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The $\mathcal{HS} = \mathcal{SH}$ problem for coalgebras

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Abstract. H. P. Gumm and T. Schröder stated a hypothesis that the preservation of preimages by a functor T for which $|T1| = 1$ is equivalent to the satisfaction of the class equality $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras. Although we were not able to find a full solution, our paper gives a positive answer to this problem for a very wide class of Set-endofunctors.

1. Introduction

H. P. Gumm and T. Schröder [4] stated the following problem:

> Is the satisfaction of the equation $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras equivalent to T preserving preimages?

The authors conjectured a positive answer to this question. The conjecture is supported by the following two theorems.

Theorem 1.1 ([4])**.** *Assume that a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ preserves preimages. Then for any class K of T-coalgebras $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$.*

Theorem 1.2 ([4])**.** *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor such that $|T1| > 1$. If $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ for any class K of T-coalgebras, then T preserves preimages.*

By Theorems 1.1 and 1.2 it follows that in order to find a positive answer to the $\mathcal{HS} = \mathcal{SH}$ problem it is enough to find, for any functor $T\colon \mathsf{Set} \to \mathsf{Set}$ not preserving preimages and satisfying $|T1| = 1$, a T-coalgebra $\mathbb{X}$ such that $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$. However, as it will be shown below, the constant functor $\mathcal{C}_{0,1}$ sending the empty set to itself and all non-empty sets to a one-element set is a counterexample. The condition which should be investigated to be equivalent to $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$ is the preservation of *non-empty preimages.*

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Any Set-endofunctor T which does not preserve non-empty preimages and satisfies $|T1| = 1$ falls into exactly one of the two classes:

- functors which do not preserve non-empty preimages on undistinguished points,
- functors which preserve non-empty preimages on undistinguished points.

In the first part of this article we present a construction of a T-coalgebra $\mathbb{X}$ with $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ for any functor T not preserving non-empty preimages on undistinguished points.

The second part focuses on finding a T-coalgebra $\mathbb{X}$ that fails to satisfy $\mathcal{SH}(\mathbb{X}) = \mathcal{HS}(\mathbb{X})$ for a functor T not preserving non-empty preimages but preserving them on undistinguished points. In this case, at the time of writing the paper we were not able to give a complete solution to the problem. Instead, we present a general setting in which we are able to construct a coalgebra $\mathbb{X}$ with $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$ and give examples of families of functors that can be put in the setting.

2. Basic notions

Let Set be the category of all sets and mappings between them. Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. A *T-coalgebra* $\mathbb{X}$ is a pair $\langle X, \xi\rangle$, where X is a set and ξ is a mapping $\xi\colon X \to TX$. The set X is called a *carrier* and the mapping ξ is called a *structure* of the coalgebra $\mathbb{X} = \langle X, \xi\rangle$.

A *homomorphism* from a T-coalgebra $\mathbb{X} = \langle X, \xi\rangle$ to a T-coalgebra $\mathbb{Y} = \langle Y, \psi\rangle$ is a mapping $h\colon X \to Y$ such that $T(h) \circ \xi = \psi \circ h$.

A T-coalgebra $\mathbb{S} = \langle S, \sigma\rangle$ is said to be a *subcoalgebra* of a T-coalgebra $\mathbb{X} = \langle X, \xi\rangle$ whenever there is an injective homomorphism from $\mathbb{S}$ into $\mathbb{X}$. This fact is denoted by $\mathbb{S} \leq \mathbb{X}$.

Let $h\colon \mathbb{X} \to \mathbb{Y}$ be a homomorphism and let $\mathbb{S} \leq \mathbb{X}$. Then there is a structure $\delta\colon h(S) \to T(h(S))$ such that $h_{|S}\colon S \to h(S)$ is a homomorphism from $\mathbb{S} = \langle S, \sigma\rangle$ onto $\langle h(S), \delta\rangle$ and $\langle h(S), \delta\rangle \leq \langle Y, \psi\rangle$. In other words, a homomorphic image of a subcoalgebra of a domain of h is a subcoalgebra of a codomain of h. For the basics on the theory of coalgebras the reader is referred to [2].

A functor $T\colon \mathsf{Set} \to \mathsf{Set}$ *preserves preimages* if for any mapping $f\colon A \to B$ and a subset $C \subset B$ the following diagram is a pullback diagram.

$$\begin{array}{ccc} TA & \xrightarrow{\;Tf\;} & TB \\ {\scriptstyle T\subseteq^A_{f^{-1}(C)}}\Big\uparrow & & \Big\uparrow{\scriptstyle T\subseteq^B_C} \\ T[f^{-1}(C)] & \xrightarrow{\;Tf_|\;} & TC \end{array}$$

In other words, a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve preimages whenever there is a mapping $f\colon A \to B$ and a subset $C \subset B$ with $\vec{a} \in TA$ and $\vec{c} \in TC$ such that $[Tf](\vec{a}) = [T\subseteq^B_C](\vec{c})$ and the element $\vec{a}$ does not belong to the image of the set $T[f^{-1}(C)]$ under the map $T[\subseteq^A_{f^{-1}(C)}]$.

Let $\mathcal{C}_{0,1}$ denote the constant functor for which $\mathcal{C}_{0,1}\varnothing = \varnothing$ and $\mathcal{C}_{0,1}A = \{\perp\}$ for $A \neq \varnothing$ and for which given $f\colon A \to B$ the map $\mathcal{C}_{0,1}(f)\colon \mathcal{C}_{0,1}A \to \mathcal{C}_{0,1}B$ is an empty map in the case when $A = \varnothing$ and it is identity $id_{\{\perp\}}$ otherwise.

We will now show that $\mathcal{C}_{0,1}$ does not preserve preimages. If we consider an inclusion map $i\colon \{0\} \to \{0,1\}$ and a subset $\{1\} \subseteq \{0,1\}$, then for the preimage $i^{-1}(\{1\}) = \varnothing$ we have

$$\begin{array}{ccc}
\mathcal{C}_{0,1}\{0\} = \{\perp\} & \xrightarrow{\mathcal{C}_{0,1}[i]=id_{\{\perp\}}} & \mathcal{C}_{0,1}\{0,1\} = \{\perp\} \\
\uparrow & & \uparrow {\scriptstyle \mathcal{C}_{0,1}[\subseteq^{\{0,1\}}_{\{1\}}]=id_{\{\perp\}}} \\
\mathcal{C}_{0,1}[i^{-1}(\{1\})] = \varnothing & \longrightarrow & \mathcal{C}_{0,1}\{1\} = \{\perp\}
\end{array}$$

The functor $\mathcal{C}_{0,1}$ does not preserve preimages in the case when they are empty. At the same time, for any class K of $\mathcal{C}_{0,1}$-coalgebras we have $\mathcal{HS}(\mathsf{K}) = \mathcal{SH}(\mathsf{K})$. That is why the notion that should be investigated to be equivalent to $\mathcal{HS} = \mathcal{SH}$ is the notion of preservation of *non-empty* preimages.

We say that a natural transformation $\mu\colon G \to T$ from a functor G to a functor T is injective whenever for any set A the mapping $\mu_A\colon GA \to TA$ is an injective mapping. We say that a functor G *is a subfunctor* of a functor T whenever there exists an injective natural transformation from G to T. We denote this fact by $G \leq T$. We say that $\vec{a} \in TA$ is a *distinguished point* (see [7]) if there is a natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$ such that $\mu(\perp) = \vec{a}$. An element $\vec{a} \in TA$ is *undistinguished* if it is not a distinguished point. Note that by the definition of $\mathcal{C}_{0,1}$ any natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$ is injective. Thus, $\mathcal{C}_{0,1} \leq T$ if and only if for a set A there is a distinguished point $\vec{a} \in TA$.

Definition 2.1 ([1]). A distinguished point $\mu_X(\perp) \in TX$, where $\mu\colon \mathcal{C}_{0,1} \to T$, is called *standard* whenever $T\varnothing$ contains precisely one element, say $\vec{s}$, such that for every non-empty set X it is sent by the T-image of the empty map $\epsilon_X\colon \varnothing \to X$ exactly on the image $\mu_X(\perp)$.

Definition 2.2 ([1]). A functor $T\colon \mathsf{Set} \to \mathsf{Set}$ is called a *standard functor* provided that

(1) it preserves inclusions, i.e., for any inclusion map $\subseteq^B_A\colon A \to B$, the map $T(\subseteq^B_A)$ is an inclusion map, that is, $T(\subseteq^B_A) = \subseteq^{TB}_{TA}$,
(2) all of its distinguished points are standard.

