s-selfdual if and only if $a_1 = c_{a_1}(x) = s^{-1}(c_{a_1}(s(x))) = s^{-1}(a_1)$ holds (for all $x \in D$), that is, if a_1 is fixed by s. Thus, by assumption, we have $c_{a_1} \notin \operatorname{Pol}_D \operatorname{graph} s$. □

To deal with affine and at least ternary central and h-regular relations, we need the following observation.

Lemma 3.2. *For all $a, b \in D$ we have $\{\vee_D, \wedge_D\} \subseteq \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)}$, where $\vee_D, \wedge_D$ denote the binary maximum and minimum relating to $\leq_D$, respectively.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathrm{R}_{(b)}^{(a)}$. If $x_1 \vee_D x_2 \geq a$ we are done. Otherwise, we have $x_1, x_2 \leq x_1 \vee_D x_2 < a$, so $y_1, y_2 \leq b$, whence $y_1 \vee_D y_2 \leq b$. The argument for $\wedge_D$ is dual. □

Central and h-regular relations share the common properties of total symmetry and total reflexivity; hence they can be dealt with in one lemma.

Lemma 3.3. *For any $m \in \mathbb{N}_{\geq 3}$ and any totally reflexive non-full m-ary relation $\varrho \in \mathrm{R}_D^{(m)}$, we have $\vee_D, \wedge_D \notin \mathrm{Pol}_D\, \varrho$.*

Proof. Since ϱ is non-full, we have $\varrho \subsetneq D^m$, and hence there exists some tuple $\mathbf{x} := (x_1, \dots, x_m) \in D^m \setminus \varrho$. By total reflexivity, the entries $x_1, \dots, x_m$ are pairwise distinct. Choose the unique $i \in \{1, \dots, m\}$ such that x_i is the least element among $x_1, \dots, x_m$ with respect to $\leq_D$ and pick indices $j, \ell \in \{1, \dots, m\}$ such that $|\{i, j, \ell\}| = 3$. This is possible due to $m \geq 3$. Define $\mathbf{y}, \mathbf{z} \in D^m$ by $y_k := x_i$ for $k = j$ and $y_k := x_k$ else; $z_k := x_k$ for $k = j$ and $z_k := x_i$ else. It follows $y_k \vee_D z_k = x_k$ for all $1 \leq k \leq m$, so $\vee_D \circ (\mathbf{y}, \mathbf{z}) = \mathbf{x} \notin \varrho$. This proves $\vee_D \not\rhd \varrho$ due to $y_j = x_i = y_i$, $z_i = x_i = z_\ell$ and total reflexivity of ϱ. For $\wedge_D$ one chooses $1 \leq i \leq m$ such that x_i is largest among $x_1, \dots, x_m$. □

Corollary 3.4. *If $h \in \mathbb{N}_{\geq 3}$ and $\varrho \subsetneq D^h$ is a central or an h-regular relation, then the clone $\mathrm{Pol}_D\, \varrho$ does not contain any maximal C-clone.*

Proof. By Theorem 2.3 maximal C-clones have the form $\mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)}$ for certain $a, b \in D$. By definition, central relations are totally reflexive, and it is not hard to see that the same also holds for h-regular relations. Using Lemmas 3.2 and 3.3, we have $\vee_D \in (\mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)}) \setminus (\mathrm{Pol}_D\, \varrho)$, so $\mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)} \nsubseteq \mathrm{Pol}_D\, \varrho$. □

It remains to discuss the case of affine relations. In fact, for any elementary Abelian p-group $\mathbf{G} = \langle G; +, -, o\rangle$ the clone $\mathrm{Pol}_G\, \varrho_{\mathbf{G}}$ contains precisely all *quasilinear* (sometimes *affine linear*) maps with regard to the canonical affine space induced by $\mathbf{G}$ over the field $\mathsf{GF}(p)$. Such clones never contain maximal C-clones as the following lemma proves, even under slightly more general assumptions.

Lemma 3.5. *For a finite set D, a group $\mathbf{G}$ on $D = \{0, \dots, n-1\}$ and any two elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$, the inclusion $\mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$ always fails.*

Proof. Let $\mathbf{G} = \langle D; +, -, o\rangle$ be any group and let $\varrho_{\mathbf{G}} \in \mathrm{R}_D^{(4)}$ be the associated affine relation, that is to say, $\varrho_{\mathbf{G}} := \{(x, y, u, v) \in D^4 \mid x + y = u + v\}$. From Lemma 3.2 we can infer that $\vee_D \in \mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)}$, however, we shall demonstrate below that $\vee_D \notin \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$, which makes an inclusion impossible.

Indeed, it is obvious that $(0, -0, 1, -1), (1, 0, 1, 0) \in \varrho_{\mathbf{G}}$. If the maximum operation preserved $\varrho_{\mathbf{G}}$, we would get, due to 0 being the least element with respect to $\leq$, that $(1, -0, 1, -1) = (0 \vee_D 1, -0 \vee_D 0, 1 \vee_D 1, -1 \vee_D 0) \in \varrho_{\mathbf{G}}$. By definition of $\varrho_{\mathbf{G}}$ this would imply $1 + (-0) = 1 + (-1) = o$, i.e., $1 = 0$, an evident contradiction. □

In the next section, we will attack possible inclusions $\mathrm{Pol}_D\, \mathrm{R}_{(b)}^{(a)} \subseteq M$ for maximal clones $M = \mathrm{Pol}_D\, \varrho$ given by non-trivial unary relations $\emptyset \subsetneq \varrho \subsetneq D$.

4. Non-trivial unary relations

The following lemma gives sufficient conditions for binary operations to belong to a given maximal C-clone.

Lemma 4.1. *Let $a, b \in D$ and suppose $f \in \mathrm{O}_D^{(2)}$ satisfies $f(x,y) \leq b$ for all pairs $(x,y) \in D^2$ where $x \leq b$ or $y \leq b$, and $f(x,y) \geq a$ for all $(x,y) \in D^2$ where $x, y \geq a$. Then $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$.*

Dually, if $f(x,y) \geq a$ for all $(x,y) \in D^2$ such that $x \geq a$ or $y \geq a$, and $f(x,y) \leq b$ for those pairs $(x,y) \in D^2$ where $x, y \leq b$, then $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$, too.

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathrm{R}_{(b)}^{(a)}$. If $f(y_1, y_2) \leq b$, then $(f(x_1, x_2), f(y_1, y_2))$ belongs to $\mathrm{R}_{(b)}^{(a)}$ and we are done. Else, by the assumption on f we must have $y_1, y_2 > b$, which implies $x_1, x_2 \geq a$ due to $(x_1, y_1), (x_2, y_2) \in \mathrm{R}_{(b)}^{(a)}$. Therefore, $f(x_1, x_2) \geq a$, which implies again $(f(x_1, x_2), f(y_1, y_2)) \in \mathrm{R}_{(b)}^{(a)}$. This proves that $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$. The proof of the second claim is by dualisation. □

We can use this type of functions to witness non-inclusions of maximal C-clones in maximal clones given by a non-trivial unary relation ϱ whenever there exists some $x \in \varrho$ respecting $b < x < a$.

