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Rigidity of unary algebras and its application to the $\mathcal{HS} = \mathcal{SH}$ problem

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Abstract. H. P. Gumm and T. Schröder stated a conjecture that the preservation of preimages by a functor T for which $|T1| = 1$ is equivalent to the satisfaction of the class equality $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras. Although T. Brengos and V. Trnková gave a positive answer to this problem for a wide class of Set-endofunctors, they were unable to find the full solution. Using a construction of a rigid unary algebra we prove $\mathcal{HS} \neq \mathcal{SH}$ for a class of Set-endofunctors not preserving non-empty preimages; these functors have not been considered previously.

1. Introduction

H. P. Gumm and T. Schröder in [4] observed that there is a strong relation between the satisfaction of the equation $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras and preimage preservation of the functor T. It is summarized by the following two theorems.

Theorem 1.1 ([4]). *Assume that a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ preserves preimages. Then for any class K of T-coalgebras $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$.*

Theorem 1.2 ([4]). *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor such that $|T1| > 1$. If $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras then T preserves preimages.*

The following problem has been stated: is the satisfaction of the equation $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras equivalent to T preserving preimages?

By Theorem 1.1 and Theorem 1.2 it follows that in order to find a positive answer to the $\mathcal{HS} = \mathcal{SH}$ problem it is enough to find, for any functor $T\colon \mathsf{Set} \to \mathsf{Set}$ not preserving preimages and satisfying $|T1| = 1$, a T-coalgebra $\mathbb{X}$ such that $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$. However, as has been shown in [2], the constant functor $\mathcal{C}_{0,1}$ sending the empty set to itself and all non-empty sets to a one-element set is a counterexample. The condition to be investigated to be equivalent to $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ should be the preservation of *non-empty preimages*. T. Brengos and V. Trnková in [2] succeeded in finding a wide class of functors not preserving non-empty preimages and satisfying $|T1| = 1$ for which the

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equality $\mathcal{HS} = \mathcal{SH}$ does not hold. Any Set-endofunctor T which does not preserve non-empty preimages and satisfies $|T1| = 1$ falls into exactly one of the two classes:

- functors which do not preserve non-empty preimages on undistinguished points,
- functors which preserve non-empty preimages on undistinguished points.

The authors of [2] have managed to find a construction of a T-coalgebra $\mathbb{X}$ with $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ for any functor T belonging to the first class listed above.

In the case of functors belonging to the second class, the authors were not able to give a complete solution to the problem. Instead, they presented a general setting in which it was possible to construct a T-coalgebra $\mathbb{X}$ with $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ and listed examples of families of functors preserving non-empty preimages on undistinguished points that can be put in this setting.

This paper shows how to apply the general setting from [2] and use a rigid unary algebra of type $(1, 1)$ over a set of arbitrary cardinality to construct a coalgebra $\mathbb{X}$ satisfying $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ for a class of functors not considered before. This is summarized by Theorem 3.18.

2. Basic notions

Let Set be the category of all sets and mappings between them. Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. A *T-coalgebra* $\mathbb{X}$ is a pair $\langle X, \xi\rangle$, where X is a set and ξ is a mapping $\xi\colon X \to TX$. The set X is called a *carrier* and the mapping ξ is called a *structure* of the coalgebra $\mathbb{X} = \langle X, \xi\rangle$.

A *homomorphism* from a T-coalgebra $\mathbb{X} = \langle X, \xi\rangle$ to a T-coalgebra $\mathbb{Y} = \langle Y, \psi\rangle$ is a mapping $h\colon X \to Y$, such that $T(h) \circ \xi = \psi \circ h$.

A T-coalgebra $\mathbb{S} = \langle S, \sigma\rangle$ is said to be a *subcoalgebra* of a T-coalgebra $\mathbb{X} = \langle X, \xi\rangle$ whenever there is an injective homomorphism from $\mathbb{S}$ into $\mathbb{X}$. This fact is denoted by $\mathbb{S} \leq \mathbb{X}$.

Let $h\colon \mathbb{X} \to \mathbb{Y}$ be a homomorphism and let $\mathbb{S} \leq \mathbb{X}$. Then there is a structure $\delta\colon h(S) \to T(h(S))$ such that $h|_S\colon S \to h(S)$ is a homomorphism from $\mathbb{S} = \langle S, \sigma\rangle$ onto $\langle h(S), \delta\rangle$ and $\langle h(S), \delta\rangle \leq \langle Y, \psi\rangle$. In other words, a homomorphic image of a subcoalgebra of a domain of h is a subcoalgebra of a codomain of h. For the basics on the theory of coalgebras the reader is reffered to [3].

A functor $T\colon \mathsf{Set} \to \mathsf{Set}$ *preserves preimages* if for any mapping $f\colon A \to B$ and a subset $C \subset B$, the following diagram is a pullback diagram.

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle T\subseteq^A_{f^{-1}(C)}}\uparrow & & \uparrow{\scriptstyle T\subseteq^B_C} \\ T[f^{-1}(C)] & \xrightarrow{Tf|} & TC \end{array}$$

Let $\mathcal{C}_{0,1}$ denote the constant functor for which $\mathcal{C}_{0,1}\varnothing = \varnothing$ and $\mathcal{C}_{0,1}A = \{\perp\}$ for $A \neq \varnothing$, and for which, given $f\colon A \to B$, the map $\mathcal{C}_{0,1}(f)\colon \mathcal{C}_{0,1}A \to \mathcal{C}_{0,1}B$ is an empty map in the case when $A = \varnothing$ and is the identity $id_{\{\perp\}}$ otherwise.

We say that a natural transformation $\mu\colon G \to T$ from a functor G to a functor T is *injective* whenever for any set A, the mapping $\mu_A\colon GA \to TA$ is an injective mapping. We say that a functor G is a *subfunctor* of a functor T whenever there exists an injective natural transformation from G to T. We denote this fact by $G \leq T$. We say that $\vec{a} \in TA$ is a *distinguished point* (see [8]) if there is a natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$ such that $\mu(\perp) = \vec{a}$. An element $\vec{a} \in TA$ is *undistinguished* if it is not a distinguished point. Note that by the definition of $\mathcal{C}_{0,1}$, any natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$ is injective. Thus, $\mathcal{C}_{0,1} \leq T$ if and only if for a set A there is a distingushed point $\vec{a} \in TA$.

Definition 2.1 ([1])**.** A distinguished point $\mu_X(\perp) \in TX$, where $\mu\colon \mathcal{C}_{0,1} \to T$, is called *standard* whenever $T\varnothing$ contains precisely one element, say $\vec{s}$, such that for every non-empty set X, $\vec{s}$ is sent by the T-image of the empty map $\epsilon_X\colon \varnothing \to X$ exactly to the image $\mu_X(\perp)$.

Definition 2.2 ([1])**.** A functor $T\colon \mathsf{Set} \to \mathsf{Set}$ is called a *standard functor* provided that:

(1) it preserves inclusions, i.e., for any inclusion map $\subseteq_A^B\colon A \to B$, the map $T(\subseteq_A^B)$ is an inclusion map, that is, $T(\subseteq_A^B) = \subseteq_{TA}^{TB}$;
(2) all distinguished points are standard.

Remark 2.3. Every functor $T\colon \mathsf{Set} \to \mathsf{Set}$ preserves non-empty intersections of finitely many sets (see [8] for a proof), but a standard functor also preserves the empty interesections of finitely many sets. Hence, e.g., $\mathcal{C}_{0,1}$ is not a standard functor, but $\mathcal{C}_1$ sending any set to $\{\perp\}$ is standard.

Theorem 2.4 ([1], p. 132)**.** *For each* Set*-endofunctor* T*, there exists a standard* Set*-endofunctor* T' *such that the restrictions of* T *and* T' *to all non-empty sets and non-empty maps are naturally isomorphic.*

If we only investigate the preservation of non-empty preimages, then by Theorem 2.4, we may restrict ourselves to standard functors. Therefore, throughout this paper, without loss of generality, we assume that every functor T we deal with is standard.

We have the following.