Remark 2.3. Every functor $T\colon \mathsf{Set} \to \mathsf{Set}$ preserves non-empty intersections of finitely many sets (see [7]), but a standard functor preserves also the empty intersections of finitely many sets. Hence, e.g., $\mathcal{C}_{0,1}$ is not a standard functor but $\mathcal{C}_1$ sending any set to $\{\perp\}$ is standard.

Theorem 2.4 ([1, p. 132]). *For each* Set*-endofunctor* T *there exists a standard* Set*-endofunctor* T' *such that the restrictions of* T *and* T' *to all non-empty sets and non-empty maps are naturally isomorphic.*

If we only investigate the preservation of non-empty preimages, then by Theorem 2.4 we may restrict ourselves to standard functors. Therefore, throughout this paper without loss of generality we will assume that a functor T we deal with is standard.

We will now translate the notion of a functor that does not preserve non-empty preimages to the language of standard functors. A functor $T\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve non-empty preimages whenever there is a mapping $f\colon A \to B$, a subset $C \subset B$ such that $f^{-1}(C) \neq \varnothing$ and an element $\vec{a} \in TA$ such that $Tf(\vec{a}) \in TC$ and $\vec{a} \notin T[f^{-1}(C)]$.

The notion of a subcoalgebra may also be stated more easily in the language of standard functors. Namely, suppose $\mathbb{X} = \langle X, \xi \rangle$ is a T-coalgebra and let $S \subseteq X$. There is a structure $\sigma\colon S \to TS$ turning $\langle S, \sigma \rangle$ into a subcoalgebra of $\mathbb{X}$ if and only if $\xi(S) \subseteq TS$ (see [2] for details). The structure $\sigma\colon S \to TS$ is defined by $\sigma := \xi_{|S}$. Therefore, subcoalgebras of a given coalgebra $\mathbb{X}$ can be understood as subsets of X 'closed' under the structure map ξ.

We have the following.

Lemma 2.5. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor and let $f\colon A \to B$ be a mapping, and $S \subseteq A$. Then*

$$[Tf](TS) = T[f(S)].$$

Proof. It is clear that $[Tf](TS) \subseteq T[f(S)]$. To see that the equality between the two sets holds, notice that the restriction $f_{|S}\colon S \to f(S)$ of the mapping f is onto and therefore an epimorphism. Any Set-endofunctor preserves epis and hence $T(f_{|S})\colon TS \to T[f(S)]$ is also onto. That is why $[Tf_{|S}](TS) = [Tf](TS) = T[f(S)]$. □

A functor T is said to be *connected* if $|T1| = 1$. Let $\mathcal{I}d$ denote the identity functor.

Lemma 2.6 ([10])**.** *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard connected functor. Then either T contains exactly one isomorphic copy of $\mathcal{I}d$ as a subfunctor or it contains exactly one isomorphic copy of $\mathcal{C}_{0,1}$ as a subfunctor.*

If T is a connected functor and if $\mathcal{C}_{0,1} \leq T$, then by Lemma 2.6 we know that the set TA contains precisely one distinguished point. Therefore, whenever we speak of a connected functor with $\mathcal{C}_{0,1} \leq T$, we denote the distinguished element of TA by $\perp$. Moreover, if T is a connected functor with $\mathcal{I}d \leq T$, then by Lemma 2.6 the functor $\mathcal{C}_{0,1}$ is not a subfunctor of T. This means that all elements from TX are undistinguished.

Lemma 2.7 ([8, Prop. II.4])**.** *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor and let $\vec{a} \in TA$. The following conditions are equivalent:*

- *there are two non-empty disjoint subsets $U_1, U_2 \subset A$ with $\vec{a} \in TU_1$ and $\vec{a} \in TU_2$;*
- *the element $\vec{a}$ is a distinguished point.*

Let T be an arbitrary Set-endofunctor and A be a set. For any $\vec{a} \in TA$ define a collection $\mathsf{Flt}_A(\vec{a})$ of subsets of A as follows:

$$\mathsf{Flt}_A(\vec{a}) := \begin{cases} \{U \subseteq A \mid \vec{a} \in TU\} & \text{if } \vec{a} \text{ is not distinguished,} \\ \mathcal{P}(A) & \text{otherwise.} \end{cases}$$

It has been shown in [8] that for an undistinguished point $\vec{a} \in TA$ the collection $\mathsf{Flt}_A(\vec{a})$ forms a filter on A. By Lemma 2.7 it follows that $\varnothing \in \mathsf{Flt}_A(\vec{a})$ if and only if $\vec{a}$ is a distinguished point.

Let $\mathcal{F}$ denote the functor which assigns to every set A the set

$$\{\mathcal{U} \mid \mathcal{U} \text{ is a filter on } A\} \cup \{\mathcal{P}(A)\}$$

and to every mapping $f\colon A \to B$ the mapping

$$\mathcal{F}(f)\colon \mathcal{F}A \to \mathcal{F}B; \quad \mathcal{G} \mapsto \{V \subseteq B \mid f(W) \subseteq V \text{ for some } W \in \mathcal{G}\}.$$

For any set A we may now treat Flt_A as a mapping

$$\mathsf{Flt}_A\colon TA \to \mathcal{F}A.$$

The collection $\mathsf{Flt} = \{\mathsf{Flt}_A\}_{A \in \mathsf{Set}}$ is a transformation from T to $\mathcal{F}$. It is worth noting that Flt need not be a natural transformation from T to $\mathcal{F}$. We have the following two theorems.

Theorem 2.8 ([8]). *For any injective mapping $f\colon A \to B$ and any $\vec{a} \in TA$, we have*

$$\mathsf{Flt}_B(Tf(\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a})).$$

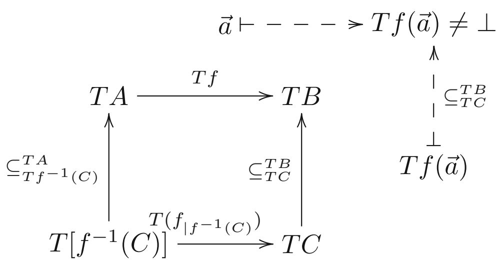
Theorem 2.9 ([8, Prop. VII.5]). *The transformation $\mathsf{Flt}\colon T \to \mathcal{F}$ is a natural transformation, i.e., for any $\vec{a} \in TA$ and any mapping $f\colon A \to B$ we have*

$$\mathsf{Flt}_B(Tf(\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a})),$$

if and only if T preserves non-empty preimages.

3. Functors not preserving non-empty preimages on undistinguished points

Definition 3.1. We say that a Set-endofunctor T *does not preserve non-empty preimages on undistinguished points* whenever there exist a mapping $f\colon A \to B$, a subset $C \subset B$ with $f^{-1}(C) \neq \varnothing$ and an element $\vec{a} \in TA$ such that $Tf(\vec{a})$ is an undistinguished point, $Tf(\vec{a}) \in TC$ and $\vec{a} \notin T[f^{-1}(C)]$.



Note that the element $\vec{a}$ in the above definition cannot be a distinguished point. Because if $\vec{a} \in TA$ was distinguished, then for any mapping $f\colon A \to B$ the element $Tf(\vec{a})$ would also be a distinguished point.

We stress that the only difference between the notion of a functor not preserving non-empty preimages and a functor not preserving non-empty preimages on undistinguished points is the additional assumption in the formulation of the latter which says that the element $Tf(\vec{a}) \neq \bot$.

The following lemma is a consequence of Lemma 2.6.

Lemma 3.2. *If $T\colon \mathsf{Set} \to \mathsf{Set}$ is a connected functor with $\mathcal{I}d \leq T$ which does not preserve non-empty preimages, then T does not preserve non-empty preimages on undistinguished points.*

Proof. By Lemma 2.6 we know there is no natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$. This means that for any set B the set TB does not contain distinguished points. Hence, for any set A, any element $\vec{a} \in TA$ and a mapping $f\colon A \to B$, the element $Tf(\vec{a}) \in TB$ is undistinguished. $\square$

The example below shows that there are connected functors with $\mathcal{C}_{0,1} \leq T$ that do not preserve non-empty preimages on undistinguished points.