Corollary 4.2. *Consider $a, b \in D$ and suppose $\varrho \subsetneq D$ contains an element $x \in \varrho$ such that $b < x < a$. Every binary function $f \in \mathrm{O}_D^{(2)}$ satisfying one of the conditions from Lemma 4.1 and mapping $f(x,x) = y$ where $y \in D \setminus \varrho$ fulfils $f \in \mathrm{Pol}_D^{(2)} \mathrm{R}_{(b)}^{(a)} \setminus \mathrm{Pol}_D \varrho$. Such functions actually exist, whence we have $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \nsubseteq \mathrm{Pol}_D \varrho$.*

Proof. As $f \in \mathrm{O}_D^{(2)}$ fulfils the conditions of Lemma 4.1, we get $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$; further, the assumption $f(x,x) = y$ where $x \in \varrho$ and $y \notin \varrho$ ensures $f \notin \mathrm{Pol}_D \varrho$.

For the existence of such operations, verify that the following function is well-defined due to $b < x < a$: we put $f(u,v) := a$ if $u, v \geq a$, $f(x,x) := y \notin \varrho$ and $f(u,v) := 0 \leq b$ everywhere else. So f satisfies the first condition from Lemma 4.1. □

In the next step we derive a necessary condition concerning the form of the unary relation ϱ that has to hold if $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$.

Lemma 4.3. *For $a, b \in D$ and a non-empty unary relation $\emptyset \subsetneq \varrho \subseteq D$, the inclusion $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$ implies $\{0, \ldots, b\} \cup \{a, \ldots, n-1\} \subseteq \varrho$.*

Proof. If there existed some $x \leq b$ such that $x \notin \varrho$, then the unary constant operation c_x with value x would belong to $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \setminus \mathrm{Pol}_D \varrho$ in contradiction to the assumption $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$. For $x \geq a$ not belonging to ϱ we use a similar argument. □

As a partial converse the next result establishes a sufficient condition for an inclusion of a maximal C-clone in a maximal clone given by a non-trivial unary relation.

Lemma 4.4. *Let $a, b \in D$ such that $a > b$. Then we have*

$$\mathrm{R}^{(a)}_{(b)} \cap \left(\mathrm{R}^{(a)}_{(b)}\right)^{-1} = \{0, \ldots, b\}^2 \cup \{a, \ldots, n-1\}^2 \text{ and}$$

$$\left\{ x \in D \;\middle|\; (x, x) \in \mathrm{R}^{(a)}_{(b)} \cap \left(\mathrm{R}^{(a)}_{(b)}\right)^{-1} \right\} = \{0, \ldots, b\} \cup \{a, \ldots, n-1\},$$

whence $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D\{0, \ldots, b\} \cup \{a, \ldots, n-1\}$.

Proof. The second equality stated in the lemma will follow by variable identification from $\mathrm{R}^{(a)}_{(b)} \cap \left(\mathrm{R}^{(a)}_{(b)}\right)^{-1} = \{0, \ldots, b\}^2 \cup \{a, \ldots, n-1\}^2$. In this equality the inclusion "$\supseteq$" is evident, so let us now consider $(x, y) \in \mathrm{R}^{(a)}_{(b)} \ni (y, x)$. If $x \geq a > b$, then $(y, x) \in \mathrm{R}^{(a)}_{(b)}$ implies $y \geq a$, thus, $(x, y) \in \{a, \ldots, n-1\}^2$. Otherwise, we have $x < a$, such that $y \leq b < a$ due to $(x, y) \in \mathrm{R}^{(a)}_{(b)}$. So it follows $x \leq b$ as $y < a$ and $(y, x) \in \mathrm{R}^{(a)}_{(b)}$. Hence, $(x, y) \in \{0, \ldots, b\}^2$.

The second equality above implies $\{0, \ldots, b\} \cup \{a, \ldots, n-1\} \in \left[\mathrm{R}^{(a)}_{(b)}\right]_{\mathrm{R}_D}$, and thus, $\operatorname{Pol}_D(\{0, \ldots, b\} \cup \{a, \ldots, n-1\}) \supseteq \operatorname{Pol}_D \left[\mathrm{R}^{(a)}_{(b)}\right]_{\mathrm{R}_D} = \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$. □

The following lemma solves the task for non-trivial unary relations.

Lemma 4.5. *Let $a, b \in D$ and $\emptyset \subsetneq \varrho \subsetneq D$ be a unary non-trivial relation. Then $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \varrho$ holds if and only if $\varrho = \{0, \ldots, b\} \cup \{a, \ldots, n-1\}$ and $a - b \geq 2$.*

Proof. If $\varrho = \{0, \ldots, b\} \cup \{a, \ldots, n-1\}$ and $a - b \geq 2 > 0$, then Lemma 4.4 implies the inclusion $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \varrho$. Conversely, if we assume this condition, then Lemma 4.3 entails $\{0, \ldots, b\} \cup \{a, \ldots, n-1\} \subseteq \varrho$. If this inclusion were proper, then there would exist some $x \in \varrho$ such that $x \not\leq b$ and $x \not\geq a$, i.e., $b < x < a$. Since $\varrho \subsetneq D$, Corollary 4.2 yields a contradiction to the assumption $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \varrho$. Therefore, we have $\{0, \ldots, b\} \cup \{a, \ldots, n-1\} = \varrho$. Moreover, supposing $a - b \leq 1$ would imply $\varrho = D$, violating our assumption. □

5. The case of bounded order relations

A *bounded* (partial) order relation is an order relation having both, a largest (top) element $\top$, and a least (bottom) element $\bot$. If $\preceq\, \subseteq D^2$ is an order relation on D, considered to be clear from the context, and $a, b \in D$ are any two elements, we occasionally use the notation $[a, b] := \{x \in D \mid a \preceq x \preceq b\}$ and call it the *interval from a to b*. Clearly, if $a \not\preceq b$, then $[a, b] = \emptyset$.

In the first step we construct binary functions witnessing non-inclusions of certain maximal C-clones in maximal clones described by non-trivial binary reflexive relations.

Lemma 5.1. *Assume that $a - b \geq 2$. Any $g \in \mathrm{O}^{(2)}_D$ satisfying $g(x, y) \leq b$ whenever $y \leq b$ and $g(x, y) \geq a$ for all $(x, y) \in D^2$ where $y \geq a$, preserves $\mathrm{R}^{(a)}_{(b)}$.*

Moreover, let $\varrho \subsetneq D^2$ be reflexive, $(x,y) \in \varrho \setminus \Delta$, $(u,v) \in D^2 \setminus \varrho$, $b < z < a$, and suppose, in addition to the above, that $g(x,z) = u$ and $g(y,z) = v$. Then we have $g \in \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \setminus \operatorname{Pol}_D \varrho$.

Proof. First, we check that $g \in \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)}$. Namely, if $(x_1,y_1),(x_2,y_2) \in \mathrm{R}_{(b)}^{(a)}$ and $x_2 \geq a$, then $g(x_1,x_2) \geq a$. Otherwise, we have $x_2 < a$ and $y_2 \leq b$, which implies $g(y_1,y_2) \leq b$. In both cases we have $(g(x_1,x_2), g(y_1,y_2)) \in \mathrm{R}_{(b)}^{(a)}$.

Furthermore, we have $(x,y),(z,z) \in \varrho$, but $(g(x,z), g(y,z)) = (u,v) \notin \varrho$, proving $g \not\rhd \varrho$. □

If $a-b \geq 2$, the many requirements on the binary function in the previous lemma are actually satisfiable.