Lemma 2.5 ([2])**.** *Let* $T\colon \mathsf{Set} \to \mathsf{Set}$ *be a functor; let* $f\colon A \to B$ *be a mapping, and let* $S \subseteq A$*. Then* $[Tf](TS) = T[f(S)]$*.*

A functor T is said to be *connected* if $|T1| = 1$. Let $\mathcal{I}d$ denote the identity functor.

Lemma 2.6 ([11])**.** *Let* $T\colon \mathsf{Set} \to \mathsf{Set}$ *be a standard connected functor. Then either* T *contains exactly one isomorphic copy of* $\mathcal{I}d$ *as a subfunctor or it contains exactly one isomorphic copy of* $\mathcal{C}_{0,1}$ *as a subfunctor.*

If T is a connected functor and if $\mathcal{C}_{0,1} \leq T$, then by Lemma 2.6, we know that the set TA contains precisely one distinguished point. Therefore, whenever we speak of a connected functor with $\mathcal{C}_{0,1} \leq T$, we denote the distinguished element of TA by $\perp$. Moreover, if T is a connected functor with $\mathcal{I}d \leq T$, then by Lemma 2.6, the functor $\mathcal{C}_{0,1}$ is not a subfunctor of T. This means that all elements from TX are undistinguished.

Lemma 2.7 ([9])**.** *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard connected functor such that $\mathcal{C}_{0,1} \leq T$. The functor T preserves non-empty preimages if and only if the restrictions of T and $\mathcal{C}_{0,1}$ to non-empty sets are naturally equivalent.*

Let $\mathcal{F}$ denote the functor which assigns to every set A the set

$$\{\mathcal{G} \mid \mathcal{G} \text{ is a filter on } A\} \cup \{\mathcal{P}(A)\}$$

and to every mapping $f\colon A \to B$ the mapping

$$\mathcal{F}(f)\colon \mathcal{F}A \to \mathcal{F}B; \quad \mathcal{G} \mapsto \{V \subseteq B \mid f(W) \subseteq V \text{ for some } W \in \mathcal{G}\}.$$

Functor $\mathcal{F}$ is called the *filter functor*.

Lemma 2.8. *Let $\mathcal{G}$ be a filter on a set A and let $f\colon A \to B$ be any function. Moreover, let A_1 be a subset of A such that $A_1 \in \mathcal{G}$ and let B_1 be a subset of B for which $B_1 \in \mathcal{F}(f)(\mathcal{G})$. Then there is a subset $V \subseteq A_1$ satisfying $V \in \mathcal{G}$ for which $h(V) \subseteq B_1$.*

Proof. Because $h(A_1) \in \mathcal{F}(f)(\mathcal{G})$ and $B_1 \in \mathcal{F}(f)(\mathcal{G})$, it follows that

$$h(A_1) \cap B_1 \in \mathcal{F}(f)(\mathcal{G}).$$

By the definition of the filter $\mathcal{F}(f)(\mathcal{G})$, it follows that there is a subset $V_1 \in \mathcal{G}$ for which $h(V_1) \subseteq h(A_1) \cap B_1$. Put $V := V_1 \cap A_1$. Clearly, $V \in \mathcal{G}$ and

$$h(V) \subseteq h(V_1) \subseteq h(A_1) \cap B_1 \subseteq B_1. \qquad \square$$

Lemma 2.9 ([9], Proposition II.4)**.** *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor and let $\vec{a} \in TA$. The following conditions are equivalent:*

- *there are two non-empty disjoint subsets $U_1, U_2 \subset A$ with $\vec{a} \in TU_1$ and $\vec{a} \in TU_2$;*
- *the element $\vec{a}$ is a distinguished point.*

Let T be an arbitrary Set endofunctor and A be a set. For any $\vec{a} \in TA$, define a collection $\mathsf{Flt}_A(\vec{a})$ of subsets of A as follows:

$$\mathsf{Flt}_A(\vec{a}) := \begin{cases} \{U \subseteq A \mid \vec{a} \in TU\} & \text{if } \vec{a} \text{ is not distinguished,} \\ \mathcal{P}(A) & \text{otherwise.} \end{cases}$$

It has been shown in [9] that for an undistinguished point $\vec{a} \in TA$, the collection $\mathsf{Flt}_A(\vec{a})$ forms a filter on A. By Lemma 2.9, it follows that $\varnothing \in \mathsf{Flt}_A(\vec{a})$ if and only if $\vec{a}$ is a distinguished point.

For any set A we may now consider Flt_A as a mapping

$$\mathsf{Flt}_A\colon TA \to \mathcal{F}A; \vec{a} \mapsto \mathsf{Flt}_A(\vec{a}).$$

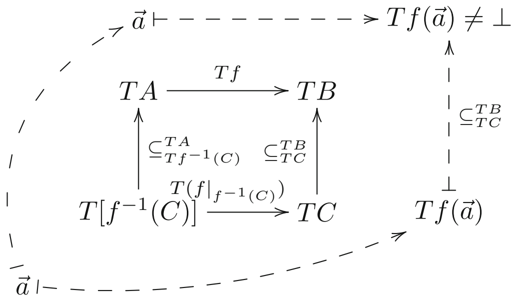
The collection $\mathsf{Flt} = \{\mathsf{Flt}_A\}_{A \in \mathsf{Set}}$ is a transformation from T to $\mathcal{F}$. It is worth noting that Flt need not be a natural transformation from T to $\mathcal{F}$. We have the following two theorems.

Theorem 2.10 ([9])**.** *For any injective mapping $f\colon A \to B$ and any $\vec{a} \in TA$, we have $\mathsf{Flt}_B(Tf(\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a}))$.*

Theorem 2.11 ([9], Proposition VII.5)**.** *The transformation $\mathsf{Flt}\colon T \to \mathcal{F}$ is a natural transformation, i.e., for any $\vec{a} \in TA$ and any mapping $f\colon A \to B$ we have $\mathsf{Flt}_B(Tf(\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a}))$ if and only if T preserves preimages.*

Let $\mathcal{G}$ be a filter on a set A. A subset $S \subseteq A$ is called *$\mathcal{G}$-stationary* if any member of $\mathcal{G}$ has a non-empty intersection with S. A filter $\mathcal{G}$ is called *uniform* if it only contains subsets of A of the same cardinality as A.

2.1. Functors preserving non-empty preimages on undistinguished points. We say that a Set endofunctor T *preserves non-empty preimages on undistinguished points*, whenever for any mapping $f\colon A \to B$, any subset $C \subset B$ with $f^{-1}(C) \neq \varnothing$, and any element $\vec{a} \in TA$ such that $Tf(\vec{a})$ is an undistinguished point, we have $Tf(\vec{a}) \in TC$ implies $\vec{a} \in T[f^{-1}(C)]$.



Note that the element $\vec{a}$ in the above definition cannot be a distinguished point. If $\vec{a} \in TA$ was distinguished, then for any mapping $f\colon A \to B$, the element $Tf(\vec{a})$ would also be a distinguished point.

We stress that the only difference between the notion of a functor preserving non-empty preimages and a functor preserving non-empty preimages on undistinguished points is the additional assumption in the formulation of the latter which says that the element $Tf(\vec{a}) \neq \bot$. A functor which preserves non-empty preimages on undistinguished points does not have to preserve non-empty preimages in general.

Remark 2.12. If a connected functor $T\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve non-empty preimages but preserves them on undistinguished points, then it has to be the case that $\mathcal{C}_{0,1}$ is a proper subfunctor of T, i.e., $\mathcal{C}_{0,1} < T$. See [2] for details.

