Algebra Univers. 65 (2011) 331–340
DOI 10.1007/s00012-011-0133-4
Published online June 19, 2011

Algebra Universalis

The commutator in equivalential algebras and Fregean varieties

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Abstract. A class $\mathcal{K}$ of algebras with a distinguished constant term $\mathbf{0}$ is called Fregean if congruences of algebras in $\mathcal{K}$ are uniquely determined by their $\mathbf{0}$-cosets and $\Theta_{\mathbf{A}}(\mathbf{0}, a) = \Theta_{\mathbf{A}}(\mathbf{0}, b)$ implies $a = b$ for all $a, b \in \mathbf{A} \in \mathcal{K}$. The structure of Fregean varieties was investigated in a paper by P. Idziak, K. Słomczyńska, and A. Wroński. In particular, it was shown there that every congruence permutable Fregean variety consists of algebras that are expansions of equivalential algebras, i.e., algebras that form an algebraization of the purely equivalential fragment of the intuitionistic propositional logic. In this paper we give a full characterization of the commutator for equivalential algebras and solvable Fregean varieties. In particular, we show that in a solvable algebra from a Fregean variety, the commutator coincides with the commutator of its purely equivalential reduct. Moreover, an intrinsic characterization of the commutator in this setting is given.

1. Introduction

Following our earlier paper [6], a variety $\mathcal{V}$ with a distinguished constant $\mathbf{0}$ is called Fregean if every algebra $\mathbf{A} \in \mathcal{V}$ is

- congruence $\mathbf{0}$-regular, i.e.,
  $\mathbf{0}/\alpha = \mathbf{0}/\beta$ implies $\alpha = \beta$ for all congruences $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$,
- congruence orderable, i.e.,
  $\Theta_{\mathbf{A}}(\mathbf{0}, a) = \Theta_{\mathbf{A}}(\mathbf{0}, b)$ implies $a = b$ for all $a, b \in A$.

These two properties of congruences allows us to introduce a natural partial order on the universe of every $\mathbf{A} \in \mathcal{V}$ by putting, for $a, b \in A$,

$$a \leqslant b \quad \text{iff} \quad \Theta_{\mathbf{A}}(\mathbf{0}, a) \subseteq \Theta_{\mathbf{A}}(\mathbf{0}, b).$$

We refer the reader to [6], where a discussion of the name *Fregean* is given. Here we only recall that it comes from Frege's idea that sentences should denote their logical values. This idea was formalized by R. Suszko in [11, 12] and was an inspiration for D. Pigozzi [8] for skillfully transferring the distinction between Fregean and non-Fregean to the field of universal algebra and abstract algebraic logic. Actually, the concept of Fregean varieties was defined for the first time by W. Blok, P. Köhler and D. Pigozzi [1, page 356].

Presented by J. Berman.
Received April 5, 2010; accepted in final form July 23, 2010.
2010 *Mathematics Subject Classification*: Primary: 08A30; Secondary: 03G25, 06D20, 03B50, 03B55.
Key words and phrases: commutator theory, Fregean varieties, equivalential algebras.
Research supported by Polish MNiSW grant N206 2106 33.

Birkhäuser

Note that among Fregean varieties there are Boolean algebras, Boolean groups, Brouwerian semilattices (with $\mathbf{0}$ interpreted as the largest element), Hilbert algebras, Heyting algebras, and many other algebras that arise as algebralizations of classical, intuitionistic and intermediate logics. In fact, equivalential algebras, introduced by J. K. Kabziński and A. Wroński in [7] as an algebraic counterpart of the purely equivalential fragment of the intuitionistic logic, constitute a natural example of a Fregean variety of special importance. Namely, every congruence permutable Fregean variety has a binary term that turns each of its members into an equivalential algebra. In [7], equivalential algebras were defined as algebras of the form $\mathbf{A} = (A, \leftrightarrow)$ that satisfy all identities $\mathbf{t} = \mathbf{s}$ where $\mathbf{t} \leftrightarrow \mathbf{s}$ is a tautology of intuitionistic logic. It was shown there that the variety $\mathcal{E}$ of equivalential algebras consists of all $(\leftrightarrow)$-subreducts of Heyting algebras (or Brouwerian semilattices), where $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. They also showed that the variety $\mathcal{E}$ is definable by the identities $xxy = y$, $xyzz = xz\,(yz)$, and $xy(xzz)(xzz) = xy$, where the convention of associating to the left and ignoring the equivalence operation symbol is used.

