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A note on ideal C^*-completions and amenability

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Abstract. For a discrete group G, we consider certain ideals $\mathcal{I} \subset c_0(G)$ of sequences with prescribed rate of convergence to zero. We show that the equality between the full group C^*-algebra of G and the C^*-completion $C^*_{\mathcal{I}}(G)$ in the sense of Brown and Guentner (Bull. London Math. Soc. 45:1181–1193, 2013) implies that G is amenable.

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1. Introduction

By a classical result of Hulanicki [4], amenable groups can be characterized by the fact that their full and reduced group C^*-algebras coincide. In [2], Brown and Guentner obtained several far reaching generalizations of this fact by introducing a new C^*-completion of any discrete group G induced by an algebraic ideal $\mathcal{D}$ of $\ell_\infty(G)$. Namely, the corresponding group C^*-algebra, denoted by $C^*_{\mathcal{D}}(G)$, is the completion of the group ring $\mathbb{C}[G]$ with respect to the norm

$$\|x\|_{\mathcal{D}} = \sup\big\{\|\pi(x)\|\colon \pi \text{ is a } \mathcal{D}\text{-representation}\big\},$$

where by a $\mathcal{D}$-*representation* we mean a unitary representation π of G on a Hilbert space H such that the matrix coefficient functions $\pi_{\xi,\eta}$ belong to $\mathcal{D}$ for all ξ, η from a dense subspace of H. Using this idea, Brown and Guentner provided new C^*-algebraic characterizations of a-T-menability and Kazhdan's property (T) and, among other things, they showed that the equality $C^*_{\ell^p}(G) = C^*(G)$ is equivalent to G being amenable.

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In this note, we consider the ideals of $c_0(G)$ consisting of sequences with prescribed rate of convergence. Namely, for $f \in c_0(G)$ and $\varepsilon > 0$, we set

$$\nu(f,\varepsilon) = \#\big\{s \in G\colon |f(s)| \geq \varepsilon\big\},$$

and define

$$\mathcal{I}_{(a_n)} = \big\{f \in c_0(G)\colon \nu(f, \tfrac{1}{n}) = O(a_n)\big\}.$$

We show that the condition

$$\mathrm{C}^*_{\mathcal{I}_{(a_n)}}(G) = \mathrm{C}^*(G) \tag{*}$$

is equivalent to (or implies) amenability, provided that (a_n) does not grow too fast.

Amenability is strictly connected to the famous and widely studied stability property arising from a problem posed by Ulam [7] whether any quasimorphism can be uniformly approximated by a homomorphism, the problem first solved for commutative groups by Hyers [5]. We say that a group G has the *Hyers–Ulam property* provided that for every map $\phi\colon G \to \mathbb{R}$ satisfying

$$\sup\big\{|\phi(xy) - \phi(x) - \phi(y)|\colon x, y \in G\big\} < \infty$$

we have $\operatorname{dist}(\phi, \operatorname{Hom}(G,\mathbb{R})) < \infty$. It is known (see [6]) that every amenable group has the Hyers–Ulam property, but the converse is not true which is witnessed e.g. by the groups $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$. Although there is an algebraic characterization of the Hyers–Ulam property, due to Bavard [1], no C^*-algebraic characterization is known.

Hence, a natural question concerning Ulam stability reads as follows: Is there an increasing sequence $(a_n) \subset \mathbb{R}_+$ such that for any discrete group G the following characterization holds true: G has the Hyers–Ulam property if and only if condition $(*)$ holds true? Our result reduces the size of the set of possible candidates for (a_n).

2. Results

In what follows, G stands for a general discrete group. We will need the following two results proved by Brown and Guentner.

Proposition 1. (see [2, Remark 2.5]) *For any ideal $\mathcal{D} \subset \ell^\infty(G)$, $\mathrm{C}^*_{\mathcal{D}}(G)$ has a faithful $\mathcal{D}$-representation.*

Theorem 2. ([2, Thm. 3.2]) *Let $\mathcal{D} \subset \ell^\infty(G)$ be a translation-invariant ideal. Then, we have $\mathrm{C}^*_{\mathcal{D}}(G) = \mathrm{C}^*(G)$ if and only if there exists a sequence $(h_n) \subset \mathcal{D}$ of positive-definite functions converging pointwise to the constant one function.*

Our main result reads as follows.

Theorem 3. **(a)** *If* $(a_n) = O(n^{2+\delta})$ *for every* $\delta > 0$*, then*

$$\mathrm{C}^*_{\mathcal{I}_{(a_n)}}(G) = \mathrm{C}^*_r(G)$$

and hence condition $(*)$ *is equivalent to* G *being amenable.*

(b) *Suppose a sequence* $(a_n) \subset \mathbb{R}_+$ *is such that for some* $k > 0$*, we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^k} < \infty. \tag{**}$$

Then, condition $(*)$ *implies that* G *is amenable.*

Lemma 4. *Let* $f \in c_0(G)$ *and* $p \geq 1$*. We have* $f \in \ell^p(G)$ *if and only if the series*

$$\sum_{n=1}^{\infty} \nu(f, \tfrac{1}{n}) n^{-(p+1)} \tag{2.1}$$

converges.