Example 2.13. Let F denote the functor $\mathsf{Hom}(2, \_)$, i.e., $FA = \{(a_0, a_1) \mid a_i \in A\}$ for any set A, and let $\nu\colon F \to T$ be an epitransformation given by the following equation:

$$\nu_X(x,x) = \nu_X(y,y), \text{ for any } X \text{ and for any } x,y \in X.$$

The functor T is connected and contains a copy of $\mathcal{C}_{0,1}$ as a subfunctor. Let $f\colon A \to B$ be any mapping and C a subset of B such that $f^{-1}(C) \neq \varnothing$. Choose any element $\vec{a}$ from TA such that $Tf(\vec{a}) \neq \bot$ and $Tf(\vec{a}) \in TC$. By our assumptions, since $Tf(\vec{a})$ is undistinguished and is an element of TC, we have $Tf(\vec{a}) = \nu(c_1,c_2)$ for $c_1,c_2 \in C$ and $c_1 \neq c_2$. By the definition of T, it follows that there exist $a_1,a_2 \in TA$ with $a_1 \neq a_2$ such that $\vec{a} = \nu(a_1,a_2)$ and $Tf(\vec{a}) = Tf(\nu(a_1,a_2)) = \nu(c_1,c_2)$. Hence, $f(a_1) = c_1$ and $f(a_2) = c_2$. This means that $a_1,a_2 \in f^{-1}(C)$ and $\vec{a} = \nu(a_1,a_2) \in Tf^{-1}(C)$. Therefore, T does not preserve non-empty preimages (see Lemma 2.7) but does preserve non-empty preimages on undistinguished points.

The next example shows that if a connected functor $T\colon \mathsf{Set} \to \mathsf{Set}$ satisfies $\mathcal{C}_{0,1} < T$, it does not mean that T must preserve non-empty preimages on undistinguished points.

Example 2.14 ([2]). Let G be $\mathsf{Hom}(4, \_)$, i.e., $GA = \{(a_0,a_1,a_2,a_3) \mid a_i \in A\}$ for any set A, and let $\nu\colon G \to T$ be an epitransformation given by the following equations:

$$\nu_Y(y,y,y,y) = \nu_Y(z,z,z,z), \text{ for any } Y \text{ and for any } y,z \in Y,$$
$$\nu_Y(w_1,y,y,z) = \nu_Y(w_2,y,y,z), \text{ for any } Y \text{ and for any } w_1,w_2,y,z \in Y.$$

For $(v,w,y,z) \in Y^4$ let us introduce the following notation

$$[v,w,y,z]^Y_\nu := \{(v',w',y',z') \in Y^4 \mid \nu_Y(v,w,y,z) = \nu_Y(v',w',y',z')\}.$$

The functor T is connected and contains a copy of $\mathcal{C}_{0,1}$ as a subfunctor. Hence, T does not preserve non-empty preimages. Moreover, the functor T is an example of a functor which does not preserve non-empty preimages on undistinguished points. See [2] for a proof.

2.2. Standing Hypothesis. We will now present the general setting, which has been introduced in [2], in which it is possible to construct a T-coalgebra $\mathbb{X}$ such that $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ for a functor T preserving non-empty preimages on undistinguished points. We will use this setting in Section 3 to prove $\mathcal{HS} \neq \mathcal{SH}$ for Set-endofunctors not considered previously.

In the rest of the paper, we assume that the functor $T\colon \mathsf{Set} \to \mathsf{Set}$ is *connected, standard*, it *does not preserve non-empty preimages*, but it *does preserve non-empty preimages on undistinguished points*, and it has a *distinguished point* $\bot \in TX$ for a non-empty set X. By Lemma 2.7, the functor T is distinct from a constant functor. This means that there is a cardinal λ for which $|T\lambda| > 1$.

We have the following lemma.

Lemma 2.15 ([2]). *Let T be as above. Let $f\colon A \to B$ be a mapping and $\vec{a} \in TA$ such that $[Tf](\vec{a}) \neq \bot$. Then $\mathsf{Flt}_B([Tf](\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a}))$.*

Assume there is a T-coalgebra $\mathbb{B} = \langle B, \beta\rangle$ and two undistinguished elements $\vec{a}_{B,1} \in TB$ and $\vec{a}_{B,2} \in TB$ satisfying the following properties.

(1) $\bot \notin \beta(B)$ and $\vec{a}_{B,1}, \vec{a}_{B,2} \notin \beta(B)$.

(2) If $\mathbb{S} \leq \langle B, \beta\rangle \leq \langle C, \gamma\rangle$ with $\vec{a}_{B,1}, \vec{a}_{B,2} \in TS$ and $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \bot\}$, then given a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ such that $Th(\vec{a}_{B,1}) = \vec{a}_{B,1} \in TB \subseteq TC$, we have $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\bot\}$.

Theorem 2.16 ([2]). *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a connected functor that preserves non-empty preimages on undistinguished points; let $\mathcal{C}_{0,1} \leq T$, and let λ be a cardinal number for which $|T\lambda| > 1$. Assume that there is a T-coalgebra $\mathbb{B}$ with $|B| \geq \lambda$ and elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ satisfying properties* (1) *and* (2) *listed above. Then there is a T-coalgebra $\mathbb{X}$ for which $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$.*

3. Rigidity of unary algebras and its applications

A careful analysis of the methods used in [2] leads to a conclusion that rigid structures play an important role in proving $\mathcal{HS} \neq \mathcal{SH}$ for some classes of functors not preserving non-empty preimages. Indeed, though it was not stated explicitly, a form of an "almost rigid" digraph (a digraph all of whose endomorphisms are injective) of cardinality $\aleph_0$ was applied in the proof of the following theorem.

Theorem 3.1 ([2]). *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be as in the Standing Hypothesis. Assume that $|T\aleph_0| > 1$. Let A be a set of cardinality $\aleph_0$ and let there be an undistinguished point $\vec{a} \in TA$ for which the filter $\mathsf{Flt}_A(\vec{a})$ is free. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

Moreover, a rigid digraph of cardinality $\lambda > \aleph_0$ was used to prove the following.

Theorem 3.2 ([2]). *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be as in the Standing Hypothesis and let λ is an uncountable cardinal for which $|T\lambda| > 1$. Let A be a set of cardinality λ. Finally, let there be an undistinguished point $\vec{a} \in TA$ and a partition $\{A_l\}_{l \leq \lambda}$ of A into λ-many non-empty disjoint $\mathsf{Flt}_A(\vec{a})$-stationary subsets of A. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

The additional assumption stated in Theorem 3.2 about the existence of an element $\vec{a} \in TA$ and a partition $\{A_l\}_{l \in \lambda}$ of the set A into λ-many subsets each being $\mathsf{Flt}_A(\vec{a})$-stationary is required because all known rigid graphs of arbitrary cardinality $\lambda > \aleph_0$ require that at least one vertex v have λ-many different neighbours. The obvious question that arises is the following: are there any different rigid structures that may be used instead of graphs in the solution to the $\mathcal{HS} = \mathcal{SH}$ problem to weaken (or alter) the assumptions? The answer to the question is positive. We will now demonstrate how to make use of rigidity of unary algebras with two operations.

Before we go any further, we recall some basic definitions from the set theory. Given an infinite limit ordinal α, the *cofinality* of α, denoted cfα, is the least limit ordinal β such that there exists an increasing β-sequence $(\alpha_\xi \mid \xi < \beta)$ with $\lim_{\xi\to\beta} \alpha_\xi = \alpha$. An infinite cardinal $\aleph_\alpha$ is called *singular* if $\mathrm{cf}\omega_\alpha < \omega_\alpha$. In ZFC there are arbitrary large singular cardinals of any given cofinality. For each ordinal α the cardinal $\aleph_{\alpha+\omega}$ is such a cardinal. For more details on singular ordinals and cardinals the reader is referred to e.g., [6].

Let T, $\perp$, λ be as in Standing Hypothesis. We additionally assume that λ is a singular cardinal with cofinality ω. The assumption about singularity implies that $\lambda > \aleph_0$. Moreover, the fact that the cardinal λ has cofinality ω is equivalent to the fact that there is a countable family $\{S_n\}_{n\in\omega}$ of non-empty subsets of λ, each of cardinality less than λ, satisfying $\lambda = \bigcup_{n\in\omega} S_n$.