Supplementing the axioms of equivalential algebras by the identity $xyy = x$, we obtain the smallest non-trivial subvariety $\mathcal{E}_2$ of $\mathcal{E}$, which consists of all associative equivalential algebras, also known as Boolean groups. It is easy to show that the term $\mathbf{0} := xx$ is constant and that, with $\mathbf{0}$ being distinguished, the variety $\mathcal{E}$ is congruence permutable and Fregean, where $(xyz)(xzzx)$ serves as the Mal'cev term. In fact, the equivalential algebras form a paradigm of congruence permutable Fregean varieties, as the following result, taken from [6], shows.

Theorem 1.1. *Let $\mathcal{V}$ be a congruence permutable Fregean variety. Then there exists a binary term* $\mathbf{e}$ *such that*

(1) $\mathbf{e}$ *is a principal congruence term of every* $\mathbf{A} \in \mathcal{V}$, *i.e.,*
$\Theta_{\mathbf{A}}(a, b) = \Theta_{\mathbf{A}}(\mathbf{0}, \mathbf{e}(a, b))$ *for all* $a, b \in A$,
(2) $(A, \mathbf{e})$ *is an equivalential algebra.*

2. Preliminaries

Fregean varieties, being $\mathbf{0}$-regular, are congruence modular (see e.g. [4]). This allows us to apply modular commutator theory as described in [2]. In particular, in [6] we have shown that Fregean varieties satisfy the condition (SC1) introduced and discussed in [5]:

Proposition 2.1. [6, Theorem 2.3] *In a subdirectly irreducible algebra* $\mathbf{A}$ *from a Fregean variety, the centralizer* $(0 : \mu)$ *does not exceed the monolith* μ *of* $\mathbf{A}$.

In fact, the condition (SC1) is much stronger than the condition (C1) considered by R. Freese and R. McKenzie in [2]. They observed that a congruence modular variety, $\mathcal{V}$ satisfies the congruence identity

$$\text{(C1)} \qquad [\alpha, \beta] = (\alpha \wedge [\beta, \beta]) \vee (\beta \wedge [\alpha, \alpha])$$

if and only if in every subdirectly irreducible algebra $\mathbf{A} \in \mathcal{V}$ the centralizer $(0 : \mu)$ of the monolith μ of $\mathbf{A}$ is Abelian. Therefore we have:

Proposition 2.2. [6, Corollary 2.4] *Every algebra from a Fregean variety satisfies the condition* (C1).

Another important feature of subdirectly irreducible algebras in Fregean varieties is stated in the following theorem.

Theorem 2.3. *Let* $\mathbf{A}$ *be a subdirectly irreducible algebra from a Fregean variety* $\mathcal{V}$ *with the monolith* μ. *Then*

(1) $|\mathbf{0}/\mu| = 2$ *and all other* μ*-cosets are one element,*
(2) *if* $\mathcal{V}$ *is congruence permutable and* $\star$ *is the unique non zero element in* $\mathbf{0}/\mu$, *then* $\mathbf{e}(a, \star) = a$ *for every* $a \in A - \{\mathbf{0}, \star\}$.

Proof. For the proof of (1), see [6, Lemma 2.1]. To show (2), suppose that $a \neq \mathbf{0}, \star$, so that $\mu \subseteq \Theta_{\mathbf{A}}(\mathbf{0}, a)$ and, by Theorem 1.1(1), $\mu \subseteq \Theta_{\mathbf{A}}(a, \star) = \Theta_{\mathbf{A}}(\mathbf{0}, \mathbf{e}(a, \star))$. Then we have $\mathbf{e}(a, \star) \equiv_{\Theta_{\mathbf{A}}(\mathbf{0}, a)} \mathbf{e}(\mathbf{0}, \star) = \star \equiv_\mu \mathbf{0}$. On the other hand, $a = \mathbf{e}(a, 0) \equiv_\mu \mathbf{e}(a, \star) \equiv_{\Theta_{\mathbf{A}}(\mathbf{0}, \mathbf{e}(a, \star))} \mathbf{0}$. This gives $\Theta_{\mathbf{A}}(\mathbf{0}, a) = \Theta_{\mathbf{A}}(\mathbf{0}, \mathbf{e}(a, \star))$, and from congruence orderability, we get $\mathbf{e}(a, \star) = a$, as required. □