Example 3.3. Let G be the functor $\mathsf{Hom}(4, -)$, i.e., $GA = \{(a_0, a_1, a_2, a_3) \mid a_i \in A\}$ for any set A, and let $\nu\colon G \to T$ be an epitransformation given by the following equations:

$$\nu_Y(y, y, y, y) = \nu_Y(z, z, z, z), \qquad \text{for any } Y \text{ and for any } y, z \in Y,$$
$$\nu_Y(w_1, y, y, z) = \nu_Y(w_2, y, y, z), \quad \text{for any } Y \text{ and for any } w_1, w_2, y, z \in Y.$$

For $(v, w, y, z) \in Y^4$ let us introduce the following notation:

$$[v, w, y, z]^Y_\nu := \{(v', w', y', z') \in Y^4 \mid \nu_Y(v, w, y, z) = \nu_Y(v', w', y', z')\}.$$

The functor T is connected and contains a copy of $\mathcal{C}_{0,1}$ as a subfunctor. We will show that T does not preserve non-empty preimages of undistinguished points: let $A = \{a, b, c, d\}$ be a 4-element set, let $B = \{a, c, d\}$, $C = \{c, d\}$ and let $f\colon A \to B$ be a mapping sending b to c and being an identity on a, c, d. Then T does not preserve the preimage of $[c, c, c, d]^C_\nu$. Indeed, the inclusion $i\colon C \to B$ sends $[c, c, c, d]^C_\nu$ to $[c, c, c, d]^B_\nu = [a, c, c, d]^B_\nu$ and $T(f)$ sends $[a, b, c, d]^A_\nu$ to $[a, c, c, d]^B_\nu \in TB$. But $f^{-1}(C) = \{b, c, d\}$ and $[a, b, c, d]^A_\nu$ cannot be an image of any element from $Tf^{-1}(C) = T\{b, c, d\}$.

Throughout this section we will assume that $T\colon \mathsf{Set} \to \mathsf{Set}$ is a standard connected functor not preserving non-empty preimages on undistinguished points. Our aim is to find a T-coalgebra $\mathbb{X} = \langle X, \xi \rangle$ for which $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$. In order to construct such a coalgebra we need to introduce some ingredients and show some of their properties.

Take sets A, B and C, a mapping $f\colon A \to B$ and an element $\vec{a} \in TA$ as in Definition 3.1. Therefore, $Tf(\vec{a}) \in TC$, $\vec{a} \in TA \setminus T[f^{-1}(C)]$ and $Tf(\vec{a}) \neq \bot$. We have the following lemmata.

Lemma 3.4. *The following is true:*

$$\vec{a} \notin T[A \setminus f^{-1}(C)].$$

Proof. If $\vec{a} \in T[A \setminus f^{-1}(C)]$, then by Lemma 2.5, $Tf(\vec{a}) \in [Tf](A \setminus f^{-1}(C))$. Because $Tf(\vec{a}) \in [Tf](A \setminus f^{-1}(C)) \cap TC$ and because $f[A \setminus f^{-1}(C)] \cap C = \varnothing$ by [7] we imply that $Tf(\vec{a})$ has to be a distinguished point. This contradicts the assumptions. □

Lemma 3.5. *We have*

$$f(A) \cap C \neq \varnothing.$$

Proof. Assume that $f(A) \cap C = \varnothing$. By [7] the set $Tf(A) \cap TC$ contains only the distinguished point. This is a contradiction since $Tf(\vec{a}) \in Tf(A) \cap TC$ and $Tf(\vec{a}) \neq \perp$. □

By Lemma 3.5 and by the fact that any Set-endofunctor preserves non-empty intersections (see [7]) we may assume without a loss of generality that $f\colon A \to B$ is onto. Indeed, if $f\colon A \to B$ is not a surjection, then the map f may be replaced with $f'\colon A \to f(A)$ and the set C with $C' = f(A) \cap C$.

We will denote the only element of the set $T\{v\}$ by $*_v$, i.e., $T\{v\} = \{*_v\}$. Note that if $\mathcal{C}_{0,1} \leq T$, then $*_v = \perp$.

Define the set $X := A + A = A \times \{1\} \cup A \times \{2\}$. Let $\vec{a}_1 \in TX$ be defined as $\vec{a}_1 := Ti_1(\vec{a})$, where $i_1\colon A \to A \times \{1\} \cup A \times \{2\};\ a \mapsto (a, 1)$. Similarly, we define $\vec{a}_2 \in TX$. That is, we take $\vec{a}_2 = Ti_2(\vec{a})$, where $i_2\colon A \to A \times \{1\} \cup A \times \{2\};\ a \mapsto (a, 2)$.

Take the set X to be the carrier of our coalgebra $\mathbb{X}$ and let the structure $\xi\colon X \to TX$ of the coalgebra $\mathbb{X}$ be defined as

$$\xi(v) = \begin{cases} \vec{a}_2 & \text{for } v \in A \times \{1\}, \\ \vec{a}_1 & \text{for } v \in f^{-1}(C) \times \{2\}, \\ *_v & \text{for } v \in [A \setminus f^{-1}(C)] \times \{2\}. \end{cases}$$

We are now going to list some basic facts about subcoalgebras of the coalgebra $\mathbb{X}$.

Lemma 3.6. *If $\mathbb{S} \leq \mathbb{X}$ with $S \cap A \times \{1\} \neq \varnothing$, then $S \cap f^{-1}(C) \times \{2\} \neq \varnothing$ and $S \cap [A \setminus f^{-1}(C)] \times \{2\} \neq \varnothing$. If $S \cap f^{-1}(C) \times \{2\} \neq \varnothing$, then $S \cap A \times \{1\} \neq \varnothing$.*

Proof. This follows directly from the definition of $\vec{a}_1$, $\vec{a}_2$ and the fact that $\vec{a} \in TA$, $\vec{a} \notin T[f^{-1}(C)]$ and $\vec{a} \notin T[A \setminus f^{-1}(C)]$. □

Lemma 3.7. *For any non-empty subcoalgebra $\mathbb{S} \leq \mathbb{X}$ we have*

$$S \cap [A \setminus f^{-1}(C) \times \{2\}] \neq \varnothing.$$

Proof. Indeed, if $v \in S$ and $v \in f^{-1}(C) \times \{2\}$, then by Lemma 3.6 we have $S \cap [A \times \{1\}] \neq \varnothing$. Now, if $v \in S \cap [A \times \{1\}]$, then again by Lemma 3.6 it implies that $S \cap [A \setminus f^{-1}(C) \times \{2\}] \neq \varnothing$. □

Notice that each element from the subset $[A \setminus f^{-1}(C)] \times \{2\}$ forms a one-element subcoalgebra of $\mathbb{X}$. In other words, Lemma 3.7 states that each non-empty subcoalgebra of $\mathbb{X}$ contains a one-element subcoalgebra.

Let $Y = A + B = A \times \{1\} \cup B \times \{2\}$. Firstly, define $h \colon X \to Y$ as

$$h(v) = \begin{cases} v & \text{for } v \in A \times \{1\}, \\ (f(a), 2) & \text{for } v = (a, 2) \in A \times \{2\}. \end{cases}$$

Now, we are ready to define the structure $\psi \colon Y \to TY$ as

$$\psi(v) = \begin{cases} Th(\vec{a}_2) & \text{for } v \in A \times \{1\}, \\ \vec{a}_1 & \text{for } v \in C \times \{2\}, \\ *_v & \text{for } v \in [B \setminus C] \times \{2\}. \end{cases}$$

It is not hard to prove that $h \colon X \to Y$ is a homomorphism between $\mathbb{X}$ and $\mathbb{Y} = \langle Y, \psi \rangle$. Moreover, the following properties of $\mathbb{Y}$ hold.

Lemma 3.8. *For any subcoalgebra $\mathbb{S} \leq \mathbb{Y}$ the condition $A \times \{1\} \cap S \neq \varnothing$ is equivalent to $C \times \{2\} \cap S \neq \varnothing$. Moreover, $\psi(A + C) \subseteq T[A + C]$.*

Proof. This follows directly from the definition of $\mathbb{Y}$ and the properties of the element $Th(\vec{a}_2)$, which is isomorphic to the element $Tf(\vec{a})$. □

By Lemma 3.8 it follows that $\langle A + C, \psi_{|A+C} \rangle$ forms a subcoalgebra of $\mathbb{Y}$. We will now show the following theorem.