Without loss of generality, we may assume that each member of the family $\{S_n\}_{n\in\omega}$ is infinite. Let A be any set of cardinality λ. Finally, we suppose that for each $n \in \omega$ there is an undistinguished element $\vec{a}_n \in TA$ and a partition $\{A^n_j\}_{j\in S_n}$ of the set A into $|S_n|$-many non-empty disjoint $\mathsf{Flt}_A(\vec{a}_n)$-stationary subsets. To see an example of a Set-endofunctor satisfying above assumptions but not satisfying assumptions of Theorem 3.2, the reader is reffered to Example 3.5. Presenting Example 3.5 requires the following intermediate results.

Lemma 3.3. *Let κ_1, κ_2 be infinite cardinal numbers satisfying $\kappa_1 < \kappa_2$. Let X be a set of cardinality κ_2. There exists a uniform filter $\mathcal{G}_{\kappa_1}$ on X for which*

(a) there is a partition $\{A_i\}_{i\leq\kappa_1}$ of the set X into κ_1-many non-empty disjoint $\mathcal{G}_{\kappa_1}$-stationary subsets;

(b) there is no partition $\{A'_i\}_{i\leq\kappa}$ of the set X into κ-many non-empty disjoint $\mathcal{G}_{\kappa_1}$-stationary subsets for $\kappa > \kappa_1$.

Proof. Consider a partition $X = X_1 \cup X_2$ of the set X into two non-empty disjoint subsets such that $|X_1| = \kappa_1$. Since $\kappa_1 < \kappa_2$ it follows that $|X_2| = |X| = \kappa_2$. Let F_{X_1} be the Fréchet filter on X_1, i.e., $F_{X_1} = \{Y \subseteq X_1 \mid |X_1 \setminus Y| < \aleph_0\}$. Now consider any uniform ultrafilter U on X_2 and define a family $\mathcal{G}_{\kappa_1}$ of subsets of $X = X_1 \cup X_2$ by

$$\mathcal{G}_{\kappa_1} = \{W \subseteq X \mid Y_1 \cup Y_2 \subseteq W \text{ for some } Y_1 \in F_{X_1} \text{ and } Y_2 \in U\}.$$

We will prove that $\mathcal{G}_{\kappa_1}$ is the desired uniform filter on X. The family $\mathcal{G}_{\kappa_1}$ is a filter. Indeed, if $A, B \in \mathcal{G}_{\kappa_1}$, then $A \supseteq Y_1 \cup Y_2$ and $B \supseteq Y'_1 \cup Y'_2$ for some $Y_1, Y'_1 \in F_{X_1}$ and $Y_2, Y'_2 \in U$. Since $Y_1 \cap Y_2 = Y_1 \cap Y'_2 = Y'_1 \cap Y_2 = Y'_1 \cap Y'_2 = \varnothing$, it follows that

$$A \cap B \supseteq (Y_1 \cup Y_2) \cap (Y'_1 \cup Y'_2) = (Y_1 \cap Y'_1) \cup (Y_2 \cap Y'_2).$$

Hence, $A \cap B \in \mathcal{G}_{\kappa_1}$. Moreover, if $A \in \mathcal{G}_{\kappa_1}$ and $A \subseteq B$, then by the definition of $\mathcal{G}_{\kappa_1}$, it follows that $B \in \mathcal{G}_{\kappa_1}$. Since any member of $\mathcal{G}_{\kappa_1}$ is of cardinality κ_2, it follows that $\mathcal{G}_{\kappa_1}$ is a uniform filter.

Consider any partition $\{A_i\}_{i\leq\kappa_1}$ of the set X_1 into κ_1-many disjoint subsets each of cardinality κ_1. Since the filter F_{X_1} on X_1 is the Fréchet filter on the set X_1, it follows that for any $i \leq \kappa_1$, the subset $A_i \subseteq X_1$ is F_{X_1}-stationary. This means that any member of the family $\{A_i\}_{i\leq\kappa_1}$ is also $\mathcal{G}_{\kappa_1}$-stationary. The subset $X_2 \subseteq X$ is $\mathcal{G}_{\kappa_1}$-stationary. Thus, the partition $X = X_2 \cup \bigcup_{i\leq\kappa_1} A_i$ is a partion of X satisfying the property (a).

Assume that $\{A'_i\}_{i\leq\kappa}$ is a partition of X into κ-many non-empty disjoint subsets for $\kappa > \kappa_1$. We will now show that there is an index $i \leq \kappa$ for which the set A'_i is not $\mathcal{G}_{\kappa_1}$-stationary. Since $\kappa > \kappa_1$ and since $|X_1| = \kappa_1$, it follows that there is $i \leq \kappa$ for which $A'_i \cap X_1 = \varnothing$. Therefore, $A'_i \subseteq X_2 \subseteq X$. If $A'_i \notin U$, then since U is an ultrafilter on X_2, it follows that $X_2 \setminus A'_i \in U$. Since $X_1 \in F_{X_1}$, it follows that $X_1 \cup (X_2 \setminus A'_i) \in \mathcal{G}_{\kappa_1}$ and $A'_i \cap [X_1 \cup (X_2 \setminus A'_i)] = \varnothing$. This means that A'_i is not $\mathcal{G}_{\kappa_1}$-stationary. Now, assume that $A'_i \in U$. If we exclude the set A'_i from the family $\{A'_i\}_{i\leq\kappa}$, we will obtain a partition of the set $X = X_1 \cup (X_2 \setminus A'_i)$ into κ-many non-empty disjoint subsets. Using a similar argument as before, we claim that there is an index $j \leq \kappa$ such that $A'_j \subseteq X_2 \setminus A'_i$. Since $A'_i \in U$ it follows that $X_1 \cup A'_i \in \mathcal{G}_{\kappa_1}$ and $A'_j \cap [X_1 \cup A'_i] = \varnothing$. Hence, A'_j is not $\mathcal{G}_{\kappa_1}$-stationary. □

Lemma 3.4. *Let $\kappa_1 < \kappa_2$ be two infinite cardinals and let X be a set of cardinality κ_2. Let $\mathcal{G}_{\kappa_1}$ be a filter on X satisfying property (b) from Lemma 3.3 and let $h\colon X \to Y$ be a function. Then the filter $\mathcal{F}(h)(\mathcal{G}_{\kappa_1})$ on Y also satisfies this property, i.e., there is no partition $\{A'_i\}_{i\leq\kappa}$ of the set Y into κ-many non-empty disjoint $\mathcal{F}(h)(\mathcal{G}_{\kappa_1})$-stationary subsets for $\kappa > \kappa_1$.*

Proof. To the contrary, assume that there is a cardinal $\kappa > \kappa_1$ for which there is a partition $\{A'_i\}_{i\leq\kappa}$ of the set Y into κ-many non-empty disjoint $\mathcal{F}(h)(\mathcal{G}_{\kappa_1})$-stationary subsets. Consider the family $\{h^{-1}(A'_i)\}_{i\leq\kappa}$. It is a family of disjoint subsets. Moreover, for any $i \leq \kappa$, the set $h^{-1}(A'_i) \subseteq X$ is non-empty. Indeed, if $h^{-1}(A'_i) = \varnothing$ for some $i \leq \kappa$, then $h(X) \subseteq Y \setminus A'_i$. Since $h(X) \in \mathcal{F}(h)(\mathcal{G}_{\kappa_1})$, it follows that the set A'_i is not $\mathcal{F}(h)(\mathcal{G}_{\kappa_1})$-stationary, which contradicts the assumptions. Hence, the family $\{h^{-1}(A'_i)\}_{i\leq\kappa}$ is a partition of X into κ-many non-empty disjoint subsets.

Observe that for any $i \leq \kappa$, the set $h^{-1}(A'_i)$ is $\mathcal{G}_{\kappa_1}$-sationary. Indeed, if for some $i \leq \kappa$ the preimage $h^{-1}(A'_i)$ is not $\mathcal{G}_{\kappa_1}$-sationary, then there is a set $X' \in \mathcal{G}_{\kappa_1}$ for which $h^{-1}(A'_i) \cap X' = \varnothing$. This means that for $h(X') \in \mathcal{F}(h)(\mathcal{G}_{\kappa_1})$, we have $A'_i \cap h(X') = \varnothing$. Hence, A'_i is not $\mathcal{F}(h)(\mathcal{G}_{\kappa_1})$-stationary.