Since algebras from Fregean varieties are $\mathbf{0}$-regular, their congruences can be identified with $\mathbf{0}$-cosets via $\varphi \mapsto \mathbf{0}/\varphi$. In fact, there is a stronger connection here between congruences φ and the ideals of the form $\mathbf{0}/\varphi$ they determine. Such a connection was carefully studied by H.P. Gumm and A. Ursini in [3] in a much more general setting. Although it is hard to give an intrinsic characterization of ideals in algebras even from congruence permutable varieties, there is one in the case of equivalential algebras. (The reader should be warned here that usually the term *filter* rather than *ideal* is used, because in Brouwerian semilattices, and therefore in equivalential algebras, traditionally the dual to the natural order determined by congruences is considered.) Namely, we have

Proposition 2.4. [7] *A subset* F *of an equivalential algebra* $\mathbf{A}$ *is a* $\mathbf{0}$*-coset of a congruence of* $\mathbf{A}$ *if and only if for all* $a, b \in A$ *it satisfies*

- $a, ab \in F$ *implies* $b \in F$,
- $a \in F$ *implies* $abb \in F$.

The family of all ideals of an equivalential algebra $\mathbf{A}$ is to be denoted by $\Phi(\mathbf{A})$. In the following, we will need to consider special mappings in equivalential algebras. In particular, they will help us to describe what ideals generated by particular sets look like.

For $x \in \mathbf{A} \in \mathcal{E}$, define a mapping $\chi_x \colon A \ni a \to axx \in A$ and observe that for $x, y \in A$,

- χ_x is a retraction (idempotent homomorphism) of $\mathbf{A}$,
- $\chi_x \circ \chi_y = \chi_y \circ \chi_x = \chi_x \circ \chi_{xy}$.

These two items show that for a subset $X = \{x_1, \ldots, x_n\} \subseteq A$, the mapping $\&X := \chi_{x_1} \circ \cdots \circ \chi_{x_n}$ is a well defined retraction of $\mathbf{A}$ (in particular, it does not depend on permuting or repeating the elements of X). For our further convenience, we define $\&\emptyset := \mathrm{id}_A$.

Now we are ready to describe how an ideal $(M]$ of an equivalential algebra $\mathbf{A}$ is generated from a subset $M \subseteq A$.

Proposition 2.5. [7] *For an equivalential algebra $\mathbf{A}$ and $\{a\} \cup M \subseteq A$, we have $a \in (M]$ iff $a(c_1\&X_1)\cdots(c_n\&X_n) = \mathbf{0}$ for some $c_1, \ldots, c_n \in M$ and some finite sets $X_1, \ldots, X_n \subseteq A$.*

We conclude this section with a couple of notions that proved themselves to be useful in studying of equivalential algebras.

We say that $x, y \in A$ are *orthogonal*, and write $x \perp y$, if $xyy = x$ and $yxx = x$. Furthermore, $X, Y \subseteq A$ are called *orthogonal* ($X \perp Y$) if $x \perp y$ for all $x \in X$ and $y \in Y$. The following fact reveals the meaning of this concept: if a subset of an equivalential algebra consists of pairwise orthogonal elements, then it generates an associative subalgebra (see [9]). For every equivalential algebra $\mathbf{A}$, there exists the smallest ideal $D_{\mathbf{A}}$ such that $\mathbf{A}/D_{\mathbf{A}} \in \mathcal{E}_2$. We call its elements dense. An element $a \in A$ is dense in $\mathbf{A}$ iff there exists a finite subset X of A such that $a\&X = \mathbf{0}$.

3. The commutator in equivalential algebras

We assume that the reader is familiar with modular commutator theory as presented in the book [2]. However, for the readers convenience, we sometimes recall some notions from this book and adapt the notation for our use.

By a commutator $[\alpha, \beta]$ of two congruences α, β of an algebra $\mathbf{A}$, we mean the smallest congruence η such that for every $n \geqslant 1$, all $(n+1)$-ary terms $\mathbf{t}$ and elements $a, b, c_1, \ldots, c_n, d_1, \ldots, d_n$ of A with $(a, b) \in \alpha$ and $(c_1, d_1), \ldots, (c_n, d_n) \in \beta$, the following term condition holds:

$$\begin{array}{ccc} \mathbf{t}(a, c_1, \ldots, c_n) & \equiv_\eta & \mathbf{t}(a, d_1, \ldots, d_n) \\ & \Updownarrow & \\ \mathbf{t}(b, c_1, \ldots, c_n) & \equiv_\eta & \mathbf{t}(b, d_1, \ldots, d_n). \end{array}$$