Theorem 3.9. *The coalgebra $\langle A + C, \psi_{|A+C} \rangle$ belongs to the class $\mathcal{SH}(\mathbb{X})$ but it does not belong to the class $\mathcal{HS}(\mathbb{X})$.*

Proof. It is clear that $\langle A + C, \psi_{|A+C} \rangle \in \mathcal{SH}(\mathbb{X})$. To show that the coalgebra $\langle A + C, \psi_{|A+C} \rangle \notin \mathcal{HS}(\mathbb{X})$ observe that by Lemma 3.7 any non-empty subcoalgebra $\mathbb{S}$ of $\mathbb{X}$ contains some elements from the subset $[A \setminus f^{-1}(C)] \times \{2\}$. That is, for each subcoalgebra $\mathbb{S} \leq \mathbb{X}$ there is $v \in S$ such that $\langle \{v\}, \xi_{|\{v\}} \rangle$ forms a one-element subcoalgebra of $\mathbb{S}$. Assume by contradiction that $\langle A + C, \psi_{|A+C} \rangle \in \mathcal{HS}(\mathbb{X})$. This means that there is $\mathbb{S} \leq \mathbb{X}$ whose homomorphic image is the coalgebra $\langle A + C, \psi_{|A+C} \rangle$. This would mean that the image of the element $v \in S$ forms a one-element subcoalgebra of $\langle A + C, \psi_{|A+C} \rangle$. This is impossible, since by Lemma 3.8 the coalgebra $\langle A + C, \psi_{|A+C} \rangle$ does not contain any one-element subcoalgebras. A contradiction. □

4. Functors preserving non-empty preimages on undistinguished points

In this section we will present a general setting which allows us to construct many examples of coalgebras for functors that fail to satisfy $\mathcal{SH} = \mathcal{HS}$. At the time of writing this paper we were unable to find a complete solution to the $\mathcal{HS} = \mathcal{SH}$ problem.

4.1. Standing Hypothesis. In the rest of the paper we assume that the functor $T\colon \mathsf{Set} \to \mathsf{Set}$ is *connected, standard*, it *does not preserve non-empty preimages*, it is *not* the case that T *does not preserve non-empty preimages on undistinguished points* and it has a *distinguished point*, say $\mu_X(\bot) \in TX$ for a natural transformation $\mu\colon \mathcal{C}_{0,1} \to T$ and a non-empty set X. For any non-empty set X, we will denote the unique element $\mu_X(\bot)$ by $\bot_X$ or simply $\bot$. We know that if T is a connected Set-endofunctor with $\mathcal{C}_{0,1} \leq T$, then T preserves non-empty preimages if and only if the restrictions of T and $\mathcal{C}_{0,1}$ to non-empty sets are naturally equivalent (see [8] for a proof). Therefore, T is distinct from a constant functor.

We have the following lemma.

Lemma 4.1. *Let T be as above. Let $f\colon A \to B$ be a mapping and $\vec{a} \in TA$ such that $[Tf](\vec{a}) \neq \bot$. Then*

$$\mathsf{Flt}_B([Tf](\vec{a})) = [\mathcal{F}f](\mathsf{Flt}_A(\vec{a})).$$

Proof. It is always the case that $[\mathcal{F}f](\mathsf{Flt}_A(\vec{a})) \subseteq \mathsf{Flt}_B([Tf](\vec{a}))$ (see [3]). By contradiction assume that $[\mathcal{F}f](\mathsf{Flt}_A(\vec{a})) \subset \mathsf{Flt}_B([Tf](\vec{a}))$. In general we have

$$\begin{aligned}[\mathcal{F}f](\mathsf{Flt}_A(\vec{a})) &= \{W \subseteq B \mid f(U) \subseteq W \text{ for } U \in \mathsf{Flt}_A(\vec{a})\}\\ &= \{W \subseteq B \mid f(U) \subseteq W \text{ for } \vec{a} \in TU\} = \{V \subseteq B \mid \vec{a} \in T[f^{-1}(V)]\}\end{aligned}$$

and

$$\mathsf{Flt}_B([Tf](\vec{a})) = \{V \subseteq B \mid [Tf](\vec{a}) \in TV\}.$$

We see that if there was a set $C \subseteq B$ such that $C \notin [\mathcal{F}f](\mathsf{Flt}_A(\vec{a}))$ and $C \in \mathsf{Flt}_B([Tf](\vec{a}))$, then $\vec{a} \notin T[f^{-1}(C)]$ and $[Tf](\vec{a}) \in TC$. Since $[Tf](\vec{a}) \neq \bot$ we get that T does not preserve non-empty preimages on undistinguished points. A contradiction. □

We define

$$\lambda := \text{ a cardinal number for which } |T\lambda| > 1,$$

and assume that A is a set with $|A| = \lambda$ and $\vec{a} \in TA$ is an undistinguished point.

In the general setting we only assume that we have a T-coalgebra $\mathbb{B} = \langle B, \beta\rangle$ and two undistinguished elements $\vec{a}_{B,1} \in TB$ and $\vec{a}_{B,2} \in TB$ satisfying the following properties.

(1) $\bot \notin \beta(B)$ and $\vec{a}_{B,1}, \vec{a}_{B,2} \notin \beta(B)$.
(2) If there are $\mathbb{S} \leq \langle B, \beta\rangle \leq \langle C, \gamma\rangle$ with
  - $\vec{a}_{B,1}, \vec{a}_{B,2} \in TS$,
  - $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \bot\}$,

  then given a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ such that $Th(\vec{a}_{B,1}) = \vec{a}_{B,1} \in TB \subseteq TC$ we have $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\bot\}$.

To see some classes of functors for which there is a coalgebra and elements satisfying conditions (1) and (2), the reader is referred to Subsections 4.2–4.5.

We will now show that given such a T-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}, \vec{a}_{B,2}$ satisfying (1), (2) we are able to construct a T-coalgebra $\mathbb{X}$ such that $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$.

Assume that the set A is disjoint from B, 2^B and 2^{2^B}, and let

$$X = B \cup A \cup 2^B \cup 2^{2^B}.$$

Finally, recall that we have $\vec{a} \in TA \subset TX$. We will define the structure $\xi\colon X \to TX$ as follows:

$$\xi(v) = \begin{cases} \beta(v) & \text{for } v \in B, \\ \vec{a}_{B,2} & \text{for } v \in A, \\ \vec{a} & \text{for } v \in 2^B, \\ \vec{a}_{B,1} & \text{for } v \in 2^{2^B}. \end{cases}$$

Now let $Y = B \cup \{s\} \cup 2^B \cup 2^{2^B}$ and let $\psi\colon Y \to TY$ be defined as follows:

$$\psi(v) = \begin{cases} \beta(v) & \text{for } v \in B, \\ \vec{a}_{B,2} & \text{for } v = s, \\ \perp & \text{for } v \in 2^B, \\ \vec{a}_{B,1} & \text{for } v \in 2^{2^B}. \end{cases}$$

It is clear that $\mathbb{Y} = \langle Y, \psi \rangle$ is a homomorphic image of $\mathbb{X}$. Moreover, the subset $B \cup 2^B \cup 2^{2^B}$ is closed under the structure map ψ of the coalgebra $\mathbb{Y}$. In other words, $\langle B \cup 2^B \cup 2^{2^B}, \psi_| \rangle \leq \mathbb{Y}$ and that is why $\langle B \cup 2^B \cup 2^{2^B}, \psi_| \rangle \in \mathcal{SH}(\mathbb{X})$. We are now going to show that $\langle B \cup 2^B \cup 2^{2^B}, \psi_| \rangle \notin \mathcal{HS}(\mathbb{X})$. Assume by contradiction that there is a subcoalgebra $\mathbb{S} = \langle S, \xi_| \rangle \leq \mathbb{X}$ whose homomorphic image under a homomorphism $h\colon S \to B \cup 2^B \cup 2^{2^B}$ is the coalgebra $\langle B \cup 2^B \cup 2^{2^B}, \psi_| \rangle$. We will derive some properties of h that will lead us to a contradiction.