Corollary 5.2. *For all $a,b \in D$ such that $a-b \geq 2$ and every non-trivial binary reflexive relation $\Delta \subsetneq \varrho \subsetneq D^2$, we have $\operatorname{Pol}_D^{(2)} \mathrm{R}_{(b)}^{(a)} \not\subseteq \operatorname{Pol}_D \varrho$.*

Proof. Since $a - b \geq 2$, functions g fulfilling the assumptions of Lemma 5.1 are indeed constructible. Choosing pairs $(x,y) \in \varrho \setminus \Delta$ and $(u,v) \in D^2 \setminus \varrho$, we may, for instance, define $g(w,z) := 0 \leq b$ for $z \leq b$, $g(w,z) := n-1 \geq a$ for $z \geq a$, $g(w,z) := u$ for $b < z < a$ and $w = x$, and $g(w,z) := v$ else, i.e., for all $(w,z) \in D^2$ satisfying $b < z < a$ and $w \neq x$. Since $y \neq x$, this ensures that $g(y,z) = v$ for all $b < z < a$, and thus g fulfils the conditions of Lemma 5.1. □

So the preceding result shows that inclusions $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \operatorname{Pol}_D \varrho$ are impossible whenever $a - b \geq 2$ and ϱ is a non-trivial equivalence, bounded order relation or binary central relation. In order to exclude more inclusions, we shall use the following trivial observation.

Lemma 5.3. *If for $a,b \in D$ an operation $f \in \mathrm{O}_D^{(1)}$ preserves $\{0,\dots,b\}$ and $\{a,\dots,n-1\}$, then $f \rhd \mathrm{R}_{(b)}^{(a)}$. In particular this follows, if $a \leq b$ and f preserves the sets $\{x \in D \mid x < a\}$, $\{x \in D \mid a \leq x \leq b\}$ and $\{x \in D \mid b < x\}$.*

Proof. If $(x,y) \in \mathrm{R}_{(b)}^{(a)}$ and $x \geq a$, then $f(x) \geq a$, otherwise, $x < a$ and $y \leq b$, whence $f(y) \leq b$. In both cases we have $(f(x), f(y)) \in \mathrm{R}_{(b)}^{(a)}$. The additional remark follows since for $a \leq b$ the union of the first two mentioned sets is $\{0,\dots,b\}$, the union of the last two sets is $\{a,\dots,n-1\}$, and invariant relations of unary operations are closed under arbitrary unions of relations of identical arity. □

In Proposition 5.5 we shall use transpositions that preserve the subsets $\{0,\dots,b\}$ and $\{a,\dots,n-1\}$ from Lemma 5.3. However, first, we shall deal with a few exceptional cases. They are actually variations of one case up to different dualisations, but we list all of them explicitly here.

Lemma 5.4. *Let $n \geq 3$, $a,b \in D$ and $\preceq \subseteq D^2$ be a bounded order relation with least element $\bot$ and greatest element $\top$. If*

(a) $0 = \bot < 1 = a = b = \top$, *or*
(b) $0 = \bot < 1 = a$, $n-2 = b < n-1 = \top$, *or*

(c) $n-1=\bot>n-2=b$, $1=a>0=\top$, *or*
(d) $n-1=\bot>n-2=b=a=\top$, *or*
(e) $a=\bot=b=1>0=\top$, *or*
(f) $a=\bot=b=n-2<n-1=\top$,

then there exists some $f\in \mathrm{Pol}_D^{(1)} \mathrm{R}_{(b)}^{(a)} \setminus \mathrm{Pol}_D \preceq$, *whence* $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \preceq$ *is impossible.*

Proof. In each case one can explicitly define a unary operation $f\in \mathrm{O}_D^{(1)}$ not preserving $\preceq$ but satisfying $f\in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$ upon application of Lemma 2.5. It is possible to use the same function in (a) and (e), and in (d) and (f), respectively. Moreover, the arguments for statements (a) and (d), and for (b) and (c) are very similar, so we choose to present only two of the cases. A complete proof can be found in [4].

(b): Define $f\in \mathrm{O}_D^{(1)}$ by $f(n-1):=0$ and $f(x):=x$ for $x\in D\setminus\{n-1\}$. Since $\mathrm{im}(f)=D\setminus\{n-1\}=\{0,\dots,b\}$, we get $f\rhd \mathrm{R}_{(b)}^{(a)}$. Also $1\preceq\top=n-1$ and $1<n-1$ due to $n\geq 3$, so supposing $1=f(1)\preceq f(n-1)=0=\bot$ would imply the contradiction $1=\bot=0$. Hence, $f\not\rhd\preceq$.

(d): Define $f\in \mathrm{O}_D^{(1)}$ by $f(n-1):=n-2$ and $f(x):=x$ for $x\in D\setminus\{n-1\}$. As $n\geq 3$, there is some $x\in D\setminus\{n-1,n-2\}$. We have $n-1=\bot\preceq x$, but assuming $\top=n-2=f(n-1)\preceq f(x)=x$ would certainly imply the contradiction $x=\top=n-2$, wherefore $f\not\rhd\preceq$. Moreover, the image of f equals $D\setminus\{n-1\}=\{0,\dots,b\}$, which ensures that $f\in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$. □

In Corollary 5.2 we excluded inclusions $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \preceq$ for bounded orders $\preceq$, whenever $a,b\in D$ satisfy $a-b\geq 2$. In the previous lemma, a few special cases were considered. Now we deal with the rest using transpositions fulfilling the criterion from Lemma 5.3.

Proposition 5.5. *Let* $n\geq 3$ *and* $\preceq\,\subseteq D^2$ *be a bounded order relation on* D *with bottom element* $\bot$ *and top* $\top$. *There do not exist parameters* $a,b\in D$ *such that* $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \preceq$.

Proof. Corollary 5.2 excludes inclusions for $a-b\geq 2$. For the remainder of the proof let us suppose $a-b\leq 1$, i.e., $a\leq b+1$. We shall exhibit unary operations (mostly transpositions) that obviously do not preserve $\preceq$, but preserve $\mathrm{R}_{(b)}^{(a)}$ (usually due to Lemma 5.3). For this we distinguish three cases regarding $\bot$. First assume $\bot<a$. If there exists $x<a$ such that $x\neq\bot$, then we use the transposition $(x,\bot)$. Else all $x<a$ satisfy $x=\bot$, i.e., $\bot=0<a=1$. In this case we have $\top\neq\bot=0$, so $\top\geq 1=a$. First consider the situation that $\top\leq b$. If there exists some $x\in[a,b]\setminus\{\top\}$, we use the transposition $(x,\top)$. Otherwise, $[a,b]\subseteq\{\top\}$, thus $1=a=\top=b$ and $0=\bot$, which is handled by Lemma 5.4(a). The complementary case is that $\top>b$. If there exists $x>b$ such that $x\neq\top$, then we can use $(x,\top)$, else every $x>b$ equals $\top$, and so we have $\top=n-1>b=n-2$ together with $a=1>0=\bot$. This is dealt with in Lemma 5.4(b).

The second main case is when $a \leq \bot \leq b$. If there is some $a \leq x \leq b$ such that $x \neq \bot$, then we use $(x, \bot)$. Otherwise, $[a, b] \subseteq \{\bot\}$, and so $a = \bot = b$. Due to $n \geq 3$, we have again $\top \neq \bot = a = b$. Let us consider the situation $\top < a$. If there exists some $x < a$, $x \neq \top$, then we may use $(x, \top)$, else every $x < a$ equals $\top$, so $\top = 0 < a = 1 = b = \bot$. This possibility is treated in Lemma 5.4(e). The opposite situation is that $\top > a = b$. If there exists some $x > b$, $x \neq \top$, then we use $(x, \top)$, otherwise every $x > b$ equals $\top$, and so $\top = n - 1 > b = n - 2 = a = \bot$, which is solved in case (f) of Lemma 5.4.