We have shown that the family $\{h^{-1}(A'_i)\}_{i\leq\kappa}$ is a partition of X into κ-many non-empty disjoint $\mathcal{G}_{\kappa_1}$-stationary sets. This contradicts our assumptions. □

Example 3.5. Consider a set A of cardinality $\aleph_\omega$. The cardinal $\aleph_\omega$ is the smallest uncountable cardinal of cofinality ω. We have $\aleph_\omega = \bigcup_{n\in\omega} \aleph_n$. By Lemma 3.3, it follows that for any number $n \in \omega$, there is a uniform filter $\mathcal{G}_{\aleph_n}$ on the set A satisfying the the following properties:

(a) there is a partition $\{A_i\}_{i\leq\aleph_n}$ of the set A into $\aleph_n$-many non-empty disjoint $\mathcal{G}_{\aleph_n}$-stationary subsets;
(b) there is no partition $\{A'_i\}_{i\leq\kappa}$ of the set A into κ-many non-empty disjoint $\mathcal{G}_{\aleph_n}$-stationary subsets for $\kappa > \aleph_n$.

Define a family M of filters on A by $M = \{\mathcal{G}_{\aleph_n} \mid n \in \omega\}$. Let $\bar{\mathcal{F}}^{\aleph_\omega}$ denote a factor functor of the filter functor $\mathcal{F}$ obtained by identifying all filters on a set B containing subsets of cardinality less than $\aleph_\omega$ with the distinguished point $\perp\in \bar{\mathcal{F}}^{\aleph_\omega}(B)$. Observe that the filters from M do not contain sets of cardinality less than $\aleph_\omega$. Hence, they are not identified with the distinguished point and $M \subseteq \bar{\mathcal{F}}^{\aleph_\omega}(A)$. Now, consider a subfunctor $\bar{\mathcal{F}}^{\aleph_\omega}_M$ of the functor $\bar{\mathcal{F}}^{\aleph_\omega}$ determined by the set $M \subseteq \bar{\mathcal{F}}^{\aleph_\omega}(A)$ as follows:

$$\bar{\mathcal{F}}^{\aleph_\omega}_M(B) = \{\bar{\mathcal{F}}^{\aleph_\omega}(f)(\mathcal{G}) \mid \mathcal{G} \in M \text{ and } f\colon A \to B\} \cup \{\perp\}.$$

Clearly, $\bar{\mathcal{F}}^{\aleph_\omega}_M$ is a connected functor with $\mathcal{C}_{0,1} < \bar{\mathcal{F}}^{\aleph_\omega}_M$. Hence, it does not preserve non-empty preimages. However, the functor $\bar{\mathcal{F}}^{\aleph_\omega}_M$ preserves non-empty preimages on undistinguished points. For any $n \in \omega$, let $\vec{a}_n \in \bar{\mathcal{F}}^{\aleph_\omega}_M(A)$ be defined by $\vec{a}_n := \mathcal{G}_{\aleph_n}$. Clearly, $\mathsf{Flt}_A(\vec{a}_n) = \mathsf{Flt}_A(\mathcal{G}_{\aleph_n}) = \mathcal{G}_{\aleph_n}$. By the property (a) satisfied by $\mathcal{G}_{\aleph_n}$, it follows that for any $n \in \omega$, there is a partition $\{A^n_j\}_{j\leq\aleph_n}$ of the set X into $\aleph_n$-many non-empty disjoint $\mathcal{G}_{\aleph_n}$-stationary subsets. Moreover, by property (b) and Lemma 3.4, it follows that for any element $\vec{a} \in \bar{\mathcal{F}}^{\aleph_\omega}_M(A)$, there is no partition $\{A^\omega_j\}_{j\leq\aleph_\omega}$ of the set A into $\aleph_\omega$-many non-empty disjoint $\mathsf{Flt}_A(\vec{a})$-stationary subsets. This means that $\bar{\mathcal{F}}^{\aleph_\omega}_M$ does not satisfy the assumptions of Theorem 3.2 but satisfies the assumptions stated in the beginning of this section.

We will now construct a coalgebra $\mathbb{B}$ together with elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ satisfying the assumptions of the Standing Hypothesis, which will prove the existence of a T-coalgebra $\mathbb{X}$ for which $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ for a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ satisfying the assumptions stated in the beginning of this section.

The following result plays a crucial role in this paper.

Theorem 3.6 ([7], p. 59-60)**.** *For any infinite cardinality κ, there exists a rigid unary algebra (W, f, g) of type $(1,1)$ such that $|W| = \kappa$.*

Consider a rigid unary algebra (W, f, g) of cardinality λ. Let $\underline{W}$ denote a copy of W disjoint from W. To distinguish elements from the first and the second copy, we will denote them w and $\underline{w}$, respectively. Since $|W\cup\underline{W}| = \lambda$ and since λ is singular with cofinality ω, we may assume that there is a countable partition $\{V_i\}_{i\in\omega}$ of the set $W \cup \underline{W}$ into sets of cardinality $|V_i| = |S_i| < |W \cup \underline{W}| = \lambda$. Without loss of generality, we assume that for any $i \in \omega$, the set V_i is "symmetric", i.e., $w \in V_i \iff \underline{w} \in V_i$.

Let $B := W\times\lambda \ \cup \ \underline{W}\times\lambda \ \cup \ \omega\times\lambda$ (where $\times$ binds more strongly than $\cup$ or $\cap$). The set B is a disjoint union of three parts. For any $w \in W$, let $\vec{a}_w \in TB$ denote an undistinguished element for which the three subsets: $\{\underline{w}\}\times\lambda \subset \underline{W}\times\lambda \subset B$, $\{f(w)\}\times\lambda \subset W\times\lambda \subset B$, and $\{g(w)\}\times\lambda \subset W\times\lambda \subset B$

are $\mathsf{Flt}_B(\vec{a}_w)$-stationary and

$$\vec{a}_w \in T[\{\underline{w}\}\times\lambda \ \cup \ \{f(w)\}\times\lambda \ \cup \ \{g(w)\}\times\lambda] \subset TB.$$

The existence of an element $\vec{a}_w$ for any $w \in W$ is guaranteed by our assumptions. Observe that $\vec{a}_w \neq \vec{a}_{w'}$ for $w \neq w'$.

For any $w \in W$, let $\vec{b}_w \in TB$ be any undistinguished element such that $\vec{b}_w \in T[\{f(w)\} \times \lambda]$. For any $i \in \omega$, assume that $\vec{c}_i$ is an undistinguished element from TB for which the sets $\{i+1\}\times\lambda \subset B$ and $\{v\}\times\lambda \subset B$, for any $v \in V_i$, are $\mathsf{Flt}(\vec{c}_i)$-stationary and $\vec{c}_i \in T\,[V_i\times\lambda \ \cup \ \{i+1\}\times\lambda]$.

Let $\beta\colon B \to TB$ be defined as follows:

$$\beta(v) = \begin{cases} \vec{a}_w & \text{for} \quad v = (w,l) \in W\times\lambda, \\ \vec{b}_w & \text{for} \quad v = (\underline{w},l) \in \underline{W}\times\lambda, \\ \vec{c}_n & \text{for} \quad v \in \{n\}\times\lambda \subset \omega\times\lambda. \end{cases}$$

In order to work with the Standing Hypothesis, we need two more elements, namely $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$. Let $\vec{a}_{B,1}$ be any undistinguished element from TB such that $\vec{a}_{B,1} \in T[\{0\}\times\lambda] \subset TB$. Finally, let $\vec{a}_{B,2}$ be any undistinguished element which satisfies $\vec{a}_{B,2} \in T[W\times\{l\}] \subset TB$ for some $l \in \lambda$.