If $\alpha \in \mathsf{Con}(\mathbf{A})$, then by $\mathbf{A}(\alpha)$ we denote the subalgebra of $\mathbf{A}^2$ with universe α. The elements of $\mathbf{A}(\alpha)$ and the pairs of such elements will be denoted by $\binom{x}{y}$ and $\left(\begin{smallmatrix} x & w \\ y & z \end{smallmatrix}\right)$, instead of (x, y) and $((x, y), (w, z))$. The congruence of the algebra $\mathbf{A}(\alpha)$ generated by the set of all pairs of the form $\left(\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right)$ with $(u, v) \in \beta$ is denoted by $\Delta_{\alpha,\beta}$.

Now suppose that $\mathbf{A}$ is an equivalential algebra and that F, G are ideals in $\mathbf{A}$. According to the natural correspondence between ideals and congruences (described in Section 2), we can define their commutator $[F, G]$ as the ideal corresponding to the congruence $[\equiv_F, \equiv_G]$. Moreover, we will write $\mathbf{A}(F)$ for the subalgebra $\mathbf{A}(\equiv_F)$ and $\Delta_{F,G}$ for the ideal on $\mathbf{A}(F)$ corresponding to the congruence $\Delta_{\equiv_F,\equiv_G}$.

The next proposition easily follows from the general theory of commutators in congruence modular varieties.

Proposition 3.1. *For $F, G \in \Phi(\mathbf{A})$ and $a \in A$ we have*

$$a \in [F,G] \quad \textit{iff} \quad \left(\begin{smallmatrix} a \\ \mathbf{0} \end{smallmatrix}\right) \in \left(\left\{\left(\begin{smallmatrix} w \\ w \end{smallmatrix}\right) \;:\; w \in G\right\}\right]_{\mathbf{A}(F)} .$$

Proof. Translating a part of Theorem 4.9 from [2] saying that

$$(x,y) \in [\alpha, \beta] \quad \text{iff} \quad \left(\begin{smallmatrix} x & y \\ y & y \end{smallmatrix}\right) \in \Delta_{\alpha,\beta}$$

into the language of ideals, we get

$$a \in [F,G] \quad \text{iff} \quad \left(\begin{smallmatrix} a \\ \mathbf{0} \end{smallmatrix}\right) \in \Delta_{F,G}.$$

Now, it is enough to observe that $\Delta_{F,G} = \left(\left\{\left(\begin{smallmatrix} w \\ w \end{smallmatrix}\right) \;:\; w \in G\right\}\right]_{\mathbf{A}(F)}$. This however follows immediately from the fact that for each equivalential algebra $\mathbf{B}$ and $C \subseteq B \times B$, the ideal $(\{xy : (x,y) \in C\}]_{\mathbf{B}}$ corresponds to the congruence generated by C. □

Now we are ready for a characterization of the commutator in equivalential algebras.

Theorem 3.2. *For two ideals F, G of an equivalential algebra $\mathbf{A}$, we have*

$$[F,G] = (\{abba, baab : a \in F, b \in G\}] .$$

Proof. Let $H = (\{abba, baab : a \in F, b \in G\}]$. To see that the generators of H, i.e., elements of the form $abba$ are in $[F,G]$, consider the term $\mathbf{t}(x,y) = xyyx$ and apply the term condition to $\mathbf{t}(\mathbf{0},b) = \mathbf{0} = \mathbf{t}(\mathbf{0},\mathbf{0})$ to get $\mathbf{t}(a,b) = \mathbf{t}(a,\mathbf{0}) = \mathbf{0}$. This obviously gives $abba \in [F,G]$.

For the converse inclusion, suppose that $a \in [F,G]$. By Proposition 3.1, this means that $\left(\begin{smallmatrix} a \\ \mathbf{0} \end{smallmatrix}\right)$ is in the ideal of $\mathbf{A}(F)$ generated by elements of the form $\left(\begin{smallmatrix} w \\ w \end{smallmatrix}\right)$ with $w \in G$. By Proposition 2.5, this has to be witnessed by $b_1, \ldots, b_n \in G$ and finite subsets $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ of A with

$$X_i = \left\{x^i_1, \ldots, x^i_{k_i}\right\},\ Y_i = \left\{y^i_1, \ldots, y^i_{k_i}\right\},\ Z_i = \left\{x^i_1 y^i_1, \ldots, x^i_{k_i} y^i_{k_i}\right\} \subseteq F,$$

satisfying

$$a(b_1 \& X_1) \cdots (b_n \& X_n) = \mathbf{0} \text{ and } \mathbf{0}(b_1 \& Y_1) \cdots (b_n \& Y_n) = \mathbf{0}.$$