Third, let us deal with the possibility that $\bot > b$. If there exists some $x > b$, $x \neq \bot$, then we can use the transposition $(x, \bot)$. Otherwise, every $x > b$ equals $\bot$, so $\bot = n - 1 > b = n - 2$. Due to $n \geq 3$, we have $\top \neq \bot = n - 1$, i.e., $\top \leq n-2 = b$. The first subcase is that $\top < a$. If there exists some $x < a$, $x \neq \top$, we use the transposition $(x, \top)$. Else, all $x < a$ satisfy $x = \top$, so we obtain $\top = 0 < a = 1$, $b = n-2 < \bot = n-1$, which is treated in Lemma 5.4(c). The remaining subcase is that $a \leq \top \leq b$. If there exists some $a \leq x \leq b$, $x \neq \top$, we use $(x, \top)$, else $[a, b] \subseteq \{\top\}$, so $a = \top = b = n - 2 < n - 1 = \bot$, which has been dealt with in Lemma 5.4(d).

So in the case that $a - b \leq 1$, we have always found a transposition or a unary operation as constructed in Lemma 5.4 that preserves $\mathrm{R}^{(a)}_{(b)}$, but does not preserve the order $\preceq$. Therefore, we have $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \nsubseteq \mathrm{Pol}_D \preceq$. □

6. The case of non-trivialequivalence relations

Throughout this section, we shall employ the notation $\mathrm{Eq}\, D$ for the set of all equivalence relations on D. It is our aim to show that maximal C-clones $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ are contained in a maximal clone given by a non-trivial equivalence relation if and only if $a = b+1$. In this case the equivalence relation is uniquely determined.

As our first result, we provide a simple sufficient condition for an inclusion in a maximal clone described by an equivalence relation.

Lemma 6.1. *Let $a, b \in D$ satisfy $a = b + 1$ and $\theta \in \mathrm{Eq}\, D$ be the equivalence relation on D having the partition $D/\theta = \{\{0, \ldots, b\}, \{a, \ldots, n - 1\}\}$. Then we have $\theta = \mathrm{R}^{(a)}_{(b)} \cap \big(\mathrm{R}^{(a)}_{(b)}\big)^{-1} \in \Big[\mathrm{R}^{(a)}_{(b)}\Big]_{\mathrm{R}_D}$, and so $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \theta$ holds.*

Proof. For all $(x, y) \in D^2$ we have $(x, y) \in \theta$ if and only if $x, y \leq b$ or $x, y \geq a$, that is, exactly if $(x, y) \in \{0, \ldots, b\}^2 \cup \{a, \ldots, n - 1\}^2 = \mathrm{R}^{(a)}_{(b)} \cap \big(\mathrm{R}^{(a)}_{(b)}\big)^{-1}$ (see Lemma 4.4). □

In the remainder of this section we shall prove that the situation described in Lemma 6.1 is the only one, where a maximal C-clone can be contained in a maximal clone given by a non-trivial equivalence relation.

As a first step, we establish a few necessary conditions.

Lemma 6.2. *Let $a, b \in D$ and $\theta \in \operatorname{Eq} D \setminus \{\Delta, \nabla\}$ be a non-trivial equivalence relation such that $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta$. Then the following conditions are fulfilled:*

(a) $0 < a \leq b+1 \leq n-1$.

(b) *For every set $I \in \{\{0, \ldots, a-1\}, \{a, \ldots, b\}, \{b+1, \ldots, n-1\}\}$ we have*
$$\forall x, y \in I \colon (x, y) \notin \theta \implies |[x]_\theta| = 1 = |[y]_\theta|.$$

(c) *For all $x, y, z \in D$ where $(x, y) \in \theta \setminus \Delta$, we have the implication*
$$((x, z \geq a) \vee (x, z \leq b) \vee (y, z \geq a) \vee (y, z \leq b)) \implies (x, z) \in \theta.$$

(d) $\forall x \leq b \, \forall y \geq a \colon (x, y) \in \theta \implies b \geq x = y \geq a$.

(e) $\forall a \leq x \leq b \colon [x]_\theta = \{x\}$.

(f) $\forall x < a \colon [x]_\theta \subseteq \{0, \ldots, a-1\}$.

(g) $\forall y > b \colon [y]_\theta \subseteq \{b+1, \ldots, n-1\}$.

(h) *If $[0]_\theta \neq \{0, \ldots, a-1\}$, then we have $a-1 > 0$, $b+1 < n-1$, $[x]_\theta = \{x\}$ for all $x \leq b$, and $[n-1]_\theta = \{b+1, \ldots, n-1\}$.*

(i) *If $[n-1]_\theta \neq \{b+1, \ldots, n-1\}$, then we have $a-1 > 0$, $b+1 < n-1$, $[y]_\theta = \{y\}$ for all $y \geq a$, and $[0]_\theta = \{0, \ldots, a-1\}$.*

Proof. (a): If $a = 0$, or $b > n-2$, i.e., $b = n-1$, then we would have a trivial clausal relation $\mathrm{R}^{(a)}_{(b)} = D^2$, and so $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} = \mathrm{O}_D$ would make the inclusion $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta$ impossible. Moreover, if we had $a - b > 1$, then Corollary 5.2 would imply the contradiction $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \nsubseteq \operatorname{Pol}_D \theta$. Therefore, it follows $0 \neq a \leq b+1 \leq n-1$.

(b): Suppose, for a contradiction, that there exists a set
$$I \in S := \{\{0, \ldots, a-1\}, \{a, \ldots, b\}, \{b+1, \ldots, n-1\}\}$$
and $x, y \in I$ such that the stated implication fails. So we have $(x, y) \notin \theta$, and since this assumption is symmetric, no generality is lost in assuming that $|[x]_\theta| > 1$. Let $z \in [x]_\theta \setminus \{x\}$, and define $f \in \mathrm{O}^{(1)}_D$ by $f(x) := y$ and $f(u) = u$ for $u \neq x$. Obviously, $(z, x) \in \theta$, but $(f(z), f(x)) = (z, y) \notin \theta$, as otherwise $(x, z) \in \theta$ and transitivity would imply $(x, y) \in \theta$. Thus, $f \not\triangleright \theta$. Moreover, as $x, y \in I$, we have $f \in \operatorname{Pol}_D S$, which implies that $f \triangleright \mathrm{R}^{(a)}_{(b)}$ by Lemma 5.3 and statement (a). This contradicts the inclusion $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta$ since $f \in \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \setminus \operatorname{Pol}_D \theta$. Thus, our initial assumption was false and the claim holds.