Observe that condition (1) from the Standing Hypothesis is satisfied for $\mathbb{B} = \langle B, \beta\rangle$ and elements $\vec{a}_{B,1}, \vec{a}_{B,2}$. Indeed,

$$\beta(B) = \{\vec{a}_w \mid w \in W\} \cup \{\vec{b}_w \mid w \in W\} \cup \{\vec{c}_n \mid n \in \omega\}$$

is a union of three sets consisting of undistinguished elements only. Hence, the distinguished point $\perp$ is not an element of $\beta(B)$. None of the elements from $\beta(B)$ are members of the subset $T[\{0\}\times\lambda] \subset TB$. Since $\vec{a}_{B,1} \in T[\{0\}\times\lambda]$, it follows that $\vec{a}_{B,1} \notin \beta(B)$. We will now prove that none of the following three cases holds:

(I) $\vec{a}_{B,2} = \vec{a}_w$ for some $w \in W$,
(II) $\vec{a}_{B,2} = \vec{b}_w$ for some $w \in W$,
(III) $\vec{a}_{B,2} = \vec{c}_i$ for some $i = 0, 1, 2, \ldots$.

If (I) holds, then $\mathsf{Flt}_B(\vec{a}_{B,2}) = \mathsf{Flt}_B(\vec{a}_w)$. Because the set $\{\underline{w}\}\times\lambda$ is $\mathsf{Flt}_B(\vec{a}_w)$-stationary, it follows that it is also $\mathsf{Flt}_B(\vec{a}_{B,2})$-stationary. This cannot be the case since by the inclusion $\vec{a}_{B,2} \in T[W\times\{l\}]$, it follows that $W\times\{l\} \in \mathsf{Flt}_B(\vec{a}_{B,2})$ and $W\times\{l\}\cap\{\underline{w}\}\times\lambda = \varnothing$. Hence, $\vec{a}_{B,2} \neq \vec{a}_w$ for any $w \in W$. If (II) holds, then $\mathsf{Flt}_B(\vec{a}_{B,2}) = \mathsf{Flt}_B(\vec{b}_w)$. Because $\vec{b}_w \in T[\{f(w)\}\times W]$ and $\vec{a}_{B,2} \in T[W\times\{l\}]$, it follows that

$$\{f(w)\}\times W \in \mathsf{Flt}_B(\vec{a}_{B,2}) \ \text{ and } \ W\times\{l\} \in \mathsf{Flt}_B(\vec{a}_{B,2}).$$

Therefore,

$$\{f(w)\}\times W \ \cap \ W\times\{l\} = \{(f(w),l)\} \in \mathsf{Flt}_B(\vec{a}_{B,2}).$$

This means that $\vec{a}_{B,2} \in T\{(f(w),l)\}$. The set $T\{(f(w),l)\}$ consists of the distinguished point only, so $\vec{a}_{B,2} = \perp$. This contradicts the assumptions. Hence, $\vec{a}_{B,2} \neq \vec{b}_w$ for any $w \in W$. Finally, if (III) holds, then $\mathsf{Flt}_B(\vec{a}_{B,2}) =$

$\mathsf{Flt}_B(\vec{c}_i)$. This implies that the set $\{i+1\}\times\lambda$ is $\mathsf{Flt}_B(\vec{a}_{B,2})$-stationary. This cannot be true since the inclusion $\vec{a}_{B,2} \in T[W\times\{l\}]$ implies

$$W\times\{l\} \in \mathsf{Flt}_B(\vec{a}_{B,2}) \text{ and } W\times\{l\} \cap \{i+1\}\times\lambda = \varnothing.$$

Hence, $\vec{a}_{B,2} \notin \beta(B)$.

To prove that condition (2) is satisfied, pick a subcoalgebra $\mathbb{S} \leq \mathbb{B}$ and a supercoalgebra $\mathbb{C} = \langle C, \gamma\rangle \geq \mathbb{B}$ such that

$$\vec{a}_{B,1}, \vec{a}_{B,2} \in TS \quad \text{and} \quad \gamma(C\setminus B) = \{\vec{a}_{B,1}, \perp\}.$$

Assume we are given a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ such that $Th(\vec{a}_{B,1}) = \vec{a}_{B,1} \in TB \subseteq TC$. We want to show that $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\perp\}$. In order to do this we need some additional lemmas.

Lemma 3.7. *The subcoalgebra $\mathbb{S} = \langle S, \beta|_S\rangle$ satisfies the following:*

(1) *For any $n \in \omega$, we have $\vec{c}_n \in TS$; hence, $\{n+1\}\times\lambda \cap S \neq \varnothing$.*
(2) *For any $v \in W \cup \underline{W}$, we have $\{v\}\times\lambda \cap S \neq \varnothing$.*
(3) *$\beta(B) = \beta(S)$.*

Proof. By our assumptions, $\vec{a}_{B,1} \in TS$. Since $\vec{a}_{B,1}$ is an undistinguished point lying in $T[\{0\}\times\lambda]$, it follows that $S \cap \{0\}\times\lambda \neq \varnothing$. For any $x \in S \cap \{0\}\times\lambda$, we have $\beta(x) = \vec{c}_0$. The set $\{1\}\times\lambda$ is $\mathsf{Flt}(\vec{c}_0)$-stationary. This means that $S \cap \{1\}\times\lambda \neq \varnothing$. By induction, it follows that for any $n \in \omega$, we have $\vec{c}_n \in TS$ and $S \cap \{n+1\}\times\lambda \neq \varnothing$.

Recall that for each $\vec{c}_n$ and any $v \in V_n$, the set $\{v\}\times\lambda$ is $\mathsf{Flt}(\vec{c}_n)$-stationary, and the following holds: $\bigcup_{n\in\omega} V_n = W \cup \underline{W}$. Therefore, since $\{\vec{c}_n \mid n \in \omega\} \subseteq TS$, for any $v \in W\cup\underline{W}$, we have $S \cap \{v\}\times\lambda \neq \varnothing$. Hence, for any $w \in W$, the elements $\vec{a}_w$ and $\vec{b}_w$ are elements of TS. This means that $\beta(B) = \beta(S)$. □

We will prove some basic facts about the homomorphism $h\colon \mathbb{S} \to \mathbb{C}$.

Lemma 3.8. *$Th(\vec{c}_0) = \vec{c}_0$.*

Proof. By our assumptions, $Th(\vec{a}_{B,1}) = \vec{a}_{B,1}$. By Lemma 2.15, it follows that $\mathcal{F}(h)(\mathsf{Flt}_S(\vec{a}_{B,1})) = \mathsf{Flt}_C(\vec{a}_{B,1})$. Since $\vec{a}_{B,1} \in T[\{0\}\times\lambda]$, $\vec{a}_{B,1} \in TS$, and $\vec{a}_{B,1} \in TC$, it follows that

$$\{0\}\times\lambda \cap S \in \mathsf{Flt}_S(\vec{a}_{B,1}) \text{ and } \{0\}\times\lambda \cap C \in \mathsf{Flt}_S(\vec{a}_{B,1}).$$

By Lemma 2.8, there is a subset $V \subseteq \{0\}\times\lambda \cap S$ satisfying $V \in \mathsf{Flt}_S(\vec{a}_{B,1})$ for which $h(V) \subseteq (\{0\}\times\lambda \cap C$. Since h is a homomorphism, it follows that $Th(\vec{c}_0) = \vec{c}_0$. □

Lemma 3.9. *$Th(\ \beta(S)\setminus\{\vec{c}_n \mid n\in\omega\}\ ) \cap \{\vec{c}_n \mid n \in \omega\} = \varnothing$.*