First of all observe that $|A| \leq |B| < |2^B| < |2^{2^B}|$. Because of cardinality issues and because h is onto, the subcoalgebra $\mathbb{S}$ has to contain some elements from 2^{2^B} which are mapped to 2^{2^B} by h. Hence, there is an element $c \in S \cap 2^{2^B}$ such that $h(c) \in 2^{2^B} \subset Y$. Since h is a homomorphism we have $Th(\xi(c)) = Th(\vec{a}_{B,1}) = \psi(h(c)) = \vec{a}_{B,1}$. Therefore, for any $v \in S \cap 2^{2^B}$ the following is true: $\psi(h(v)) = Th(\xi(v)) = Th(\xi(c)) = \vec{a}_{B,1}$. Hence for any $v \in S \cap 2^{2^B}$ it has to be the case that $h(v) \in 2^{2^B}$ because otherwise $\psi(h(v)) = Th(\xi(v))$ would not be equal to $\vec{a}_{B,1}$. Similarly, by cardinality issues $2^B \cap h(S \cap 2^B) \neq \varnothing$. Using an argument similar to the one above we show that $h(S \cap 2^B) \subseteq 2^B$. To sum up:

$$S \cap 2^{2^B} \neq \varnothing \text{ with } h(S \cap 2^{2^B}) \subseteq 2^{2^B},$$
$$S \cap 2^B \neq \varnothing \text{ with } h(S \cap 2^B) \subseteq 2^B.$$

From the construction of $\mathbb{X}$ it follows that $\mathbb{S}$ also contains some elements from the subset $A \subset X$ and some elements from B since $\vec{a} \in TS$ and $\vec{a}_{B,1} \in TS$.

Moreover, because $S \cap A \neq \varnothing$ we also have $\vec{a}_{B,2} \in TS$. That is,

$$S \cap A \neq \varnothing, \quad S \cap B \neq \varnothing \quad \text{and} \quad \mathbb{S} \cap \mathbb{B} \leq \mathbb{B} < \mathbb{X}.$$

For any $v \in S \cap 2^{2^B}$ we have $h(v) \in 2^{2^B}$ and therefore, the following equality is true:

$$Th(\xi(v)) = Th(\vec{a}_{B,1}) = \psi(h(v)) = \vec{a}_{B,1} \in TB \subset T[B \cup 2^B \cup 2^{2^B}].$$

Therefore, if we consider the coalgebras $\mathbb{S} \cap \mathbb{B} \leq \mathbb{B} < \langle B \cup 2^B \cup 2^{2^B}, \psi_{|}\rangle$, a homomorphism $h_{|S\cap B}\colon \mathbb{S} \cap \mathbb{B} \to \langle B \cup 2^B \cup 2^{2^B}, \psi_{|}\rangle$ which is a restriction of h, then it is true that $Th(\vec{a}_{B,1}) = Th_{S\cap B}(\vec{a}_{B,1}) = \vec{a}_{B,1}$. This means that we may apply property (2) and get

$$Th[\xi(S \cap B)] = Th_{|S\cap B}[\xi(S \cap B)] = Th_{|S\cap B}[\beta(S \cap B)] \subseteq \beta(B).$$

By (2) we also see that $Th_{|S\cap B}(\vec{a}_{B,2}) = Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}, \bot\}$. This is a contradiction, because for any $v \in S \cap A$ the following is true:

$$Th(\vec{a}_{B,2}) = Th(\xi(v)) = \psi(h(v)) \in \psi(B \cup 2^B \cup 2^{2^B}) = \beta(B) \cup \{\vec{a}_{B,1}, \bot\}.$$

Theorem 4.2. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a connected functor that preserves non-empty preimages on undistinguished points and $\mathcal{C}_{0,1} \leq T$. Let λ be any cardinal number for which $|T\lambda| > 1$. Moreover, assume that there is a T-coalgebra $\mathbb{B}$ with $|B| \geq \lambda$ and elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ with properties* (1) *and* (2) *listed above. Then there is a T-coalgebra $\mathbb{X}$ for which $\mathcal{SH}(\mathbb{X}) \neq \mathcal{HS}(\mathbb{X})$.*

4.2. Non-free filters. Let T, $\bot$, A, $\vec{a} \in TA$, λ be as in the Standing Hypothesis. In this subsection we will present a construction of a T-coalgebra $\langle B, \beta\rangle$ and elements $\vec{a}_{B,1}, \vec{a}_{B,2} \in TB$ for a functor T satisfying an additional property: the intersection $\bigcap \mathsf{Flt}_A(\vec{a})$ is non-empty, i.e., the filter $\mathsf{Flt}_A(\vec{a})$ is not free.

We know that any finitary functor different from a constant functor has elements whose filters are not free. That is why the construction presented in this subsection works for finitary connected functors preserving non-empty preimages on undistinguished points and having $\mathcal{C}_{0,1}$ as a subfunctor.

Define

$$B := [A + A] \cup \{x, y\} = [A \times \{1\}] \cup [A \times \{2\}] \cup \{x, y\}.$$

Let $z \in \bigcap \mathsf{Flt}_A(\vec{a})$. Define the mapping $i_x^k\colon A \to B$ for $k = 1, 2$ as follows:

$$i_x^k(v) := (v, k) \;\; \text{for } v \neq z, \text{ and}$$
$$i_x^k(z) := x.$$

Similarly, we define $i_y^k\colon A \to B$ for $k = 1, 2$:

$$i_y^k(v) := (v, k) \;\; \text{for } v \neq z, \text{ and}$$
$$i_y^k(z) := y.$$

Moreover, let $i_1\colon A \to B;\ a \mapsto (a,1)$ and $i_2\colon A \to B;\ a \mapsto (a,2)$. We introduce the following elements:

$$\vec{a}_x^k := Ti_x^k(\vec{a}),$$
$$\vec{a}_y^k := Ti_y^k(\vec{a}),$$
$$\vec{a}_1 := Ti_1(\vec{a}),$$
$$\vec{a}_2 := Ti_2(\vec{a}).$$

It is clear that for $k = 1, 2$ we have

$$x \in \bigcap \mathsf{Flt}_B[Ti_x^k(\vec{a})] \quad \text{and} \quad y \in \bigcap \mathsf{Flt}_B[Ti_y^k(\vec{a})].$$

We will construct a coalgebra on the carrier B. We define the structure $\beta\colon B \to TB$ as follows:

$$\beta(v) = \begin{cases} \vec{a}_y^1 & \text{for } v = x,\\ \vec{a}_x^2 & \text{for } v = y,\\ \vec{a}_1 & \text{for } v \in A \times \{1\},\\ \vec{a}_2 & \text{for } v \in A \times \{2\}. \end{cases}$$

Now let $\vec{a}_{B,1} = \vec{a}_x^1$ and $\vec{a}_{B,2} = \vec{a}_y^2$. First of all observe that property (1) from the Standing Hypothesis is satisfied. To show that (2) holds take any subcoalgebra $\mathbb{S} \le \mathbb{B}$ with $\vec{a}_{B,1} \in TS$ and $\vec{a}_{B,2} \in TS$. Also consider a coalgebra $\mathbb{C}$ with $\mathbb{B} \le \mathbb{C}$, $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \perp\}$ and a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ such that $Th(\vec{a}_{B,1}) = \vec{a}_{B,1}$. The fact $\vec{a}_{B,1} = \vec{a}_x^1 \in TS$ and $\vec{a}_{B,2} = \vec{a}_y^2 \in TS$ implies that $x, y \in S$ and

$$S \cap A \times \{1\} \neq \varnothing \neq S \cap A \times \{2\}.$$

Lemma 4.3.

$$h(S \cap A \times \{1\}) \subseteq S \cap A \times \{1\},$$
$$h(S \cap A \times \{2\}) \subseteq S \cap A \times \{2\},$$
$$h(x) = x,$$
$$h(y) = y.$$

Proof. If $Th(\vec{a}_x^1) = Th(\vec{a}_{B,1}) = \vec{a}_{B,1} = \vec{a}_x^1$, then by Lemma 4.1 we have

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}_x^1)) = \{W \mid f(U) \subseteq W \text{ and } U \in \mathsf{Flt}_S(\vec{a}_x^1)\} = \mathsf{Flt}_C(\vec{a}_x^1).$$

Therefore, there is a set $U \subseteq S \cap [A \times \{1\}] \cup \{x\}$ such that $h(U) \subseteq [A \times \{1\}] \cup \{x\}$ and there is a point $v \in S \cap [A \times \{1\}] \cup \{x\}$ which satisfies $h(v) = x$. By the fact $\beta(A \times \{1\}) = \{\vec{a}_1\} \neq \{\vec{a}_x^1\} = \beta(\{x\})$ and the fact that $\vec{a}_x^1 \notin \gamma(C \setminus \{x\})$ if there was an element $v \in S \cap [A \times \{1\}]$ such that $v \neq x$ and $h(v) = x$, then $h(S \cap [A \times \{1\}]) = \{x\}$ and

$$Th(\vec{a}_1) = Th(\beta(v)) = \beta(h(v)) = \beta(x) = \vec{a}_y^1.$$

This clearly implies that

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}_1)) = \mathsf{Flt}_C(\vec{a}_y^1).$$

Thus, there is $V \subseteq S \cap A \times \{1\}$ such that $h(V) \subseteq A \times \{1\} \cup \{y\}$, which contradicts the fact that $h(S \cap [A \times \{1\}]) = \{x\}$. Therefore, $h(S \cap [A \times \{1\}]) \subseteq A \times \{1\}$ and $h(x) = x$.