(c): Let $x, y, z \in D$ where $(x, y) \in \theta$ and $x \neq y$. Moreover, the assumption of the implication is that we can find $w \in \{x, y\}$ such that $w, z \geq a$ or $w, z \leq b$. We define $f \in \mathrm{O}^{(1)}_D$ by $f(w) := w$ and $f(u) := z$ for $u \neq w$. Clearly, we have $\operatorname{im}(f) = \{w, z\}$, so at least one of the conditions $\operatorname{im}(f) \subseteq \{a, \ldots, n-1\}$ and $\operatorname{im}(f) \subseteq \{0, \ldots, b\}$ must hold. This implies $f \in \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta$ by Lemma 2.5 and the assumption of this lemma. So we get $(f(x), f(y)) \in \theta$ from $(x, y) \in \theta$. If $w = x$, this means $(x, z) \in \theta$. Else, if $w = y$, we obtain $(z, y) \in \theta$, which together with $(x, y) \in \theta$ yields $(x, z) \in \theta$.

(d): Let us assume, for a contradiction, that there exists $x \le b$ and $y \ge a$, where the stated implication fails, i.e., where $(x, y) \in \theta$, but $x \ne y$. Now for every $z \ge a$, statement (c) implies $(x, z) \in \theta$, so $\{a, \dots, n-1\} \subseteq [x]_\theta$. Any other element $z \in D$ satisfies $z < a \le b+1$ by item (a), i.e., $z \le b$. Then again statement (c) implies $(x, z) \in \theta$. In conclusion, we have $D \subseteq [x]_\theta$, which means $\theta = \nabla$. As this was excluded beforehand, the claim holds.

(e): Let us consider any $x \in D$ where $a \le x \le b$. For $y \in [x]_\theta$ such that $y \ge a$, we get $y = x$ by item (d). Any other $y \in [x]_\theta$ satisfies $y < a \le b+1$ by (a), i.e., $y \le b$. Again, statement (d), with roles of x and y interchanged, yields $y = x$.

(f): Let $x < a \le b+1$ (by (a)), then $x \le b$. If there existed some $y \in [x]_\theta$ such that $y \ge a$, then statement (d) would imply $a > x = y \ge a$. This contradiction proves $[x]_\theta \subseteq \{0, \dots, a-1\}$.

(g): The proof is dual to that of statement (f), using again (a) and (d).

(h): Let $[0]_\theta \ne \{0, \dots, a-1\}$. As (a) and (f) imply $[0]_\theta \subseteq \{0, \dots, a-1\}$, there must exist some $x < a$ such that $x \notin [0]_\theta$. In particular, $x \ne 0$, so $0 < x \le a-1$ yields $0 < a-1$. Since $(x, 0) \notin \theta$, we get $|[0]_\theta| = 1$ from (b). So every $0 < z < a$ satisfies $(0, z) \notin \theta$, whence (b) yields $|[z]_\theta| = 1$. Together with statement (e) we can infer $[z]_\theta = \{z\}$ for all $z \le b$. Since $\theta \ne \Delta$ by assumption, we cannot only have singleton equivalence classes for all other $y > b$. Thus, there must be some $y > b$ where $|[y]_\theta| > 1$. If there were also some $z > b$ such that $(z, y) \notin \theta$, then again (b) would imply the contradiction $|[y]_\theta| = 1$. Hence, for all $z > b$ we have $z \in [y]_\theta$, i.e., $\{b+1, \dots, n-1\} \subseteq [y]_\theta \subseteq \{b+1, \dots, n-1\}$ by (g). This means that $[y]_\theta = \{b+1, \dots, n-1\} = [n-1]_\theta$, and, because $|[y]_\theta| \ge 2$, we also get that $b+1 < n-1$.

(i): The proof of this statement works dually to the preceding one. $\square$

We have gathered now enough prerequisites to prove the following result.

Proposition 6.3. *Let $a, b \in D$ and $\theta \in \operatorname{Eq} D \setminus \{\Delta, \nabla\}$ be a non-trivial equivalence relation. Then we have*

$$\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta \iff a = b+1 \text{ and } D/\theta = \{\{0, \dots, b\}, \{a, \dots, n-1\}\}.$$

Proof. The implication "$\Longleftarrow$" is stated in Lemma 6.1. Conversely, let us assume that $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \theta$. For the remainder of the proof we can suppose $0 < a \le b+1 \le n-1$ due to Lemma 6.2 (a). We define $f \in \mathrm{O}^{(2)}_D$ by $f(b+1, 0) := 0$, $f(x, y) := a$ if $x, y > b$ and $f(x, y) := b$ else. If $x \le b$ or $y \le b$, then $f(x, y) \ne a$, so $f(x, y) \le b$. Moreover, if $x, y \ge a$, then either $x, y > b$ and $f(x, y) = a$, or else $x, y \ge a > 0$ and $a \le x \le b$ or $a \le y \le b$, whence $f(x, y) = b \ge a$. Therefore, the conditions of Lemma 4.1 are fulfilled, and so $f \in \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$.

Now, we want to prove that $[0]_\theta = \{0, \dots, a-1\}$. If this were false, then by Lemma 6.2 (h) we would get $a-1 > 0$, $b+1 < n-1$, $[x]_\theta = \{x\}$ for every $x \le b$ and $[n-1]_\theta = \{b+1, \dots, n-1\}$. Therefore, $(b+1, n-1), (0, 0) \in \theta$, but since $n-1 \ne b+1$, we obtain $(f(b+1, 0), f(n-1, 0)) = (0, b) \notin \theta$ because

$b \notin [0]_\theta = \{0\}$. Hence, $f \notin \operatorname{Pol}_D \theta$, in contradiction to the assumed inclusion $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \operatorname{Pol}_D \theta$.

Consequently, we get $[0]_\theta = \{0, \ldots, a-1\}$, and dually, one can demonstrate that $[n-1]_\theta = \{b+1, \ldots, n-1\}$. If we can show $a = b+1$, we shall be done. As we already know $a \leq b+1$, we only have to exclude $a < b+1$, i.e., $a \leq b$. So, in order to obtain a contradiction, we suppose $b \geq a$. Then we have $b \notin [0]_\theta = \{0, \ldots, a-1\}$, i.e., $(0, b) \notin \theta$. If $b+1 < n-1$, we could use the same arguments as in the previous paragraph to prove that $f \in \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \setminus \operatorname{Pol}_D \theta$. Hence, we must have $b+1 = n-1$, and so $[y]_\theta = \{y\}$ holds for all $y \geq a$ (recall Lemma 6.2(e)). Since $[0]_\theta = \{0, \ldots, a-1\}$, it follows $a-1 > 0$ due to $\theta \neq \Delta$. In this case we can use the dual version of f to get a contradiction: define $g \in \mathrm{O}_D^{(2)}$ by $g(a-1, n-1) := n-1$, $g(x, y) := b$ if $x, y < a$, and $g(x, y) := a$ else. This function preserves $\mathrm{R}_{(b)}^{(a)}$ since the conditions of Lemma 4.1 are met: if $x \geq a$ or $y \geq a$, then $g(x, y) \neq b$, so $g(x, y) \geq a$. If $x, y \leq b$, then $y < n-1$, so $g(x, y) \neq n-1$. So either $x, y < a$, whence $g(x, y) = b$, or $a \leq x \leq b$ or $a \leq y \leq b$ such that we get $g(x, y) = a \leq b$. Thus, $g \rhd \mathrm{R}_{(b)}^{(a)}$. We finish by demonstrating that $g \not\rhd \theta$. Indeed, $(0, a-1), (n-1, n-1) \in \theta$, but due to $a \leq b < n-1$, we have $a \notin \{n-1\} = [n-1]_\theta$. So we obtain that $g \not\rhd \theta$ because $(g(0, n-1), g(a-1, n-1)) = (a, n-1) \notin \theta$.