Proof. Pick any element $\vec{d} \in \beta(S)\setminus\{\vec{c}_n \mid n \in \omega\}$ and assume, to the contrary, that $Th(\vec{d}) = \vec{c}_n$ for some $n \in \omega$. Observe that the element $\vec{d}$ must be either of the form $\vec{a}_w$ or of the form $\vec{b}_w$ for some $w \in W$. If $\vec{d} = \vec{a}_w$ for some $w \in W$, then the equality $Th(\vec{a}_w) = \vec{c}_n$ implies the following equality of filters:

$$\mathcal{F}(h)(\mathsf{Flt}_S(\vec{a}_w)) = \mathsf{Flt}_C(\vec{c}_n).$$

Lemma 2.8 implies that there is a subset

$$V' \subseteq \{\underline{w}\}\times\lambda \ \cup\ \{f(w)\}\times\lambda \ \cup\ \{g(w)\}\times\lambda$$

having a non-empty intersection with each of $\{\underline{w}\}\times\lambda$, $\{f(w)\}\times\lambda$, and $\{g(w)\}\times\lambda$, such that its image $h(V')$ is a subset of $V_n\times\lambda \ \cup\ \{n+1\}\times\lambda$ and has a non-empty intersection with $\{n+1\}\times\lambda$ and $\{v\}\times\lambda$ for any $v \in V_n$. Since h is a homomorphism, it follows that

$$\begin{aligned} Th(\beta(V')) &= Th(\{\vec{a}_{f(w)},\ \vec{a}_{g(w)},\ \vec{b}_w\}) \\ &= \{\vec{a}_v \mid v \in V_n \cap W\} \cup \{\vec{b}_v \mid v \in V_n \cap W\} \cup \{\vec{c}_{n+1}\}. \end{aligned}$$

The above equality leads to a contradiction since it implies that an image of a three element set is infinite. Using a similar argument, we prove that $\vec{d}$ cannot be of the form $\vec{b}_w$ for any $w \in W$. Hence,

$$Th(\beta(S) \setminus \{\vec{c}_n \mid n \in \omega\}) \cap \{\vec{c}_n \mid n \in \omega\} = \varnothing. \qquad \square$$

Lemma 3.10. *$Th(\beta(S)) = \beta(B) \subset \gamma(C)$ and $Th(\vec{c}_n) = \vec{c}_n$, for any $n \in \omega$.*

Proof. Lemma 2.15 combined with Lemma 3.8 gives us the following equality between filters: $\mathcal{F}(h)(\mathsf{Flt}_S(\vec{c}_0)) = \mathsf{Flt}_C(\vec{c}_0)$. By Lemma 2.8, it follows that there is a subset $V' \subseteq V_0\times\lambda \cup \{1\}\times\lambda$ which is mapped onto $h(V') \subseteq V_0\times\lambda \cup \{1\}\times\lambda$. The subset V' and its image $h(V')$ both have non-empty intersections with $\{v\}\times\lambda$ for any $v \in V_0$ and have a non-empty intersection with $\{1\}\times\lambda$. The set $\beta(V' \cap (V_0\times\lambda \ \cup\ \{1\}\times\lambda))$ contains the elements $\vec{c}_1$, $\vec{a}_w$ for any $w \in V_0 \cap W$, and $\vec{b}_w$ for any $w \in V_0 \cap W$. Since h is a homomorphism, we have

$$\begin{aligned} &Th(\beta(V' \cap (V_0\times\lambda \ \cup\ \{1\}\times\lambda))\ ) \\ &\quad= Th(\{\vec{a}_w \mid w \in V_0 \cap W\} \cup \{\vec{b}_w \mid w \in V_0 \cap W\} \cup \{\vec{c}_1\}) \\ &\quad= \gamma(h(V')) \ = \ \gamma(h(V') \cap (V_0\times\lambda \ \cup\ \{1\}\times\lambda) \\ &\quad= \beta(h(V') \cap (V_0\times\lambda \ \cup\ \{1\}\times\lambda) \\ &\quad= \{\vec{a}_w \mid w \in V_0 \cap W\} \cup \{\vec{b}_w \mid w \in V_0 \cap W\} \cup \{\vec{c}_1\} \ \subseteq\ \beta(B). \end{aligned}$$

In particular, by Lemma 3.9, this means that $Th(\vec{c}_1) = \vec{c}_1$.

By induction and a similar argument to the one above, it follows that for any $n \in \omega$,

$$\begin{aligned} &Th(\{\vec{a}_w \mid w \in V_n \cap W\} \cup \{\vec{b}_w \mid w \in V_n \cap W\} \cup \{\vec{c}_{n+1}\}) \\ &\quad= \{\vec{a}_w \mid w \in V_n \cap W\} \cup \{\vec{b}_w \mid w \in V_n \cap W\} \cup \{\vec{c}_{n+1}\} \subset \beta(B), \end{aligned}$$

and $Th(\vec{c}_n) = \vec{c}_n$. Since

$$\beta(B) = \beta(S) = \bigcup_{n\in\omega} \{\vec{a}_w \mid w \in V_n \cap W\} \cup \{\vec{b}_w \mid w \in V_n \cap W\} \cup \{\vec{c}_{n+1}\},$$

it follows that $Th(\beta(S)) = \beta(B)$. $\square$

Lemma 3.11. $Th(\{\vec{b}_w \mid w \in W\}) = \{\vec{b}_w \mid w \in W\}$.

Proof. By Lemma 3.8 and Lemma 3.10, we know that

$$Th(\{\vec{b}_w \mid w \in W\}) \cap \{\vec{c}_n \mid n \in \omega\} = \varnothing.$$

Now assume that there is $w \in W$ for which $Th(\vec{b}_w) = \vec{a}_v$ for some $v \in W$. By Lemma 2.15, it follows that $\mathcal{F}(h)(\mathsf{Flt}_S(\vec{b}_w)) = \mathsf{Flt}_C(\vec{a}_v)$. Hence, by Lemma 2.8, there is a set $V \subseteq \{f(w)\}\times\lambda$ whose image $h(V)$ is a subset of $\{\underline{v}\}\times\lambda \cup \{f(v)\}\times \lambda \cup \{g(v)\}\times\lambda$ and has non-empty intersections with $\{\underline{v}\}\times\lambda$, $\{f(v)\}\times\lambda$, and $\{g(v)\}\times\lambda$. Thus, $Th(\beta(V)) = Th(\{\ \vec{a}_{f(w)}\ \}) = \{\vec{a}_{f(v)},\ \vec{a}_{g(v)},\ \vec{b}_v\}$. The last equality leads to a contradiction since the set on the right hand side has at least two elements. □

Lemma 3.12. *$Th(\{\vec{a}_w \mid w \in W\}) = \{\vec{a}_w \mid w \in W\}$.*

Proof. To the contrary, assume that there exists $\vec{a}_w$ for some $w \in W$ such that $Th(\vec{a}_w) = \vec{b}_v$ for $v \in W$. Since the elements $\vec{a}_w$ and $\vec{b}_v$ are undistinguished, we get the following equality of filters: $\mathcal{F}(h)(\mathsf{Flt}_S(\vec{a}_w)) = \mathsf{Flt}_C(\vec{b}_v)$. Lemma 2.8 implies the existence of a subset

$$V \subseteq \{\underline{w}\}\times\lambda\ \cup\ \{f(w)\}\times\lambda\ \cup\ \{g(w)\}\times\lambda$$

having non-empty intersections with $\{\underline{w}\}\times\lambda$, $\{f(w)\}\times\lambda$, and $\{g(w)\}\times\lambda$ such that its image $h(V)$ is a subset of $\{f(v)\}\times\lambda$. Therefore,

$$Th(\beta(V)) = Th(\{\vec{b}_{f(w)}, \vec{a}_{f(w)}, \vec{a}_{g(w)}\}) = \gamma(h(V)) = \beta(h(V)) = \{\vec{a}_{f(v)}\}.$$

In particular, $Th(\vec{b}_{f(w)}) = \vec{a}_{f(v)}$, which contradicts Lemma 3.11. □

Lemma 3.13. *$h(S\ \cap\ W\times\lambda) \subseteq W\times\lambda$.*

Proof. Pick any pair $(w,l) \in S\ \cap\ W\times\lambda$. Indeed, if the inclusion $h((w,l)) \in C \setminus W\times\lambda$ is true, then

$$\begin{aligned} Th(\vec{a}_w) &= Th(\beta((w,l))) \\ &= \gamma(h(w,l)) \in \{\vec{c}_n \mid n \in \omega\} \cup \{\vec{b}_v \mid v \in W\} \cup \{\bot, \vec{a}_{B,1}\}. \end{aligned}$$

This contradicts Lemma 3.12. □

Lemma 3.14. *For $w \in W$, there is $v \in W$ such that $h(S \cap \{w\}\times\lambda) \subseteq \{v\}\times\lambda$.*

Proof. Assume that for two elements $(w,l),(w,l') \in S\ \cap\ \{w\}\times\lambda$, there are $(v,k),(v',k') \in W\times\lambda$ such that $v \neq v'$, $h(w,l) = (v,l)$, and $h(w,l') = (v',l')$. Since h is a homomorphism, we get

$$\begin{aligned} Th(\vec{a}_w) &= Th(\beta(w,l)) = \gamma(h(w,l)) = \gamma(v,k) = \vec{a}_v; \\ Th(\vec{a}_w) &= Th(\beta(w,l')) = \gamma(h(w,l')) = \gamma(v',k') = \vec{a}_{v'}. \end{aligned}$$

This leads to a contradiction since $\vec{a}_v \neq \vec{a}_{v'}$ for $v \neq v'$. □

Notice that Lemma 3.14 states that the following function $h^{\#}\colon W \to W$ is well defined: $h^{\#}(w) = v \iff h(S \cap \{w\}\times\lambda) \subseteq \{v\}\times\lambda$.