Now, if $h(x) = x$, then

$$\vec{a}_y^1 = \beta(x) = \gamma(x) = \gamma(h(x)) = Th(\beta(x)) = Th(\vec{a}_y^1).$$

Therefore, by applying a similar reasoning we get $h(S \cap [A \times \{2\}]) \subseteq A \times \{2\}$ and $h(y) = y$. □

If we define a bijection $j \colon C \to C$ as

$$j(v) = \begin{cases} v & \text{for } v \notin \{x, y\}, \\ x & \text{for } v = y, \\ y & \text{for } v = x, \end{cases}$$

then

$$j \circ h = h \circ j_{|S} \quad \text{and} \quad \vec{a}_{B,2} = \vec{a}_y^2 = Tj(\vec{a}_x^2).$$

By the fact that $h(y) = y$ (see Lemma 4.3) and that $h \colon \mathbb{S} \to \mathbb{C}$ is a homomorphism we have

$$Th(\vec{a}_x^2) = Th(\beta_{|S}(y)) = \gamma(h(y)) = \gamma(y) = \vec{a}_x^2.$$

Therefore,

$$Th(\vec{a}_{B,2}) = Th(\vec{a}_y^2) = Th[Tj_{|S}(\vec{a}_x^2)] = Tj[Th(\vec{a}_x^2)] = Tj(\vec{a}_x^2) = \vec{a}_y^2 = \vec{a}_{B,2}.$$

Hence, $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{a_{B,1}\} \cup \{\perp\}$.

The existence of a T-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ for which the properties (1) and (2) are true guarantees that the following theorem is true.

Theorem 4.4. *Let $T \colon \mathsf{Set} \to \mathsf{Set}$ and A be as in the Standing Hypothesis. Moreover, let there be an undistinguished point $\vec{a} \in TA$ such that the set $\bigcap \mathsf{Flt}_A(\vec{a})$ is non-empty. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

4.3. Free filters on ω. Let T, $\perp$, A, $\vec{a} \in TA$, λ be as in the Standing Hypothesis. Throughout this subsection we assume that we deal with functors for which $\lambda = \omega$ and for which there is an undistinguished point $\vec{a} \in TA$ whose filter $\mathsf{Flt}_A(\vec{a})$ is free. We will present a construction of a T-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$, $\vec{a}_{B,2}$ satisfying properties (1) and (2).

Let $B = \omega \times \omega$. For any $n = 0, 1, 2, \ldots$ define a mapping $i_n \colon \omega \to B;\ k \mapsto (k, n)$ and an element from TB as follows:

$$\vec{a}_n = Ti_n(\vec{a}).$$

It is clear from the definition that $\vec{a}_n \in T[\omega \times \{n\}] \subset TB$. Finally, let $i_\omega \colon \omega \to X;\ k \mapsto (1, k)$ and denote

$$\vec{a}_\omega = Ti_\omega(\vec{a}).$$

We see that $\vec{a}_\omega \in T[\{1\} \times \omega] \subset T[\omega \times \omega] = TB$ and $\vec{a}_\omega \notin T[\omega \times \{n\}]$ for any $n = 0, 1, 2, \dots$. We are ready to define the structure $\beta\colon B \to TB$ and the two elements as follows:

$$\begin{aligned} &\beta(v) := \vec{a}_{n+1} \quad \text{for } v \in \omega \times \{n\}, \\ &\vec{a}_{B,1} := \vec{a}_0, \\ &\vec{a}_{B,2} := \vec{a}_\omega. \end{aligned}$$

Clearly $\mathbb{B} = \langle B, \beta \rangle$, $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ satisfy property (1).

To see that (2) is also satisfied take a subcoalgebra $\mathbb{S} \leq \mathbb{B}$ such that $\vec{a}_{B,1} = \vec{a}_0 \in TS$ and a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ where $\mathbb{C}$ has $\mathbb{B}$ as its subcoalgebra and $Th(\vec{a}_{B,1}) = \vec{a}_{B,1} \in TB \subseteq TC$. If $\vec{a}_0 \in TS$, then it means that S contains a subset of $\omega \times \{0\} \subseteq B$. That is $S \cap \omega \times \{0\} \neq \varnothing$. But then it can be easily shown by induction that $S \cap \omega \times \{n\} \neq \varnothing$ for any $n = 1, 2, \dots$. Moreover, if the following equality is true

$$Th(\vec{a}_0) = Th(\vec{a}_{B,1}) = \vec{a}_{B,1} = \vec{a}_0,$$

then there is $V \subseteq S \cap \omega \times \{0\}$ such that $h(V) \subseteq \omega \times \{0\}$. This means that for $v \in V$, $Th(\vec{a}_1) = Th(\beta(v)) = \gamma(h(v)) = \vec{a}_1$. We may apply induction and show that $Th(\vec{a}_n) = \vec{a}_n$ for $n = 2, 3, 4, \dots$.

Now, in order to show that (2) is satisfied it is enough to prove that $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\bot\}$. We will start with showing that $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\}$. Assume by contradiction that $Th(\vec{a}_{B,2}) = Th(\vec{a}_\omega) \in \beta(B) \cup \{\vec{a}_{B,1}\} = \{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots\}$. This means that there is a number $n \in \{0, 1, 2, \dots\}$ such that

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}_\omega)) = \{W \mid h(U) \subseteq W \text{ and } U \in \mathsf{Flt}_S(\vec{a}_\omega)\} = \mathsf{Flt}_B(\vec{a}_n).$$

Therefore, there exists a subset $U \subseteq \{1\} \times \omega$ with $U \in \mathsf{Flt}_S(\vec{a}_\omega)$ such that $h(U) \subseteq \omega \times \{n\}$. Because the filter $\mathsf{Flt}_S(\vec{a}_\omega)$ is free and because $U \in \mathsf{Flt}_S(\vec{a}_\omega)$, we deduce that U is infinite and $U \nsubseteq \omega \times \{n\}$ for any $n = 0, 1, 2, \dots$. This precisely means that $\beta(U)$ is an infinite subset of $\{\vec{a}_k \mid k = 0, 1, 2, \dots\}$ such that $Th(\beta(U)) = \{\vec{a}_{n+1}\}$. This is a contradiction, since the mapping Th is injective on the set $\{\vec{a}_k \mid k = 0, 1, 2, \dots\}$. Thus, $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_0\} = \{\vec{a}_k \mid k = 0, 1, 2, \dots\}$. Now, it is enough to show that $Th(\vec{a}_{B,2}) \neq \bot$. Recall that $\vec{a}_{B,2} = \vec{a}_\omega$ was defined as an element of $T[\{1\} \times \omega] \subset TB$. Because of the fact that Th is one-to-one on $\{a_k \mid k \in \omega\}$, the restriction $h_{|\{1\}\times\omega}\colon \{1\} \times \omega \to C$ has to be a one-to-one mapping. This exactly means that $Th(\vec{a}_{B,2}) = Th_{|\{1\}\times\omega}(\vec{a}_\omega) \neq \bot$. Hence, $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\} \cup \{\bot\}$.

Theorem 4.5. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ and A be as in the Standing Hypothesis. Moreover, let $\lambda = \omega$ and let there be an undistinguished point $\vec{a} \in TA$ for which the filter $\mathsf{Flt}_A(\vec{a})$ is free. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

The corollary below follows from Theorems 4.4 and 4.5.

Corollary 4.6. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ and A be as in the Standing Hypothesis. Moreover, let $\lambda \leq \omega$. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

4.4. Free filters with λ-many disjoint stationary sets. Let T, $\perp$, A, $\vec{a} \in TA$, λ be as in the Standing Hypothesis and λ uncountable. Here we suppose that we deal with a functor T for which there is an element $\vec{a} \in TA$ and a partition $\{A_l\}_{l\in\lambda}$ of the set A into λ-many non-empty subsets such that each of them is $\mathsf{Flt}_A(\vec{a})$-stationary. Our aim is to find a T-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$, $\vec{a}_{B,2}$ satisfying (1) and (2).