This contradicts $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \operatorname{Pol}_D \theta$; thus $a > b$ follows, that is, by the above we have $a = b+1$. $\square$

7. The case of central relations

Inclusions $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \operatorname{Pol}_D \varrho$ for at least ternary non-trivial central relations ϱ have already been excluded in Corollary 3.4. Moreover, unary central relations have been studied in Section 4. So further in this section, we shall only consider binary central relations ϱ. These are reflexive in the usual sense, i.e., $\Delta \subseteq \varrho$, and hence, we can apply Corollary 5.2, which states $\operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)} \not\subseteq \operatorname{Pol}_D \varrho$ for $a - b \geq 2$ and non-trivial ϱ. Next, we prove the same for $a - b = 1$.

Lemma 7.1. *Let $a \in D \setminus \{0\}$, $b \in D \setminus \{n-1\}$ be such that $a - b \leq 1$, and consider a non-trivial binary central relation $\varrho \subsetneq D^2$ having a central element $c \in D$ satisfying $c < a$ or $c > b$. Then one can construct a binary function $f \in \operatorname{Pol}_D^{(2)} \mathrm{R}_{(b)}^{(a)} \setminus \operatorname{Pol}_D \varrho$.*

Proof. If $c < a$ then choose $d > b$, e.g., $d = n-1$; else, if $c > b$, then choose $d < a$, e.g., $d = 0$. Moreover, let $(u, v) \in D^2 \setminus \varrho$. We shall consider three cases, (1) that $u, v \leq b$, (2) $u, v \geq a$, which is not disjoint from the previous case, and (3) that neither (1) nor (2) holds. In case (3) no generality is lost in assuming $u < a \leq b+1$, i.e., $u \leq b$, otherwise one can just swap u and v due to ϱ being symmetric. Since we are not in case (1), we cannot have $v \leq b$, hence $v > b \geq a-1$, i.e., $v \geq a$. So (3) means $u \leq b$ and $v \geq a$. In this case we define $z := c$. For (1) we choose $z \in \{c, d\}$ such that $z < a$, implying $z \leq a-1 \leq b$, and in case (2) we pick $z \in \{c, d\}$ such that $z > b$, i.e., $z \geq b+1 \geq a$. We define now an operation $f \in \mathrm{O}_D^{(2)}$. In case (1) we put $f(x,y) := \min(x,y)$ if $x,y \geq a$, $f(x,y) := v$ if $(x,y) = (c,z)$, and $f(x,y) := u$ else. In case (2) we set $f(x,y) := \max(x,y)$ if $x,y \leq b$, $f(x,y) := v$ if $(x,y) = (c,z)$, and $f(x,y) := u$ else. Provided that $c < a$ in (3), put $f(x,y) := \max(x,y)$ if $x,y \leq b$ and $(x,y) \neq (c,z)$, $f(x,y) := u$ if $(x,y) = (c,z)(= (c,c))$, and $f(x,y) := v$ else. Otherwise, if $c > b$ in case (3), we define $f(x,y) := \min(x,y)$ if $x,y \geq a$ and $(x,y) \neq (c,z)$, $f(x,y) := v$ if $(x,y) = (c,z)(= (c,c))$, and $f(x,y) := u$ else. It is not hard to check that always the function f is well-defined and that $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$ by Lemma 4.1. Since ϱ is reflexive and c is a central element, we have $(c,d),(z,z) \in \varrho$. However, $(f(c,z), f(d,z)) = (u,v) \notin \varrho$ for case (3) and $c < a$, and otherwise we have $(f(c,z), f(d,z)) = (v,u) \notin \varrho$ by symmetry of ϱ. This shows that $f \notin \mathrm{Pol}_D \varrho$. $\square$

Corollary 7.2. *Let $a,b \in D$ such that $a - b = 1$ and $\varrho \subsetneq D^2$ be any non-trivial binary central relation, then there exists a function $f \in \mathrm{Pol}_D^{(2)} \mathrm{R}_{(b)}^{(a)} \setminus \mathrm{Pol}_D \varrho$.*

Proof. Clearly $a - b = 1$ implies $a = b + 1 \geq 1 > 0$ and $b = a - 1 < a \leq n - 1$. Moreover, ϱ must have a central element $c \in D$. We either have $c < a$ or $c \geq a = b + 1 > b$. In both cases, Lemma 7.1 yields the result. $\square$

The following lemma states conditions for an inclusion.

Lemma 7.3. *Let $a,b \in D$ such that $0 < a \leq b < n - 1$. Then we have*

$$\left[\mathrm{R}_{(b)}^{(a)}\right]_{\mathrm{R}_D} \ni \mathrm{R}_{(b)}^{(a)} \cap \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1} = \{0,\dots,b\}^2 \cup \{a,\dots,n-1\}^2 =: \sigma_{a,b},$$

and $\sigma_{a,b} \subseteq D^2 \setminus \{(0,n-1),(n-1,0)\}$ is a non-trivial binary central relation having any $c \in \{a,\dots,b\}$ as a central element. Moreover, we have the inclusion $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \sigma_{a,b}$.

Proof. First, we demonstrate that $\sigma_{a,b}$ is a non-trivial binary central relation. It is clear that $\sigma_{a,b}$ is symmetric as a union of symmetric relations. Moreover, $\sigma_{a,b}$ is reflexive: for all $x \in D$ with $x \leq b$ we have $(x,x) \in \{0,\dots,b\}^2 \subseteq \sigma_{a,b}$, and for $x > b \geq a$ we have $(x,x) \in \{a,\dots,n-1\}^2 \subseteq \sigma_{a,b}$. Now consider any $a \leq c \leq b$ (such elements exist due to $a \leq b$); we show that it is central for $\sigma_{a,b}$. For $x \leq c$ the pairs (x,c) and (c,x) belong to $\{0,\dots,b\}^2 \subseteq \sigma_{a,b}$; otherwise, we have $x > c \geq a$, and (x,c) and (c,x) lie in $\{a,\dots,n-1\}^2 \subseteq \sigma_{a,b}$. Furthermore, as $0 < a$ and $n - 1 > b$, we have $(0,n-1) \notin \sigma_{a,b}$ and $(n-1,0) \notin \sigma_{a,b}$.

The inclusion $\sigma_{a,b} \subseteq \mathrm{R}_{(b)}^{(a)} \cap \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1}$ is straightforward. For the converse, suppose that $(x,y) \in \mathrm{R}_{(b)}^{(a)} \ni (y,x)$, which means ($x \geq a$ or $y \leq b$) and ($y \geq a$ or $x \leq b$). If $x,y \geq a$ or $x,y \leq b$, then clearly $(x,y) \in \sigma_{a,b}$. Otherwise, we have $a \leq x \leq b$ or $a \leq y \leq b$. In the first case, we can have $y \geq a$ or $y < a \leq b$, and always it follows $(x,y) \in \sigma_{a,b}$. The second case can be treated dually.