Obviously, $b_i \equiv_H b_i \& Z_i$, so that the properties of the retractions of the form χ_x, χ_y and χ_{xy} give

$$b_i \& X_i \equiv_H b_i \& Z_i \& X_i = b_i \& Y_i \& X_i = b_i \& Z_i \& Y_i \equiv_H b_i \& Y_i.$$

Consequently, as $a \in [F,G] \subseteq F$, we get

$$\begin{aligned} a &\equiv_H a\&\{b_1\&X_1,\dots,b_n\&X_n\} \\ &= a(b_1\&X_1)\cdots(b_n\&X_n)\,((b_1\&X_1)\cdots(b_n\&X_n)) \\ &= (b_1\&X_1)\cdots(b_n\&X_n) \equiv_H (b_1\&Y_1)\cdots(b_n\&Y_n)) = \mathbf{0}, \end{aligned}$$

which completes the proof. □

From the above theorem, we immediately get that the orthogonality relation developed in the theory of equivalential algebras captures centrality.

Corollary 3.3. *For two ideals F, G of an equivalential algebra $\mathbf{A}$, we have $[F,G] = 0$ iff $F \perp G$.*

In particular, we have the following:

Corollary 3.4. *The variety of Abelian equivalential algebras coincides with the variety $\mathcal{E}_2$ of Boolean groups.*

Also, the concept of density has a natural counterpart in commutator theory.

Proposition 3.5. *If F is an ideal of an equivalential algebra $\mathbf{A}$, then $[F,F] = D_{\mathbf{F}}$. In particular, $[A,A]$ is the ideal of all dense elements in $\mathbf{A}$.*

Proof. It is easy to check that $D_{\mathbf{F}}$ is an ideal in $\mathbf{A}$. Moreover, we have $(abba)bb = abbb(ab) = ab(ab) = \mathbf{0}$, so that $abba \in D_{\mathbf{F}}$ for $a,b \in F$. Consequently, $[F,F] \subseteq D_{\mathbf{F}}$.

Conversely, if $a \in D_{\mathbf{F}}$, then there is a finite $Y \subseteq F$ with $a\&Y = \mathbf{0}$. Since $a\&Y \equiv_{[F,F]} a$, we get $a \in [F,F]$, as required. □

Our description of the commutator allows us to axiomatize the subvariety of $\mathcal{V}$ consisting of all n-step solvable algebras.

Define the n-ary term $\mathbf{p}_n$ inductively, putting $\mathbf{p}_1(x_1) = x_1$ and

$$\mathbf{p}_{n+1}(x_1,\dots,x_{n+1}) = x_{n+1}\mathbf{p}_n(x_1,\dots,x_n)\mathbf{p}_n(x_1,\dots,x_n)x_{n+1}.$$

The subvariety $\mathcal{E}h_n$ of $\mathcal{E}$ determined by the identity $\mathbf{p}_n = \mathbf{0}$ consists of all algebras in which every linearly ordered subuniverse has at most n elements (see e.g. [7]). Another useful characterization of this variety is the following: $\mathbf{A} \in \mathcal{E}h_{n+1}$ iff the length of any chain of completely meet irreducible elements in $\mathsf{Con}(\mathbf{A})$ does not exceed n (see [10]).

Recall that $\mathbf{A} \in \mathcal{E}$ is *n-step solvable* if $[A]^n = 0$, where $[A]^k$ is defined inductively by $[A]^0 = A$ and $[A]^{k+1} = \big[[A]^k,[A]^k\big]$.