Recall that a binary relation (W, R) is called rigid if the only homomorphism (in the sense of relational structures) with (W, R) as a domain and codomain is the identity id_W (see [5, p. 63] and [9] for details). The following result about rigidity of binary relations will be extensively used in this subsection.

Theorem 4.7 ([9], see also [5])**.** *For any cardinal κ there is a binary rigid relation (W, R) without loops such that $|W| = \kappa$ and for any $x \in W$ there is $y \in W$ such that $(x, y) \in R$.*

Now take a rigid relation (W, R) from Theorem 4.7 for which $|W| = \lambda$. For any $x \in W$ define the set $\eta(x) = \{y \in W \mid (x, y) \in R\}$. It is plain to see that $\eta(x) \neq \varnothing$ for any $x \in W$. Let

$$B = W \times \lambda.$$

By $\vec{a}^C \in TB$ where $C \subseteq W$ we denote an isomorphic copy of $\vec{a}$ for which the set $\{x\} \times \lambda$ is $\mathsf{Flt}_B(\vec{a}^C)$-stationary if and only if $x \in C$. A construction of $\vec{a}^C$ is done by taking an injective mapping $i_C\colon A \to W \times \lambda$ for which the elements of the partition $\{A_k\}_{k\leq\lambda}$ of A are mapped into subsets $\{x\} \times \lambda$ for $x \in C$ and taking $\vec{a}^C := Ti_C(\vec{a})$. Such an injective mapping exists by the fact that any subset $C \subseteq W$ has cardinality at most λ. Let $\vec{a}_{B,1} \in TB$ be defined as $\vec{a}_{B,1} := \vec{a}^W$. Finally, choose $k \in \lambda$ and let $\vec{a}_{B,2}$ be an isomorphic copy of $\vec{a}$ such that $\vec{a}_{B,2} \in T[W \times \{k\}] \subset TB$.

Now we are ready to define the structure $\beta\colon B \to TB$ of the coalgebra $\mathbb{B}$:

$$\beta(v) := \vec{a}^{\eta(x)} \quad \text{for } v \in \{x\} \times \lambda.$$

Lemma 4.8. *Given a homomorphism $h\colon \mathbb{S} \to \mathbb{B}$ where $\mathbb{S} \leq \mathbb{B}$, for any $x \in W$ we have $h(\{x\} \times \lambda \cap S) \subseteq \{y\} \times \lambda$ for some $y \in W$.*

Proof. If there were two elements $(x, l_1) \in S$ and $(x, l_2) \in S$ such that $h(x, l_1) \in \{y\} \times \lambda$ and $h(x, l_2) \in \{z\} \times \lambda$, then since $\beta(x, l_1) = \beta(x, l_2)$ we would have $Th(\beta(x, l_1)) = \beta(h(x, l_1)) = \beta(h(x, l_2)) = Th(\beta(x, l_2))$ and therefore, $\vec{a}^{\eta(y)} = \vec{a}^{\eta(z)}$. If $y \neq z$, then it means that in (W, R) two distinct elements y and z have the same family of neighbours, i.e., $\eta(y) = \eta(z)$. This implies that a function $f\colon W \to W$ which is defined as $f(x) = x$ for $x \notin \{y, z\}$, $f(z) = y$ and $f(y) = z$ is a homomorphism from the binary relation (W, R) into (W, R) which is not an identity. This contradicts the fact that (W, R) is rigid. $\square$

Lemma 4.9. *If $\mathbb{S} \leq \mathbb{B}$ is a subcoalgebra with $S \cap \{x\} \times \lambda \neq \varnothing$ for any $x \in W$ and $h\colon \mathbb{S} \to \mathbb{B}$ is a homomorphism, then*

$$h(\{x\} \times \lambda \cap S) \subseteq \{x\} \times \lambda \ \ \textit{for any } x \in W.$$

Proof. By Lemma 4.8, given a homomorphism $h\colon \mathbb{S} \to \mathbb{B}$ for any element $x \in W$ we have $h(\{x\} \times \lambda \cap S) \subseteq \{y\} \times \lambda$ for some $y \in W$. Given a pair $(x, l) \in S$ we have

$$Th(\vec{a}^{\eta(x)}) = Th(\beta(x, l)) = \beta(h(x, l)) = \beta((y, l')) = \vec{a}^{\eta(y)}.$$

This means that

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}^{\eta(x)})) = \{W \mid h(U) \subseteq W \text{ for } U \in \mathsf{Flt}_S(\vec{a}^{\eta(x)})\} = \mathsf{Flt}_B(\vec{a}^{\eta(y)}).$$

By the equality above and by the fact that the set $\eta(x) \times \lambda \cap S$ is $\mathsf{Flt}_S(\vec{a}^{\eta(x)})$-stationary and the sets $\{t\} \times \lambda$ for $t \in \eta(y)$ are $\mathsf{Flt}_B(\vec{a}^{\eta(y)})$-stationary we have

$$h(\eta(x) \times \lambda \cap S) \subseteq \eta(y) \times \lambda, \text{ and}$$
$$h(\eta(x) \times \lambda \cap S) \cap \{z\} \times \lambda \neq \varnothing \ \text{ for any } z \in \eta(y).$$

This exactly means that if we define $\tilde{h}\colon W \to W$ as $\tilde{h}(x) = \pi_1(h((x, k)))$ for any $k \in \lambda$, then $\tilde{h}(\eta(x)) = \eta(\tilde{h}(x))$. Therefore, $\tilde{h}\colon W \to W$ is a homomorphism from (W, B) to itself. Since (B, W) is rigid this implies that $\tilde{h} = id_W$. Thus, for $(x, l) \in \{x\} \times \lambda \cap S$ we have $h(x, l) = (x, l')$ for some $l' \in \lambda$. □

We will show that $\langle B, \beta\rangle$, $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ satisfy properties (1) and (2) listed at the beginning of this section. It is clear that the property (1) is satisfied. To show (2) is true choose a subcoalgebra $\mathbb{S}$ of $\mathbb{B}$ with $\vec{a}_{B,1}, \vec{a}_{B,2} \in TS$ and a homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ where $\mathbb{C}$ has $\mathbb{B}$ as its subcoalgebra, $Th(\vec{a}_{B,1}) = \vec{a}_{B,1} \in TB \subseteq TC$ and $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \perp\}$. By the definition of $\vec{a}_{B,1} = \vec{a}^W$ and by the fact $\vec{a}_{B,1} = \vec{a}^W \in TS$ it follows that $S \cap \{x\} \times \lambda \neq \varnothing$ for any $x \in W$. The equality $Th(\vec{a}_{B,1}) = Th(\vec{a}^W) = \vec{a}_{B,1} = \vec{a}^W$ implies that

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}^W)) = \{W \mid h(U) \subseteq W \text{ and } U \in \mathsf{Flt}_S(\vec{a}^W)\} = \mathsf{Flt}_C(\vec{a}^W).$$

Any set $U \in \mathsf{Flt}_S(\vec{a}^W)$ intersects with $\{x\} \times \lambda$ for all $x \in W$. That is why $\beta(U) = \{\vec{a}^{\eta(x)} \mid x \in W\} = \beta(S)$. Moreover, for any $U \in \mathsf{Flt}_S(\vec{a}^W)$ the set $h(U) \in \mathsf{Flt}_C(\vec{a}^W)$ intersects with any $\{x\} \times \lambda \subset C$ for $x \in W$ and therefore, $\gamma(h(U)) \supseteq \gamma(B) = \beta(B) = \{\vec{a}^{\eta(x)} \mid x \in W\}$.

Since $B \in \mathsf{Flt}_C(\vec{a}^W)$, there is $V \in \mathsf{Flt}_S(\vec{a}^W)$ such that $h(V) \subseteq B$. Therefore, $Th(\beta(S)) = Th(\beta(V)) = \beta(h(V)) = \beta(B)$. By the equality $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \perp\}$ we have $h(S) \subseteq B$.

In other words, we may treat the homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ as a homomorphism from $\mathbb{S}$ to $\mathbb{B}$.