The inclusion we have just demonstrated implies that $\sigma_{a,b} \in \left[\mathrm{R}_{(b)}^{(a)}\right]_{\mathrm{R}_D}$, hence $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} = \mathrm{Pol}_D \left[\mathrm{R}_{(b)}^{(a)}\right]_{\mathrm{R}_D} \subseteq \mathrm{Pol}_D \sigma_{a,b}$. $\square$

Lemma 7.4. *Let $a, b \in D$ such that $a \leq b$ and $x_1, x_2 < a$, $y_1, y_2 > b$. Then we have $f \in \mathrm{Pol}_D^{(1)} \mathrm{R}_{(b)}^{(a)}$ for $f \in \mathrm{O}_D^{(1)}$ defined by $f(x_1) := x_2$, $f(y_1) := y_2$ and $f(z) := z$ for $z \in D \setminus \{x_1, y_1\}$.*

Proof. First, the function $f \in \mathrm{O}_D^{(1)}$ is well-defined due to $x_1 < a \leq b < y_1$. Since $x_1, x_2 < a \leq b$ and $y_1 > b$, it is evident that $f \rhd \{0, \dots, b\}$. Similarly, we obtain $f \rhd \{a, \dots, n-1\}$. Using Lemma 5.3, we can infer $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$. □

With these lemmas at hand, we can prove the following characterisation.

Proposition 7.5. *Let $a, b \in D$, $\sigma_{a,b} \subseteq D^2$ be defined as in Lemma 7.3 and $\varrho \subsetneq D^2$ be a non-trivial binary central relation. Then we have*

$$\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho \iff 0 < a \leq b < n-1 \text{ and } \varrho = \sigma_{a,b}.$$

Proof. The implication "$\Longleftarrow$" holds by Lemma 7.3. Conversely, suppose that $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$ is true. Then $a \neq 0$ and $b \neq n-1$, as otherwise $\mathrm{R}_{(b)}^{(a)} = D^2$ and then $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} = \mathrm{O}_D$, which is not contained in any maximal clone. Moreover, as ϱ is reflexive and non-trivial, Corollaries 5.2 and 7.2 allow us to infer that $a \leq b$. It remains to show that $\varrho = \sigma_{a,b}$.

First, let us consider the inclusion $\sigma_{a,b} \subseteq \varrho$. For this let $d \in D$ be a central element of ϱ. If $d < a$ or $d > b$, then this would violate the assumed inclusion $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$ due to Lemma 7.1. Hence, we have $a \leq d \leq b$. For any pair $(x, y) \in \{0, \dots, b\}^2 \cup \{a, \dots, n-1\}^2$ we can define a unary function $f \in \mathrm{O}_D^{(1)}$ by $f(0) := x$ and $f(z) := y$ if $z \in D \setminus \{0\}$. Obviously, we have $\mathrm{im}(f) = \{x, y\}$, such that $\mathrm{im}(f) \subseteq \{0, \dots, b\}$ or $\mathrm{im}(f) \subseteq \{a, \dots, n-1\}$. Thus using Lemma 2.5 we obtain $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$, and therefore $(x, y) = (f(0), f(d)) \in \varrho$ since $d \geq a > 0$ was a central element of ϱ. This demonstrates that ϱ contains $\{0, \dots, b\}^2 \cup \{a, \dots, n-1\}^2$, which equals $\sigma_{a,b}$.

To prove that $\varrho \subseteq \sigma_{a,b}$ we rule out that $\left(D^2 \setminus \mathrm{R}_{(b)}^{(a)}\right) \cap \varrho \neq \emptyset$. Namely, if there were some $(x_1, y_1) \in \left(D^2 \setminus \mathrm{R}_{(b)}^{(a)}\right) \cap \varrho$, then for any $(x_2, y_2) \in D^2 \setminus \mathrm{R}_{(b)}^{(a)}$, we could use the function $f \in \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \varrho$ constructed in Lemma 7.4 to show that $(x_2, y_2) = (f(x_1), f(y_1)) \in \varrho$. This would mean $D^2 \setminus \mathrm{R}_{(b)}^{(a)} \subseteq \varrho$, and, by symmetry of ϱ, would imply $D^2 \setminus \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1} \subseteq \varrho$. Hence, we would have the inclusion $D^2 \setminus \sigma_{a,b} = D^2 \setminus \left(\mathrm{R}_{(b)}^{(a)} \cap \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1}\right) \subseteq \varrho$. Together with $\sigma_{a,b} \subseteq \varrho$, we would get $\varrho = D^2$, in contradiction to ϱ being non-trivial.

Therefore, $\left(D^2 \setminus \mathrm{R}_{(b)}^{(a)}\right) \cap \varrho$ is empty, which means $\varrho \subseteq \mathrm{R}_{(b)}^{(a)}$. By symmetry of ϱ this implies $\varrho = \varrho^{-1} \subseteq \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1}$, and so $\varrho \subseteq \mathrm{R}_{(b)}^{(a)} \cap \left(\mathrm{R}_{(b)}^{(a)}\right)^{-1} = \sigma_{a,b}$. □

8. Theorem statement

Before we come to our main result, let us observe the following relationship concerning maximal C-clones.

Lemma 8.1. *For any choice of parameters* $a \in D \setminus \{0\}$ *and* $b \in D \setminus \{n-1\}$, *we have* $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subsetneq \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \cap \big(\mathrm{R}^{(a)}_{(b)}\big)^{-1}$.

Proof. Let us abbreviate $\sigma := \mathrm{R}^{(a)}_{(b)} \cap \big(\mathrm{R}^{(a)}_{(b)}\big)^{-1}$. It is clear that $\sigma \in \Big[\mathrm{R}^{(a)}_{(b)}\Big]_{\mathrm{R}_D}$, whence $\operatorname{Pol}_D \sigma \supseteq \operatorname{Pol}_D \Big[\mathrm{R}^{(a)}_{(b)}\Big]_{\mathrm{R}_D} = \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$. To prove that this inclusion is strict, we construct a unary function $f \in \operatorname{Pol}_D \sigma \setminus \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$. In case that $a \leq b$ we define f by $f(x) := n-1$ for $x < a$, $f(x) := x$ for $a \leq x \leq b$ and $f(x) := 0$ for $x > b$. Since $(f(n-1), f(0)) = (0, n-1) \notin \mathrm{R}^{(a)}_{(b)}$, we have $f \not\rhd \mathrm{R}^{(a)}_{(b)}$. In case that $a > b$ we let $f(x) := n-1$ for $x \leq b$, $f(x) := x$ for $b < x < a$ and $f(x) := 0$ for $x \geq a$. Again, we have $(f(a), f(b)) = (0, n-1) \notin \mathrm{R}^{(a)}_{(b)}$, which shows $f \not\rhd \mathrm{R}^{(a)}_{(b)}$.

Next, we prove for $a \leq b$ that $(f(x), f(y)) \in \mathrm{R}^{(a)}_{(b)}$ whenever $(x, y) \in \sigma$. By symmetry of σ this implies $f \rhd \sigma$. Namely, if $(x, y) \in \sigma$ and $f(y) > b$, then $f(y) \neq 0$, so $y \leq b$. Assuming $y \geq a$ would yield $b < f(y) = y \leq b$, whose absurdity entails $y < a$. Then $(y, x) \in \mathrm{R}^{(a)}_{(b)}$ implies $x \leq b$, and thus either $f(x) = x \geq a$ if $x \geq a$, or $f(x) = n-1 \geq a$ if $x < a$. Therefore, in any case, we have $f(y) \leq b$ or $f(x) \geq a$, meaning $(f(x), f(y)) \in \mathrm{R}^{(a)}_{(b)}$.

For $a > b$, the argument is even simpler: if $(x, y) \in \sigma$ and $f(x) < a$, then $f(x) \neq n-1$, so $x > b$. Since $(y, x) \in \mathrm{R}^{(a)}_{(b)}$, we get $y \geq a$, so $f(y) = 0 \leq b$. $\square$

We can now combine the previously proven results to obtain the following theorem, giving a complete description of the relationship between maximal clones and maximal clausal clones.