Proof. Let $\Gamma_n = \{s \in G \colon \frac{1}{n} \leq |f(s)| < \frac{1}{n-1}\}$ for $n \in \mathbb{N}$ with the convention $\frac{1}{0} = \infty$, and note that

$$\sum_{s \in G} |f(s)|^p = \sum_{n=1}^{\infty} \sum_{s \in \Gamma_n} |f(s)|^p.$$

Since $|\Gamma_1| = \nu(f, 1)$ and $|\Gamma_n| = \nu(f, \frac{1}{n}) - \nu(f, \frac{1}{n-1})$ for $n \geq 2$, we have

$$\sum_{s \in G} |f(s)|^p \leq \nu(f, 1) \cdot \|f\|_\infty + \sum_{n=1}^{\infty} \big(\nu(f, \tfrac{1}{n+1}) - \nu(f, \tfrac{1}{n})\big) \cdot n^{-p} \tag{2.2}$$

and

$$\sum_{s \in G} |f(s)|^p \geq \sum_{n=1}^{\infty} \big(\nu(f, \tfrac{1}{n+1}) - \nu(f, \tfrac{1}{n})\big) \cdot (n+1)^{-p}. \tag{2.3}$$

Denote $d_n = \nu(f, \frac{1}{n+1}) - \nu(f, \frac{1}{n})$; the series occurring in (2.2) is the limit of partial sums

$$\begin{aligned} &\lim_{N \to \infty} \sum_{n=1}^{N} d_n n^{-p} \\ &= \lim_{N \to \infty} \left[\sum_{n=1}^{N-1} (d_1 + \ldots + d_n)\big(n^{-p} - (n+1)^{-p}\big) + (d_1 + \ldots + d_N) \cdot N^{-p} \right]. \end{aligned}$$

Since $d_1 + \ldots + d_N = \nu(f, \frac{1}{N+1}) - \nu(f, 1)$, the above limit exists if and only if the series

$$\sum_{n=1}^{\infty} \nu(f, \tfrac{1}{n+1})(n^{-p} - (n+1)^{-p})$$

converges. By Lagrange's mean value theorem, we have $n^{-p} - (n+1)^{-p} = p\,\theta_n^{-(p+1)}$ for some $\theta_n \in (n, n+1)$, hence the above series converges if and only if (2.1) converges.

We have proved that the convergence of series (2.1) implies that $f \in \ell^p(G)$. The converse implication is proved in a similar fashion by using estimate (2.3) instead of (2.2). $\square$

Proof of Theorem 3. **(a)** Suppose that $(a_n) = O(n^{2+\delta})$ for each $\delta > 0$. Then for any $f \in \mathcal{I}_{(a_n)}$ and any $\delta > 0$ there is $C_\delta > 0$ such that

$$\nu(f, \tfrac{1}{n}) n^{-(p+1)} \leq C_\delta \cdot n^{-p+1+\delta} \quad (n \in \mathbb{N}).$$

Therefore, series (2.1) converges for every $p > 2$ and hence Lemma 4 implies that

$$\mathcal{I}_{(a_n)} \subseteq \bigcap_{\varepsilon > 0} \ell^{2+\varepsilon}(G). \tag{2.4}$$

By the Cowling–Haagerup–Howe theorem [3], if $\pi\colon G \to \mathcal{B}(H)$ is a unitary representation of G with a cyclic vector $v \in H$ such that $\pi_{v,v} \in \bigcap_{\varepsilon>0} \ell^{2+\varepsilon}(G)$, then π is weakly contained in the regular representation λ, i.e. $\|\pi(x)\| \leq \|\lambda(x)\|$ for each $x \in G$.

Now, for any fixed $x \in \mathrm{C}^*_{\mathcal{I}_{(a_n)}}(G)$ we use Proposition 1 to pick a cyclic $\mathcal{I}_{(a_n)}$-representation π with $\pi(x) \neq 0$ (the restriction of a faithful $\mathcal{I}_{(a_n)}$-representation to a cyclic subspace). Then, as explained above, inclusion (2.4) implies that π is weakly contained in the regular representation. Therefore, x is not in the kernel of the canonical map $\mathrm{C}^*_{\mathcal{I}_{(a_n)}}(G) \to \mathrm{C}^*_r(G)$, which proves that $\mathrm{C}^*_{\mathcal{I}_{(a_n)}}(G) = \mathrm{C}^*_r(G)$.

(b) This is essentially [2, Remark 2.13] by Brown and Guentner. Notice that condition $(**)$ says that for any $f \in \mathcal{I}_{(a_n)}$ we have $f^k \in \ell^1(G)$. Indeed, let $C > 0$ be such that $\nu(f, \frac{1}{n}) \leq C a_n$. Then, the inequality $|f(x)|^k \geq n^{-k}$ holds true for at most $C a_n$ elements $x \in G$, hence $\|f^k\|_1 \leq \nu(f, 1) \cdot \|f\|_\infty + C \sum_{n \geq 2} a_n n^{-k} < \infty$.

Now, by Theorem 2, condition $(*)$ implies that there exists a sequence $(h_n) \subset \mathcal{I}_{(a_n)}$ of positive-definite functions converging pointwise to the constant one function. In view of $(**)$, we have $(h_n^k) \subset \ell^1(G)$; if $f_n \subset c_{00}(G)$ approximates the square root of h_n^k in $\mathrm{C}^*_r(G)$, then h_n^k is approximated by the finitely supported positive-definite functions $f_n^* f_n$. This yields $\mathrm{C}^*_r(G) = \mathrm{C}^*(G)$, i.e. G is amenable. $\square$

We conclude our note with a corollary which shows that if there is any ideal of the form $\mathcal{I}_{(a_n)}$ characterizing the Hyers–Ulam property for discrete groups, then (a_n) must grow quite rapidly. This follows from Theorem 3 and the fact that the Hyers–Ulam property is weaker than amenability.

Corollary 5. *If there exists a sequence* $(a_n) \subset \mathbb{R}_+$ *such that condition* $(*)$ *characterizes the Hyers–Ulam property, then* (a_n) *grows faster than any polynomial.*

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Declarations

Conflict of interest I declare that there is no conflict of interest.

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