Theorem 3.6. *An equivalential algebra $\mathbf{A}$ is n-step solvable iff $\mathbf{A} \in \mathcal{E}h_{n+1}$.*

Proof. The 'only if' direction follows from the fact that for $a_1,\dots,a_{k+1} \in A$, we have $b_{k+1} := \mathbf{p}_{k+1}(a_1,\dots,a_{k+1}) \in [A]^k$. To see the last claim, we induct on k. Obviously, $\mathbf{p}_1(a_1) = a_1 \in A = [A]^0$. Now, suppose that $b_k := \mathbf{p}_k(a_1,\dots,a_k) \in [A]^{k-1}$. Then applying Proposition 2.4 to $b_{k+1}b_kb_k =$

$a_{k+1}b_kb_ka_{k+1}b_kb_k = \mathbf{0}$, we get $b_{k+1} \in (b_k] \subseteq [A]^{k-1}$. Hence, by Theorem 3.2, we have $b_{k+1} = b_{k+1}b_kb_kb_{k+1} \in \big[[A]^{k-1}, [A]^{k-1}\big] = [A]^k$.

For the 'if' direction, we induct on n to show that every algebra from $\mathcal{E}h_{n+1}$ is n-solvable. If $n = 1$, the assertion follows from Corollary 3.4. Now, if $n > 1$, it suffices to show that if $\mathbf{A}$ is a subdirectly irreducible algebra from $\mathcal{E}h_{n+1}$, then $[A]^n = 0$. If $M = \{\mathbf{0}, \star\}$ is the smallest nontrivial ideal of $\mathbf{A}$, then $\mathbf{A}/M \in \mathcal{E}h_n$, as otherwise we have $\mathbf{p}_n(a_1, \ldots, a_n) \notin M$ for some $a_1, \ldots, a_n \in A$ which, by Theorem 2.3, gives

$$\mathbf{p}_{n+1}(a_1, \ldots, a_n, \star) = \star\mathbf{p}_n(a_1, \ldots, a_n)\mathbf{p}_n(a_1, \ldots, a_n)\star = \star \neq \mathbf{0},$$

a contradiction. Now the induction hypothesis gives $[\mathbf{A}/M]^{n-1} = 0$, i.e., $[A]^{n-1} \subseteq M$. Consequently, $[A]^n = \big[[A]^{n-1}, [A]^{n-1}\big] \subseteq [M, M] = 0$, and we are done. □

Our next proposition characterizes the centralizer $(G : F)$ of ideals of an equivalential algebra $\mathbf{A}$, i.e., the largest $H \in \Phi(\mathbf{A})$ with $[F, H] \subseteq G$.

Proposition 3.7. *For $F, G \in \Phi(\mathbf{A})$, we have*

$$(G : F) = \{a \in A : acca, caac \in G \ \ \textit{for all}\ c \in F\}.$$

Proof. The only non-trivial point is to prove that the set

$$H := \{a \in A : acca, caac \in G \ \ \text{for all}\ c \in F\}$$

is an ideal. We first prove that $axx \in H$ for every $a \in H$ and $x \in A$. Let $c \in F$. Then $(axx)cc(axx) = (acca)xx \in G$. Moreover, $c(axx)(axx) \equiv_G caa(axx)(axx) = caa \equiv_G c$, so that $c(axx)(axx)c \in G$.

It remains to prove that $a, ab \in H$ gives $b \in H$. Let $c \in F$. We have $ab \equiv_G abcc = (acc)(bcc) \equiv_G a(bcc)$ and $bccb \in F$. Hence, $bccbaa \in G$ and $bccb \equiv_G bccbaa$. Thus, $bccb \in G$. Moreover, as $cbb \in F$, we deduce that $cbb \equiv_G cbbaa = caa(ba)(ba) \equiv_G caa \equiv_G c$ and, in consequence, $cbbc \in G$. This completes the proof. □

As an immediate consequence of the above proposition, we get a simple characterization of the center $Z(\mathbf{A}) = (0 : A)$ of an equivalential algebra $\mathbf{A}$. In a group, the center consists of all elements commuting with any other element of the group. This has an analogue in the theory of equivalential algebras, where the center of $\mathbf{A}$ is the set of all elements generating with any other element of A an associative subalgebra, i.e., a Boolean group.

Corollary 3.8. $Z(\mathbf{A}) = \{a \in A : a \perp A\}$.

Using Proposition 2.2 one can easily infer that the join of (arbitrary many) Abelian congruences is Abelian. Consequently, every algebra $\mathbf{A}$ from $\mathcal{E}$ has a largest Abelian congruence. This congruence will be called the *Abelian radical* of $\mathbf{A}$.