Lemma 3.15. *The mapping $h^{\#}\colon W \to W$ is a homomorphism from the unary algebra (W, f, g) into itself.*

Proof. In order to show that $h^{\#}$ is a homomorphism, it is enough to prove that for any $w \in W$, the following two equalities hold:

$$h^{\#}(f(w)) = f(h^{\#}(w)) \text{ and } h^{\#}(g(w)) = g(h^{\#}(w)).$$

Choose any $w \in W$. By the definition of $h^{\#}$, it follows that

$$h(S \cap \{w\}\times\lambda) \subseteq \{h^{\#}(w)\}\times\lambda.$$

Since h is a homomorphism, we have:

$$Th(\{\vec{a}_w\}) = Th(\beta(S \cap \{w\}\times\lambda)) \subseteq \gamma(\{h^{\#}(w)\}\times\lambda) = \{\vec{a}_{h^{\#}(w)}\}.$$

This implies the following equality between filters:

$$\mathcal{F}(h)(\mathsf{Flt}_S(\vec{a}_w)) = \mathsf{Flt}_C(\vec{a}_{h^{\#}(w)}). \tag{3.15.1}$$

Hence, by Lemma 2.8 there is a subset V of the set

$$\{\underline{w}\}\times\lambda \cup \{f(w)\}\times\lambda \cup \{g(w)\}\times\lambda$$

having a non-empty intersection with $\{\underline{w}\}\times\lambda$, $\{f(w)\}\times\lambda$, and $\{g(w)\}\times\lambda$ such that its image satisfies

$$h(V) \subseteq \{\underline{h^{\#}(w)}\}\times\lambda \cup \{f(h^{\#}(w))\}\times\lambda \cup \{g(h^{\#}(w))\}\times\lambda$$

and intersects with $\{\underline{h^{\#}(w)}\}\times\lambda$, $\{f(h^{\#}(w))\}\times\lambda$, and $\{g(h^{\#}(w))\}\times\lambda$. We see that by Lemma 3.14, the following is true:

$$h^{\#}(f(w)) = f(h^{\#}(w)) \text{ and } h^{\#}(g(w)) = g(h^{\#}(w)), \text{ or}$$
$$h^{\#}(f(w)) = g(h^{\#}(w)) \text{ and } h^{\#}(g(w)) = f(h^{\#}(w)).$$

In order to exclude the latter case, observe that the Equality 3.15.1 also implies that $Th(\vec{b}_w) = \vec{b}_{h^{\#}(w)}$. This means that there exists $V' \subseteq \{f(w)\}\times\lambda$ whose image $h(V')$ is a subset of $\{f(h^{\#}(w))\}\times\lambda$. Therefore, $h^{\#}(f(w)) = f(h^{\#}(w))$ and $h^{\#}(g(w)) = g(h^{\#}(w))$. □

Finally, the following lemma is true.

Lemma 3.16. $h(S \cap \{w\}\times\lambda) \subseteq \{w\}\times\lambda$ *for any* $w \in W$.

Proof. The statement follows directly by Lemma 3.15 and by the fact that the algebra (W, f, g) is a rigid algebra. □

Lemma 3.17. $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\bot\}$.

Proof. Recall that the element $\vec{a}_{B,2}$ is an undistinguished point from the set $T[S \cap W\times\{l\}] \subset TB \subset TC$. By Lemma 3.16, it follows that

$$Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\}.$$

Moreover, by Lemma 3.16, we see that the restriction $h|_{S \cap W\times\{l\}}$ of the mapping h is injective and that is why $Th(\vec{a}_{B,2}) = Th|_{S \cap W\times\{l\}}(\vec{a}_{B,2}) \neq \bot$. □

The existence of a T-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ for which the properties (1) and (2) are true guarantees the following.

Theorem 3.18. *Let* $T\colon \mathsf{Set} \to \mathsf{Set}$ *be a connected functor that preserves non-empty preimages on undistinguished points and* $\mathcal{C}_{0,1} < T$*. Let* λ *be an uncountable cardinal such that* $\lambda = \bigcup_{i\in\omega} S_i$ *is a countable union of the family* $\{S_i\}_{i\in\omega}$ *of sets with* $|S_i| < \lambda$ *for any* $i \in \omega$*, and let there be a set* A *of cardinality* λ *such that for each* $i \in \omega$*, there is an element* $\vec{a}_i \in TA$ *and a partition* $\{A^i_j\}_{j\in S_i}$ *of the set* A *into* $|S_i|$*-many non-empty disjoint* $\mathsf{Flt}_A(\vec{a}_i)$*-stationary subsets. Then there exists a* T*-coalgebra* $\mathbb{X}$ *for which* $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$*.*

4. Conclusion and future work

In this paper we have shown how to make use of a rigid unary algebra in order to prove $\mathcal{HS} \neq \mathcal{SH}$ for some class of functors not preserving non-empty preimages. We constructed a coalgebra $\mathbb{B}$ and two elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ which fit into the framework of the Standing Hypothesis. The procedure of finding such a coalgebra and elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ presented here can be divided into two steps:

(1) Take a rigid unary algebra (W, f, g) and "code" it as some coalgebra $\mathbb{B}'$. The resulting coalgebra $\mathbb{B}'$ has to contain all the necessary information about (W, f, g),
(2) Add some elements to $\mathbb{B}'$ and come up with good candidates for $\vec{a}_{B,1}, \vec{a}_{B,2}$ so that the final result $\mathbb{B}$ together with $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ satisfies the assumptions of the Standing Hypothesis.

In our case, the first step boils down to constructing a subcoalgebra of $\mathbb{B}$ defined on the set $W \times \lambda \,\cup\, \underline{W} \times \lambda$. Indeed, the subcoalgebra

$$\left\langle W \times \lambda \,\cup\, \underline{W} \times \lambda,\ \beta|_{W\times\lambda \,\cup\, \underline{W}\times\lambda} \right\rangle$$

contains all the crucial information about (W, f, g), namely about the definition of $f\colon W \to W$ and $g\colon W \to W$. The second step involves adding the part $\omega \times \lambda$. Observe that not all of the assumptions listed in the beginning of this paper are required to go through step 1. of the procedure. In order to construct the coalgebra $\left\langle W \times \lambda \,\cup\, \underline{W} \times \lambda,\ \beta|_{W\times\lambda \,\cup\, \underline{W}\times\lambda} \right\rangle$ we only need to assume the existence of an element $\vec{a} \in TA$ and a partition $A = A_1 \cup A_2 \cup A_3$ into three non-empty subsets, each being $\mathsf{Flt}_A(\vec{a})$-stationary. It is the second step of the procedure which requires λ to be singular with cofinality ω and which requires for any $i \in \omega$, the existence of an element $\vec{a}_i \in TA$ and a partition $\{A^i_j\}_{j\in S_i}$ of the set A into $|S_i|$-many non-empty disjoint subsets, where each is $\mathsf{Flt}_A(\vec{a}_i)$-stationary. Since the fist step is the most crucial, we ask the following question and leave it as an open problem.

Open Problem 4.1. Can we modify the Standing Hypothesis making some of the additional assumptions from step 2. redundant?

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