We will now show that $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}, \perp\}$. First of all, we will show $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\}$. By contradiction assume that there is $C \subseteq W$ such that $Th(\vec{a}_{B,2}) = \vec{a}^C$. Therefore,

$$\mathcal{F}h(\mathsf{Flt}_S(\vec{a}_{B,2})) = \{W \mid h(U) \subseteq W \text{ and } U \in \mathsf{Flt}_S(\vec{a}_{B,2})\} = \mathsf{Flt}_C(\vec{a}^C).$$

Hence, there is $U \in \mathsf{Flt}_S(\vec{a}_{B,2})$ such that

- $U \subseteq W \times \{k\}$,
- $h(U) \subseteq B$,
- $h(U)$ intersects with $\{x\} \times \lambda$ for any $x \in C$.

Fix any $x \in C$. The set $[\{x\} \times \lambda] \cap h(U)$ has cardinality λ. This means that there is $V \subseteq U$ of cardinality λ such that $h(V) \subseteq \{x\} \times \lambda$. Therefore, there are two distinct elements $(y,k),(z,k) \in V$ such that $h(y,k) \in \{x\} \times \lambda$ and $h(z,k) \in \{x\} \times \lambda$. By Lemma 4.9 this is a contradiction. Therefore, $Th(\vec{a}_{B,2}) \notin \beta(B) \cup \{\vec{a}_{B,1}\}$.

Now it is enough to show that $Th(\vec{a}_{B,2}) \neq \perp$. This is easy to see because by Lemma 4.9 the restriction $h_{|W\times\{k\}} \colon W \times \{k\} \to B$ of h is a one-to-one mapping. Thus, $Th(\vec{a}_{B,2}) = Th_{|W\times\{k\}}(\vec{a}_{B,2}) \neq \perp$. Hence, the coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$, $\vec{a}_{B,2}$ satisfy property (2).

Theorem 4.10. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a connected functor that preserves non-empty preimages on undistinguished points and $\mathcal{C}_{0,1} < T$. Let there be a set A of uncountable cardinality λ together with its partition $\{A_l\}_{l\leq\lambda}$ and an undistinguished point $\vec{a} \in TA$ for which any element A_l of the partition of A is $\mathsf{Flt}_A(\vec{a})$-stationary. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

4.5. Subfunctors of factor functors of the Stone–Czech functor $\mathcal{B}$. Let T, $\perp$, A, $\vec{a} \in TA$, λ be as in the Standing Hypothesis. Let us recall that $\mathcal{B}\colon \mathsf{Set} \to \mathsf{Set}$ is the functor sending any set X to the underlying set of the Stone–Czech compactification $\mathcal{B}X$ of X viewed as a discrete space and, for $f\colon X \to Y$, $\mathcal{B}f\colon \mathcal{B}X \to \mathcal{B}Y$ is the underlying map of the continuous extension of f. The functor $\mathcal{B}$ can be also described as follows: it is a subfunctor of the functor $\mathcal{F}$ with $\mathcal{B}X$ being the set of all ultrafilters on X. Since $\mathcal{B}$ preserves all preimages, there is no problem with $\mathcal{B}$-coalgebras, i.e., $\mathcal{HS} = \mathcal{SH}$ is valid for them.

The functor $\mathcal{B}$ is connected and faithful, the copy of $\mathcal{I}d$ in $\mathcal{B}$ is formed by all fixed ultrafilters (ultrafilters with non-empty intersections).

Let $\nu\colon \mathcal{B} \to \bar{\mathcal{B}}$ be an epitransformation which glues together all fixed ultrafilters on any set X into one element of $\bar{\mathcal{B}}X$—the distinguished point $\perp \in \bar{\mathcal{B}}X$. The functor $\bar{\mathcal{B}}$ does not preserve non-empty preimages. For $\bar{\mathcal{B}}$-coalgebras we get $\mathcal{HS} \neq \mathcal{SH}$ simply by applying the construction presented in Subsection 4.3: put $\bar{\mathcal{B}}\omega = TA$ and let $\vec{a}$ be any free ultrafilter on ω.

Interesting subfunctors $\bar{\mathcal{B}}_M$ of $\bar{\mathcal{B}}$ are determined by a fixed subset $M \subseteq \bar{\mathcal{B}}X$ as follows:

$$\bar{\mathcal{B}}_M(Y) = \{\perp\} \cup \{[\bar{\mathcal{B}}f](\mathcal{U}) \mid \mathcal{U} \in M \text{ and } f\colon X \to Y\}.$$

We already know that $\mathcal{HS} \neq \mathcal{SH}$ for $\bar{\mathcal{B}}_M$-coalgebras in the case when M contains an ultrafilter on ω.

We were not able to decide whether $\mathcal{HS} \neq \mathcal{SH}$ for $\bar{\mathcal{B}}_M$-coalgebras whenever M consists of exactly one ultrafilter on arbitrarily large set X. This problem remains open.

On the other hand, $\mathcal{HS} \neq \mathcal{SH}$ can be shown for $\bar{\mathcal{B}}_M$-coalgebras if M consists of ultrafilters on an arbitrarily large set X whenever the set M itself is "sufficiently rich" which will be explained below. We recall that a Rudin–Keisler order on ultrafilters is defined as follows: for $\mathcal{U} \in \mathcal{B}X$ and $\mathcal{V} \in \mathcal{B}Y$, we write $\mathcal{U} \leq \mathcal{V}$ if and only if there exists $f\colon Y \to X$ such that $\mathcal{B}f(\mathcal{V}) = \mathcal{U}$. We will use the fascinating result of Shelah and Rudin [6]:

> for any infinite cardinal κ there exist 2^{2^κ} ultrafilters on a set X with $|X| = \kappa$ which are pairwise incomparable in the Rudin–Keisler order.

By the above result we know that we are able to construct a family $M \subset \bar{\mathcal{B}}A$ of ultrafilters on arbitrary large A which will contain at least $|A|$-many incomparable ultrafilters in the Rudin–Keisler order. Now, if we assume that M possesses this property, then we will be able to show that $\mathcal{HS} \neq \mathcal{SH}$. Indeed, we will now construct a $\bar{\mathcal{B}}_M$-coalgebra $\mathbb{B}$ and elements $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ satisfying properties (1) and (2) from the Standing Hypothesis. Let $B = A$. Therefore, $|B| = |A| = \lambda$. For any $v \in B$ define $\vec{a}_v \in \bar{\mathcal{B}}_M B$ so that the family $\{\vec{a}_v\}_{v \in B}$ is a family of incomparable ultrafilters. Introduce incomparable $\vec{a}_{B,1}, \vec{a}_{B,2} \in \bar{\mathcal{B}}_M B$ such that both ultrafilters are incomparable with $\vec{a}_v$ for any $v \in B$. Finally, define $\beta\colon B \to \bar{\mathcal{B}}_M B$; $v \mapsto \vec{a}_v$. It is clear that $\mathbb{B} = \langle B, \beta \rangle$, $\vec{a}_{B,1}$ and $\vec{a}_{B,2}$ satisfy property (1). We will show that they also satisfy (2). In order to do this take the coalgebras $\mathbb{S}$ and $\mathbb{C} = \langle C, \gamma \rangle$ such that $\mathbb{S} \leq \mathbb{B} \leq \mathbb{C}$ with $\vec{a}_{B,1}, \vec{a}_{B,2} \in \bar{\mathcal{B}}_M S$ and $\gamma(C \setminus B) = \{\vec{a}_{B,1}, \bot\}$. Since for different $v, w \in S$ the filters $\vec{a}_v$ and $\vec{a}_w$ are incomparable, then any homomorphism $h\colon \mathbb{S} \to \mathbb{C}$ is a one-to-one mapping. Hence, Th is one-to-one and $Th(\vec{a}_{B,2}) \neq \bot$.

Since the Rudin–Keisler order may be generalized to any filters, the above result may also be easily generalized to the following one.

Theorem 4.11. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a connected functor that preserves nonempty preimages on undistinguished points and $\mathcal{C}_{0,1} < T$. Let there be a set A of cardinality λ and let there be a family $\{\vec{a}_l\}_{l \in \lambda} \subseteq TA$ for which the filters $\mathsf{Flt}_A(\vec{a}_{l_1})$ and $\mathsf{Flt}_A(\vec{a}_{l_2})$ are incomparable in the generalization of the Rudin–Keisler order for any $l_1 \neq l_2$. Then there exists a T-coalgebra $\mathbb{X}$ such that $\mathcal{HS}(\mathbb{X}) \neq \mathcal{SH}(\mathbb{X})$.*

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