Proposition 3.9. *The ideal corresponding to the Abelian radical of an equivalential algebra* $\mathbf{A}$ *consists of all elements* $a \in A$ *such that* $xaa = x$ *for all* $x \in A$*.*

Proof. Put $R := \{a \in A : xaa = x \text{ for every } x \in A\}$. First we check that R is an ideal. Indeed, $x\mathbf{00} = x$, so that $\mathbf{0} \in R$. Assume now that $a, ab \in R$. Then $xbb = xbbaa = x(ab)(ab)aa = x(ab)(ab) = x$. Finally, for $a \in R$ and $b \in A$, we have $x(abb)(abb) = xaa(abb)(abb) = xaa = x$, as required.

Now, to see that R is Abelian, assume that $x \in [R, R] = D_{\mathbf{R}}$. This means that $x\&E = \mathbf{0}$ for some finite subset $E \subseteq R$. But the definition of R then gives that $x = \mathbf{0}$.

Finally, we have to show that R is the largest Abelian ideal, i.e., if $a \notin R$, then the principal ideal $(a]$ is not Abelian. Take $x \in A$ such that $xaa \neq x$. Moreover, $xaax \equiv_{(a]} \mathbf{0}$, i.e., $xaax \in (a]$. This, by Theorem 3.2, gives $\mathbf{0} \neq xaax = xaaxaa(xaax) \in [(a], (a]]$, and so $(a]$ is not Abelian. □

4. Solvable Fregean varieties

If $\mathcal{V}$ is a congruence permutable Fregean variety, then Theorem 1.1(2) provides us with a binary term $\mathbf{e}$ such that the $\mathbf{e}$-reduct $\mathbf{A}^{\mathbf{e}}$ of an algebra $\mathbf{A} \in \mathcal{V}$ is an equivalential algebra. In particular, we have that $\mathsf{Con}(\mathbf{A})$ is a sublattice of $\mathsf{Con}(\mathbf{A}^{\mathbf{e}})$. In general for $\alpha, \beta \in \mathsf{Con}(\mathbf{A})$, the commutator $[\alpha, \beta]_{\mathbf{A}^{\mathbf{e}}}$ computed in the algebra $\mathbf{A}^{\mathbf{e}}$ is smaller than the one $[\alpha, \beta]_{\mathbf{A}}$ computed in the richer algebra $\mathbf{A}$. The aim of this section is to prove that they are equal, provided $\mathcal{V}$ is solvable.

Lemma 4.1. *If* α *is a solvable congruence of an algebra* $\mathbf{A}$ *from a congruence permutable Fregean variety, then* $[\alpha, \alpha]_{\mathbf{A}} = [\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}}$*.*

Proof. We start by proving that

(1) if $\alpha \in \mathsf{Con}(\mathbf{A})$, then $[\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}} \in \mathsf{Con}(\mathbf{A})$.

By $\mathbf{0}$-regularity, it suffices to show that

$$\Theta_{\mathbf{A}}(\mathbf{0}, x) \subseteq [\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}} \quad \text{whenever} \quad (\mathbf{0}, x) \in [\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}},$$

or in other words, that

$$x \in \mathbf{0}/[\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}} \quad \text{and} \quad \Theta_{\mathbf{A}}(\mathbf{0}, y) \subseteq \Theta_{\mathbf{A}}(\mathbf{0}, x) \quad \text{imply} \quad y \in \mathbf{0}/[\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}}.$$

Proposition 3.5 applied to $x \in \mathbf{0}/[\alpha, \alpha]_{\mathbf{A}^{\mathbf{e}}}$ gives a finite subset L of $\mathbf{0}/\alpha$ with $x\&L = \mathbf{0}$. Therefore,

$$(y\&L)x \equiv_{\Theta_{\mathbf{A}}(\mathbf{0},x)} y\&L \equiv_{\Theta_{\mathbf{A}}(\mathbf{0},x)} \mathbf{0}\&L = \mathbf{0}.$$

On the other hand, modulo $\Theta_{\mathbf{A}}(\mathbf{0}, (y\&L)x)$, we have

$$\mathbf{0} \equiv (y\&L)x = (y\&L\&L)(x\&L)x = (((y\&L)x)\&L)x \equiv (\mathbf{0}\&L)x = x.$$

Therefore, $\Theta_{\mathbf{A}}(\mathbf{0}, x) = \Theta_{\mathbf{A}}(\mathbf{0}, (y\&L)x)$, i.e., $(y\&L)x = x$. This in turn gives $y\&(L \cup \{x\}) = \mathbf{0}$. However, $x \in \mathbf{0}/\,[\alpha,\alpha]_{\mathbf{A}^{\mathbf{e}}} \subseteq \mathbf{0}/\alpha$, so that $L \cup \{x\}$ is a (finite) subset of $\mathbf{0}/\alpha$. Therefore, by Proposition 3.5, $y \in \mathbf{0}/\,[\alpha,\alpha]_{\mathbf{A}^{\mathbf{e}}}$, as required.