Theorem 8.2. *For every maximal C-clone* $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$ *on* $D = \{0, \ldots, n-1\}$, *where* $n \in \mathbb{N}$, *and* $a \in D \setminus \{0\}$ *and* $b \in D \setminus \{n-1\}$, *there exists precisely one maximal clone* M *such that* $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq M$.

More precisely, we have that

- $\operatorname{Pol}_D \mathrm{R}^{(1)}_{(0)} = \operatorname{Pol}_D \leq_2$ *for* $n = 2$;
- *for* $n \geq 3$ *the following proper inclusions hold:*
  - $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subsetneq \operatorname{Pol}_D \varrho$ *if* $a - b > 1$, *where* ϱ *is the unary non-trivial relation* $\{0, \ldots, b\} \cup \{a, \ldots, n-1\}$;
  - $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subsetneq \operatorname{Pol}_D \theta$ *if* $a - b = 1$, *where* θ *is the equivalence relation on* D *given by the partition* $D/\theta = \{\{0, \ldots, b\}, \{a, \ldots, n-1\}\}$; *and*
  - $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subsetneq \operatorname{Pol}_D \sigma_{a,b}$ *if* $a - b < 1$ *where* $\sigma_{a,b}$ *denotes the binary central relation* $\{0, \ldots, b\}^2 \cup \{a, \ldots, n-1\}^2$.

Proof. Summarising results from Section 3, inclusions $\operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \operatorname{Pol}_D \varrho$ are impossible whenever ϱ is the graph of a prime permutation (Lemma 3.1), an affine relation corresponding to an elementary Abelian p-group (Lemma 3.5), an at least ternary (non-trivial) central or h-regular relation (Corollary 3.4), or a bounded partial order relation for $n \geq 3$ (Proposition 5.5). So from the types of relations listed in Theorem 2.4 only non-trivial equivalence relations, bounded partial order relations for $n = 2$ and unary and binary central relations remain.

Lemma 4.5 and Propositions 6.3 and 7.5 confirm the inclusions claimed in the theorem for $n \geq 3$. We only have to prove that each maximal C-clone is not contained in any other maximal clone. For instance, if $a - b = 1$, then Proposition 7.5 and Lemma 4.5 show that $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ is not contained in $\mathrm{Pol}_D \varrho$ for any non-trivial unary or binary central relation ϱ. Moreover, by Proposition 6.3, an inclusion $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \theta$, where θ is a non-trivial equivalence relation, implies that θ is exactly the equivalence stated in the theorem. For the cases $a - b \gtreqless 1$ analogous arguments prove that $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ is a subset of a unique maximal clone.

If the inclusions for $n \geq 3$ were improper, then $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ would be a maximal clone in each of the three cases. Hence, from Lemma 8.1 we would get $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \subsetneq \mathrm{Pol}_D \sigma = \mathrm{O}_D$ where $\sigma := \mathrm{R}^{(a)}_{(b)} \cap \big(\mathrm{R}^{(a)}_{(b)}\big)^{-1}$. Thus, $\sigma \in \mathrm{Inv}_D \mathrm{O}_D$ would have to be a binary diagonal relation. Since $(a, a) \in \sigma$ but $(0, n-1) \notin \sigma$, this would mean $\sigma = \Delta$. However, Lemmas 4.4 and 7.3 prove that $\Delta = \sigma$ equals $\{0, \dots, b\}^2 \cup \{a, \dots, n-1\}^2$, which thus implies $b = 0$ and $a = n - 1$. This in turn gives that $\Delta = \sigma = \{0\}^2 \cup \{n-1\}^2 = \{(0, 0), (n-1, n-1)\}$, i.e., $n \leq 2$.

The statements concerning $|D| = n = 2$ have been established already in [16, Theorem 2.14] (see also [3, Theorem 6]): the clone of monotone Boolean functions is the only maximal C-clone on a two-element domain. □

From the previous theorem, we can derive a completeness criterion for clones on finite sets described by clausal relations. This will require the following additional lemma.

Lemma 8.3. *Let $n \in \mathbb{N}$, $D = \{0, \dots, n-1\}$ and $Q \subseteq C\,\mathrm{R}_D$ be a set of clausal relations. If $\mathrm{Pol}_D Q \subsetneq \mathrm{O}_D$, then there exists a maximal C-clone $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ (where $a \in D \setminus \{0\}$, $b \in D \setminus \{n-1\}$) such that $\mathrm{Pol}_D Q \subseteq \mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$.*

Proof. If every $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \in Q$ contains a 0 among $\{a_1, \dots, a_p\}$ or $n-1 \in \{b_1, \dots, b_q\}$, then $\mathrm{Pol}_D Q = \mathrm{O}_D$, so the premise of the implication is not fulfilled. This is in particular the case for $n \leq 1$, so let us further consider $n \geq 2$ and suppose that there exists some $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \in Q$ where $\mathbf{a} \in (D \setminus \{0\})^p$ and $\mathbf{b} \in (D \setminus \{n-1\})^q$. It follows that $\mathrm{Pol}_D Q \subseteq \mathrm{Pol}_D\{\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}\}$. By Lemma 6.1.3 of [17] we have the inclusion $\mathrm{Pol}_D\{\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}\} \subseteq \mathrm{Pol}_D\big\{\mathrm{R}^{(a)}_{(b)}\big\}$ for the parameters $a = \min\{a_1, \dots, a_p\} > 0$ and $b = \max\{b_1, \dots, b_q\} < n - 1$. By Theorem 2.3, $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$ is indeed a maximal C-clone; by the above, it is a superclone of $\mathrm{Pol}_D Q$. □

Corollary 8.4. *Let for $n \geq 3$ and $D = \{0, \dots, n-1\}$ a set $Q \subseteq C\,\mathrm{R}_D$ of clausal relations be given. Put $F := \mathrm{Pol}_D Q$. If for each $0 \leq b < n-1$ there is some $f \in F$ such that $f \not\triangleright \theta_b$, where θ_b is the equivalence relation belonging to the non-trivial partition $D/\theta_b = \{\{0, \dots, b\}, \{b+1, \dots, n-1\}\}$, and for each $0 < a \leq b < n-1$ there is some $f \in F$ such that f does not preserve $\{0, \dots, b\}^2 \cup \{a, \dots, n-1\}^2$, and for each pair (b, k) such that $0 \leq b \leq n-3$*

and $2 \leq k \leq n-1-b$ *we have* $f \not\rhd \{0,\ldots,b\} \cup \{b+k,\ldots,n-1\}$ *for some* $f \in F$; *then* $F = \operatorname{Pol}_D Q = \mathrm{O}_D$.

Proof. By the assumptions and Theorem 8.2, we have $F \nsubseteq \operatorname{Pol}_D \mathrm{R}_{(b)}^{(a)}$ for all parameters $a \in D \setminus \{0\}$, $b \in D \setminus \{n-1\}$. Therefore, the C-clone F is not contained in any maximal C-clone. Using Lemma 8.3, we can conclude that $\operatorname{Pol}_D Q = F$ must be the full C-clone O_D. □

Acknowledgements

Open access funding provided by TU Wien (TUW).

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Received: 23 March 2015.
Accepted: 17 September 2017.