(2) If $\mathbf{A}$ is a subdirectly irreducible algebra from a congruence permutable Fregean variety and μ is its monolith, then $\mathbf{A}^{\mathbf{e}}$ is subdirectly irreducible with monolith μ.

According to Theorem 2.3, the only nontrivial coset of μ has the form $\{\mathbf{0}, \star\}$ and $a\star = a$ for all $a \in A - \{\mathbf{0}, \star\}$. From Proposition 2.4, we get $\mu \subseteq \Theta_{\mathbf{A}^{\mathbf{e}}}(\mathbf{0}, a)$, and so μ is the monolith of $\mathbf{A}^{\mathbf{e}}$, as required.

Now to prove our Lemma, assume that $[\alpha,\alpha]_{\mathbf{A}^{\mathbf{e}}} < [\alpha,\alpha]_{\mathbf{A}}$. From (1), we know that $[\alpha,\alpha]_{\mathbf{A}^{\mathbf{e}}}$ is a congruence of $\mathbf{A}$.

Hence and since $[\alpha,\alpha]_{\mathbf{A}}$ is solvable, then the second part of [5, Proposition 16] provides a subcover β of $[\alpha,\alpha]_{\mathbf{A}}$ that contains $[\alpha,\alpha]_{\mathbf{A}^{\mathbf{e}}}$. Now pick η to be a maximal congruence of $\mathbf{A}$ that is over β but not over $[\alpha,\alpha]_{\mathbf{A}}$. Then $\mathbf{A}/\eta$, and therefore by (2), also $\mathbf{A}^{\mathbf{e}}/\eta$ is subdirectly irreducible.

Moreover, $([\alpha,\alpha]_{\mathbf{A}} \vee \eta)/\eta$ is the monolith in $\mathbf{A}^{\mathbf{e}}/\eta$. However, in $\mathbf{A}^{\mathbf{e}}$ the congruence $\alpha \vee \eta$ is Abelian over η so that Proposition 2.1 gives that $\alpha \vee \eta = [\alpha,\alpha]_{\mathbf{A}} \vee \eta$. Consequently, modularity gives us that $\alpha \wedge \eta$ is a subcover of α. Moreover, since α is solvable, then α is Abelian over $\alpha \wedge \eta$. Thus, the first part of [5, Proposition 16] gives that $[\alpha,\alpha]_{\mathbf{A}} \subseteq \alpha \wedge \eta$, a contradiction with our choice of η. □

Theorem 4.2. *Let* $\mathbf{A}$ *be a solvable algebra from a Fregean variety. Then there is a binary term* $\mathbf{e}$ *such that the* $\mathbf{e}$*-reduct* $\mathbf{A}^{\mathbf{e}}$ *of* $\mathbf{A}$ *is an equivalential algebra. Moreover, for any two congruences* α, β *of* $\mathbf{A}$*, we have* $[\alpha,\beta]_{\mathbf{A}} = [\alpha,\beta]_{\mathbf{A}^{\mathbf{e}}}$.

Proof. Since $\mathbf{A}$ is solvable and belongs to a congruence modular variety, the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ is solvable. Therefore, by Theorem 6.2 of [2], $\mathcal{V}(\mathbf{A})$ is congruence permutable, so that we have a binary term $\mathbf{e}$ that satisfies the first part of theorem.

On the other hand, both $\mathbf{A}$ and $\mathbf{A}^{\mathbf{e}}$ belong to Fregean varieties. Hence, Proposition 2.2 ensures us that in $\mathsf{Con}(\mathbf{A})$, as well as in $\mathsf{Con}(\mathbf{A}^{\mathbf{e}})$, the commutator of congruences is determined by the lattice operations and commutator 'square', i.e., $[\alpha,\beta] = (\alpha \wedge [\beta,\beta]) \vee (\beta \wedge [\alpha,\alpha])$.

However, on the congruences of $\mathbf{A}$, the lattice operations and, by Lemma 4.1, the commutator 'square' are the same in $\mathsf{Con}(\mathbf{A})$ and $\mathsf{Con}(\mathbf{A}^{\mathbf{e}})$, so that the Theorem follows. □

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